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MODELLING AN AIRCRAFT MAXIMUM ENDURANCE HORIZONTAL FLIGHT FOR AIR TRIALS

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Abstract. The paper considers theoretical preparation for the aircraft pre-air-trial. The construction of some mathematical models of a horizontal flight is based upon the material system of variable mass motion. Optimal speed of horizontal flight is obtained as a function of variable mass. This speed is a solution (extremal) of the objective functional of the flying apparatus horizontal flight endurance. The solution delivers maximal value to the objective functional. The main significant assumptions made at the problem setting are: the rate of the aircraft horizontal flight speed change is negligibly small, flying object engines thrust has the horizontal component only, the dependence between aerodynamic coefficients is simplified in approximation with a quadratic parabola; the data used in simulation are abstract, although plausible. It was shown that in spite of the speed changes during the studied flight, the rate of that change plays an unimportant role for the considered case; therefore, such supposition of the rate neglecting is properly grounded. The derived equations allow taking into account the rate when it is the matter of importance. Since the presented study is the simplified one, the obtained results could be considered as some reference values to be tested and possibly approached to.

Keywords: aircraft, air trials, horizontal flight, optimal aircraft speed, variation, objective functional, maximum flight endurance, extremal.

Introduction

Aircraft perfection has a many criteria estimation as that follows from the book by Babister (1980). One of such criteria is the flight performance characteristic that is usually endorsed with the air trials documents. Apparently, all further flying object flight operation will depend upon the economic activity model (Silberberg & Suen, 2001) prescribed by the airline managerial staff in accordance with the subjective individuals' preferences considered in reference (Kasianov, 2013).

Flight operation, parameters testing, air trials, maximum flight endurance and range, all those and many other circumstances relate with the features of the specified flying apparatus (Bunge, 2015; Kasjanov, 1999; Olejniczak & Nowacki, 2019; Babister, 1980) and subjective preferences (Kasianov, 2013; Rohacs & Kasyanov, 2011; Goncharenko, 2014). The factors of the technical-economical-human systems (Goncharenko, 2018b, 2016) create the conditions of the necessity for optimization.

Aviation fuel savings are one of the priorities in any economic model as it is clear from the book by Silberberg and Suen (2001). On the other hand new and improved aeronautical materials, their properties, and treatment (Goncharenko, 2018a) are also at the target of economical pressure. Similar economical effects could be seen in the piloted versus unmanned aerial vehicles application dilemma (Hulek & Novák, 2019).

In such context the objectives of the presented research is close to the tasks mentioned in reference (2020); however the theoretical background deals with the motion of a dynamic or material system of variable mass (Kosmodemjanskiy, 1965, 1966).

Mathematical means and apparatus are based upon theoretical mechanics (Kosmodemjanskiy, 1965, 1966), matrix algebra (Korn & Korn, 2000), and calculus of variations (Kosmodemjanskiy, 1965, 1966; Korn & Korn, 2000).

Although a great deal of effort has already been put by the aerodynamic community to model aerodynamics of flying bodies, still there is a need of predicting the aircraft performance prior to her operation. The simplified, however principal, values are required at least being modeled and simulated. Therefore, to some degree a more thorough mathematical approach has to be tried to be implemented to the important topic of maximum endurance horizontal flight.

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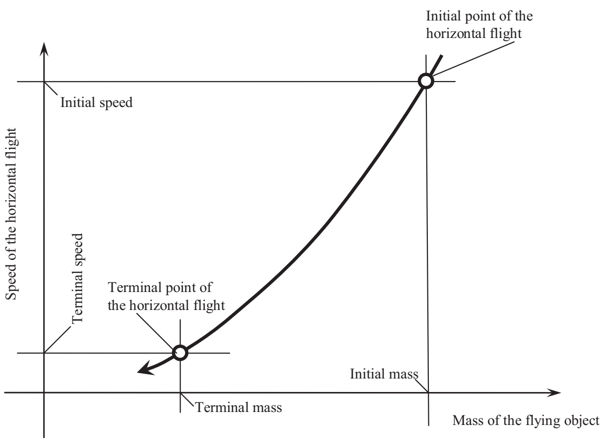

Figure 1. Infographic diagram and the related workflow

An infographic diagram of the work and the related workflow are presented in Figure 1.

As the dependence of the aerodynamic coefficients on the forces and moments are very complex to model, these non dimensional coefficients are obtained from the exhaustive wind tunnel tests. They are specific for an aircraft. It is proposed a simplified consideration allowing obtaining the optimal result in principle for a generalized aircraft. Such result could be a good reference value.

The problems of flight endurance and range optimization are not absolutely new (Sachs, 1992; Sachs & Grüter, 2020; Wang et al., 2019; Larsen et al., 2017). As it is clearly seen, the problem is constantly persuasive for the researchers all-over the world, beseeching them not quitting their endeavors. Two major aspects are noticeable: optimization of the aircraft design and operation mode. For example, in (Larsen et al., 2017) paper, design optimization of modular quadrotor UAV is performed for an operational case where the goal is to maximize the distance covered.

The presented study aimed at optimization of the flight mode. Thus, the problem remains urgent and actual.

Contribution of the article is the possibility to optimize the flight operation of the already manufactured copy of the aircraft. State-of-art in endurance computations likewise for the electrically driven flights, compared to fuel, with the elements of the environmental issues optimization (Donateoa et al., 2017) highlight the novelty aspects of the considered varied mass object motion dealing with the propulsion efficiency, calorific value of fuel, and speed explicitly. Also, proportionality of all that mentioned to the thrust is taken into account.

1. Schematic consideration of the prototypic problem statement

Maximum endurance (maximal duration) and maximum range (maximal distance) of the aircraft with a rocket engine horizontal flight has been considered in references Kosmodemjanskiy (1965, 1966). It has been obtained the following Equations in there:

$$f = \frac{v^2}{v_0^2}, \tag{1}$$

f – the function to be variated, or the free function: the law for the flying apparatus mass variation (Kosmodemjanskiy, 1966);
v – the speed of the flying object center of mass, it is also accepted by a supposition that the aircraft center of mass has no displacement with respect to the fuselage during the fuel burning out, therefore the vector differential equation of the center of mass motion will not differ from the equation of a dynamic system of variable mass motion;
v_0– the value of the initial speed of the flying apparatus horizontal flight;

and (Kosmodemjanskiy, 1966):

$$f = \left(\frac{A}{B}\right)^{\frac{1}{n}} v^2 \left[\frac{v + V_r}{(2n-1)V_r + (n-1)v}\right]^{\frac{1}{n}}, \tag{2}$$

A and B – the constant values determined by expressions of (Kosmodemjanskiy, 1966):

$$A = \frac{C_{x_0}\rho S}{2M_0}, \quad B = \frac{bg^n(2M_0)^{n-1}}{(\rho S)^{n-1}}, \tag{3}$$

C_{x_0} – the value of the drag (ram air flow resistance) aerodynamic coefficient when the value of the aerodynamic force of lift coefficient equals zero; ρ – the air density at the given altitude; S – the characterizing square area of the object; M_0 – the initial mass of the flying apparatus (at the initial moment of time when the aircraft is at the initial point of the rectilinear horizontal trajectory of its flight); b – a certain constant value determined at the specified speed diapason during blowing tests in the wind tunnels; g – the acceleration stipulated by the force of gravity; n – some constant determined analogously to the C_{x_0} and b coefficients; V_r – the effective velocity of the burning products ejection from the engine nozzle (Kosmodemjanskiy, 1966).

The Equations of (1) and (2) are extremals of the corresponding objective functionals (integrals) of the horizontal flight endurance and range (Kosmodemjanskiy, 1966) derived from the differential equations of the aircraft center of mass motion.

Thus, the results of Equations (1)–(3) are known already so the further efforts are put to find optimal speed of the horizontal flight for maximum endurance. All further derivations and formulas are the portion of the contribution. In case of the results coincidence it verifies the correctness of the independent search.

2. Proposed approach

The proposed concept implies the aircraft horizontal motion as the material point with the variable mass motion, however not in the style of reactive motion, unlike above Equations (1)–(3) (Kosmodemjanskiy, 1965, 1966). Also, some simplifications will be accepted by suppositions. It is going to be considered the maximum endurance; for further research it will be possible to continue in the style of reference Shigeru et al. (2020), for both range and endurance. Complication of the problem setting is for the future as well.

The materials in section 2 and the contributions and applications are to obtain the maximum endurance function and activate flight testing in accordance.

In fact, this concept is a combination of an approach to the endurance problem with explanations, simulations, and review elements.

2.1. The simplest consideration

The differential equations for the aircraft horizontal motion in the simplest case will get the following view of the algebraic equation:

$$0 = P - R,\ 0 = -G + Y, \tag{4}$$

P – the thrust developed by the aircraft engines; R – the aerodynamic force of the drag (ram resistance); G – the force of gravity; Y – the aerodynamic force of lift.

Here, in Equation (4), it is assumed that the speed of the flying object changes negligibly little. Hence, the linear acceleration deemed to be insignificant. In fact, this assumption is the easiest to be tested at the air trials.

Equations (4) became algebraic in the view although ideologically it derives from differential equations of motion.

In fact, it is quite obvious that for achieving an equilibrium state which is required for remaining in steady uniform flight, the main vector (resultant forces) and main moment (moments) about the aircraft's center of gravity should be zero. Such equilibrium for steady uniform flight is considered in the presented simplified problem setting. Namely, Equations of (4) are those zero conditions.

The weight/gravity terms are considered in Jones and Childers (1999), Hibbeler (2012, 2013), Galili (2001), Houghton and Carpenter (2003).

The proposed terminology and symbols are applied to be more accurate with the ideology of the paper.

It is proposed to take into account the components of the differential equations of Equation (4) roughly as

$$P = -\eta_H \frac{dm}{dt},\ R = C_x \frac{\rho v^2}{2} S, \tag{5}$$

η_H – the coefficient of the proportionality between the engine thrust and aviation fuel consumption; m – the mass of the flying object; t – time; C_x – the coefficient of the ram air flow resistance aerodynamic force;

and

$$G = mg,\ Y = C_y \frac{\rho v^2}{2} S, \tag{6}$$

C_y – the coefficient of the aerodynamic force of lift.

For the first approach, the coefficient of proportionality η_H can be evaluated as

$$\eta_H = \eta \frac{Q}{v}, \tag{7}$$

η – the coefficient of the useful action (efficiency) of the flying apparatus propulsion complex; Q – the calorific value per a unit mass of the burnt aviation fuel.

Thus, substituting the values of Equations (5)–(7) into expressions of Equations (4) one can get the system of Equations (8):

$$0 = -\eta \frac{Q}{v} \frac{dm}{dt} - C_x \frac{\rho v^2}{2} S,\ 0 = -mg + C_y \frac{\rho v^2}{2} S. \tag{8}$$

Assuming the "drag-polar" for C_x and C_y relation, which is valid only for the linear lift region, it is possible to derive the differential equation and corresponding integral from the system (Eq. (8)).

It will give the endurance of the horizontal flight (Eq. (9)):

$$T = \int_{M_0}^{M_E} -\frac{2\eta Q \rho v S}{C_{x_0}\left(\rho v^2 S\right)^2 + b\left(2mg\right)^2} dm, \tag{9}$$

M_E – the mass of the flying apparatus at the end of the active fragment of the horizontal flight.

In terms of calculus of variations, the problem of the maximum endurance of the horizontal flight formulated in the view of the integral of (Eq. (9)) extremization means the simplest problem with the solution of Equation (10), nevertheless unlike in references Kosmodemjanskiy (1965, 1966):

$$v_{opt}(m) = \sqrt[4]{\frac{4}{3} \frac{b}{C_{x_0}} \left(\frac{mg}{S\rho}\right)^2}. \tag{10}$$

The extremal (function) (Eq. (10)) can deliver extremal (maximal) value to the objective functional (integral) (Eq. (9)).

The necessary conditions for the exremal of (Eq. (10)) existence is the Euler-Lagrange equation.

2.2. Computer simulation

The solutions in the view of Eq. (10) as well as dependencies upon time satisfy the necessary conditions. Simulation allows to be sure if it is a real extremum and whether it is maximum. Then, the most important parameters and their values could be verified at the aircraft air trials.

The accepted data for the calculations are as following:

$$b = 0.045,\ g = 9.81\text{m/s}^2,\ C_{x_0} = 0.036,$$
$$S = 3.34\ \text{m}^2,\ \rho = 1.1\ \text{kg/m}^3,\ M_0 = 150\ \text{kg},\ \eta = 0.25,$$
$$Q = 32{,}000 \cdot 10^3\ \text{J/kg},\ M_E = 102\ \text{kg}. \tag{11}$$

In result, the extremal of Eq. (10) for the objective functional of Eq. (9) is plotted on the diagram in Figure 2.

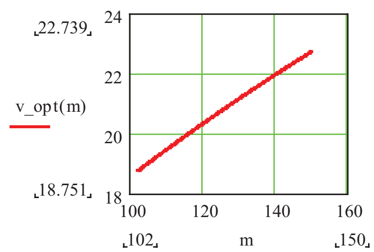

Figure 2. Optimal speed as a function of the variable mass of the flying object

The labeling of the respective axes in Figure 2 is as follows. The ordinate axis designation v_opt(m) stands for the values of $v_{opt}(m)$ obtained by Equation (10) with the use of the data of Equation (11). The abscissa axis m value is the independent variable of mass m.

As for the optimal aviation fuel consumption in a function of time, the extremal curve is presented in Figure 3.

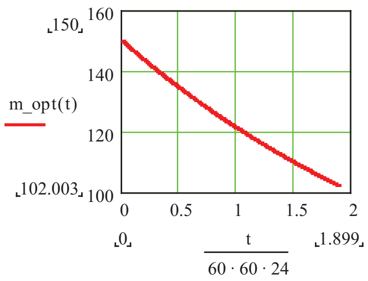

Figure 3. Optimal aviation fuel consumption as a function of the time for correspondent variable mass of the flying object

The labeling of the respective axes in Figure 3 is as follows. The ordinate axis designation m_opt(t) stands for the values of $m_{opt}(t)$ obtained by equation derived for the dependence upon time with the use of the data of Eq. (11). The abscissa axis $\frac{t}{60 \cdot 60 \cdot 24}$ value is the independent variable of time t.

In Figure 3, the time scale is reduced to 24 hours magnitude for a perception ease.

Next up in the given research is the optimal speed of the aircraft performing the horizontal flight as a function of time, too. The diagrams by formulae depending upon time are demonstrated in Figure 4.

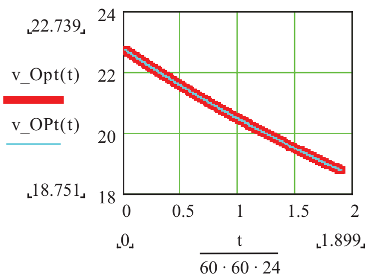

Figure 4. Optimal speed as a function of time for the corresponding variable mass of the flying object

In Figure 4, the curve of v_Opt(t) (bold red line) is plotted by the formula with the mass depending upon time and v_OPt(t) (light blue line) by the formula derived based upon the v_Opt(t) expression. Both curves coincide.

The phase portrait of $v_{opt}(t)$ and $m_{opt}(t)$ is shown in Figure 5.

In Figure 5, it is visible that both speed curves:

1. Phase trajectory of v_Opt(t) by m_opt(t) plotted by the formula with the mass depending upon time (bold red line); and
2. Extremal (Eq. (10)) of v_opt(m) shown in Figure 1 (light blue dashed line);

coincide.

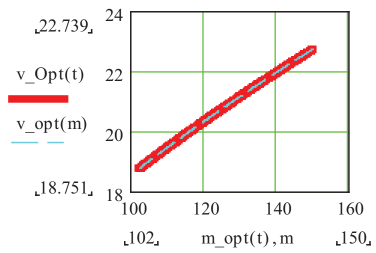

Figure 5. Phase trajectory of the optimal speed over optimal mass through the phase variable of time

3. Discussion

Since the obtained solution, suspected for delivering the extremeum at the extremal (Eq. (10)), satisfies only the necessary conditions, there is a need of checking the extremal for delivering a maximum to the objective functional of the aircraft horizontal flight endurance. It touches a complex of problems described in the introductory section, started from the purely aerodynamic (Bunge, 2015; Kasjanov, 1999; Olejniczak & Nowacki, 2019; Babister, 1980) and down to technical possibilities of materials (Goncharenko, 2018a), adjacent issues (Goncharenko, 2018b, 2016), expediency (Hulek & Novák, 2019), preferences (Kasianov, 2013; Rohacs & Kasyanov, 2011; Goncharenko, 2014), all for economic reasons (Silberberg & Suen, 2001). Also, some prospects of further research should be discussed.

3.1. The maximal extremality

Considering this variational problem with the fixed boundary points, one can give a small variation to the suspected extremal. It is proposed to variate solution (Eq. (10)) (see Figure 2) in the following way.

Using the vector-column of speeds for a variation greater than solution, the vector-column of the unknown coefficients for the variated function greater than the solution is, at the value of variation δ = 0.4 m/s (Korn & Korn, 2000):

$$\mathbf{A_M} = \begin{pmatrix} -8.604 \cdot 10^{-4} \\ 0.3 \\ -2.888 \end{pmatrix}. \qquad (12)$$

That approach applied for the variation of the solution function to be less than the exrremal (Eq. (10)) (see Figure 2) yields

$$\mathbf{A_m} = \begin{pmatrix} 5.285 \cdot 10^{-4} \\ -0.05 \\ 18.362 \end{pmatrix}. \qquad (13)$$

As a result, the functions as the variations of the solution with the solution itself are shown in Figure 6.

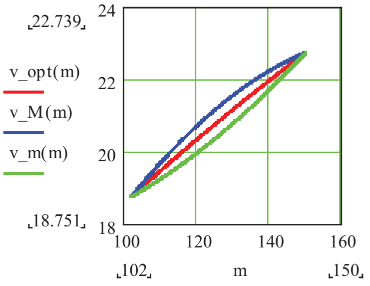

Figure 6. Extremal and its variations

The variation function greater than solution (Eq. (10)), v_opt(m) (red curve) (also shown in Figure 2), is depicted as v_M(m) (blue curve); and the variation function less than the solution is designated as v_m(m) (green curve) (see Figure 6).

Objective functional (Eq. (9)) has such expiring values for those functions illustrated in Figure 6.

$$T\left[v_{opt}\left(M_E\right)\right]=T\left[\mathrm{v\_opt}\left(\mathrm{M_E}\right)\right]=1.64113\cdot 10^5,$$

$$T\left[\mathrm{v\_M}\left(\mathrm{M_E}\right)\right]=1.64066\cdot 10^5,$$

$$T\left[\mathrm{v\_m}\left(\mathrm{M_E}\right)\right]=1.64066\cdot 10^5. \tag{14}$$

The advantages of the optimal solution can be traced in Figure 7.

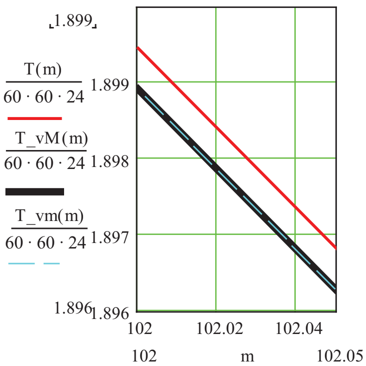

Figure 7. Optimal solution delivering the maximal value to the objective functional of the aircraft horizontal flight endurance

The labeling of the respective axes in Figure 7 is as follows. The ordinate axis designation $\frac{\mathrm{T(m)}}{60\cdot 60\cdot 24}$ stands for the values of endurance T obtained by Equation (9) with the use of the optimal speed (Eq. (10)) and data of (Eq. (11)). $\frac{\mathrm{T\_vM(m)}}{60\cdot 60\cdot 24}$ means the endurance of T obtained by the Equation of (Eq. (9)) with the use of the optimal speed (Eq. (10)) variation $\mathrm{v\_M(m)}$ whereas $\frac{\mathrm{T\_vm(m)}}{60\cdot 60\cdot 24}$ is T by (Eq. (9)) with the use of the (Eq. (10)) variation v_m(m) (see Figure 6). The abscissa axis m value is the independent variable of mass *m*.

The scales for the final stages of the modeled horizontal flight shown in Figure 7 have to be enlarged because the variation of δ is small enough. Nevertheless it is noticeable from the results of the demonstrated simulation that solution (Eq. (10)) is optimal since it delivers maximum to the objective functional (Eq. (9)).

Of course, the greater the variation of the extremal (Eq. (10)) is the more distinguishing extremum (maximum) the objective functional (Eq. (9)) has at the optimal (extremal) solution (Eq. (10)).

3.2. The contours of further research

As a development of the presented problem setting (Eqs (4)–(14)), it might be considered a step of generalization. For instance, it was assumed at the problem formulation that the speed of the flying object changes through the time of light negligibly small. However, as it is seen from the solution it changes, and apparently in some cases, it might be crucially necessary to take that fact into account or just at least to try to model such situation before the aircraft air trials.

In such circumstances, it can be proposed a modification in the system of Equations (4):

$$m\frac{dv}{dt}=P-R,\quad 0=-G+Y. \tag{15}$$

There is no doubt that this will make the problem much more difficult and harder to solve. Although, the approach will be as above:

$$m\frac{dv}{dt}=-\eta\frac{Q}{v}\frac{dm}{dt}-C_x\frac{\rho v^2}{2}S,\quad 0=-mg+C_y\frac{\rho v^2}{2}S. \tag{16}$$

From where

$$-\frac{\rho v^3 S+2\eta Q\rho v S}{C_{x_0}\left(\rho v^2 S\right)^2+b\left(2mg\right)^2}dm=dt\,. \tag{17}$$

And for extremal, likewise in case of Equations (9) and (10)

$$\begin{gathered}\left(3\rho v^2 S+2\eta Q\rho S\right)\left[C_{x_0}\left(\rho v^2 S\right)^2+b\left(2mg\right)^2\right]=\\ \left(\rho v^3 S+2\eta Q\rho v S\right)\left[4C_{x_0}\left(\rho S\right)^2 v^3\right].\end{gathered} \tag{18}$$

It is not so easy to extract the explicit dependence of speed upon mass from Eq. (18). Therefore, it is proposed a numerical method of iterations. Thus, the extremal solution can be approximated.

With the use of the abstract data and conditions for simulation (Eq. (11))

$$\begin{gathered}v_{opt}\left(150\right)^2=22.739\,,\; v_{opt}\left(140\right)^2=21.968\,,\\ v_{opt}\left(130\right)^2=21.169\,,\; v_{opt}\left(120\right)^2=20.338\,,\\ v_{opt}\left(110\right)^2=19.472\,,\; v_{opt}\left(100\right)^2=18.566\,.\end{gathered} \tag{19}$$

The very precise approximation for the extremal given at the discrete points of Eq. (19) leads to the result presented in Figure 8.

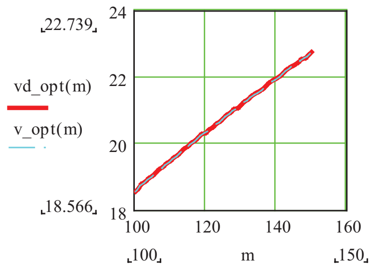

Figure 8. Practical coincidence of the extremals for the cases with taking into consideration the change of the aircraft speed and ignoring such change

In Figure 8, the curve of vd_opt(m) (red bold line) is plotted by the sixth order polynomial approximation dependence with the use of data (Eq. (19)). It can be noticed that both curves practically coincide, which says about the correctness of the supposition of the negligibly small rate of the optimal speed change in the discussed case. Although, for other circumstances it might be not so unimportant. Then the approach of Eqs (15)–(19) would be an aid in the research conduction and pre-air-trial preparation.

The other question is the angle of the thrust; in consideration of Eqs (4)–(14) it was supposed horizontal thrust. Also, the angle of attack can be taken into account. As well as, more generally, the combinations of the mentioned issues make sense to be investigated.

A few more simplifications of the presented research were the relation between aerodynamic coefficients (Eq. (9)) and the accepted initial data and conditions (Eq. (11)). All those can be more accurate for the simulation, specified for the aircraft type, for example.

3.3. Further details on the accepted data

Some extra details on the "accepted data" for the calculations include following:

$$b = 0.045,\ \eta = 0.25,\ Q = 32 \cdot 10^6,\ M_0 = 45 \cdot 10^3,$$
$$M_E = 30 \cdot 10^3,\ g = 9.8,\ C_{x_0} = 0.036,\ S = 34,\ \rho = 1.1. \quad (20)$$

It is for the generalized middle range aircraft type. The class of the simulated aircraft is close to An-26. The measurement units in Eq. (20) are set in correspondence (likewise of Eq. (11)) and just omitted. It is not for a specific aircraft. It is to illustrate the approach.

The results of computer simulation with the data of Eq. (20) are similar to those ones shown in Figures 2–8 obtained with the use of Eq. (11).

Conclusions

From the presented theoretical methods (Eqs (1)–(19)) for the aircraft maximum endurance horizontal flight pre-air-trial planning, illustrated with the examples (see Figures 1–8), there is a possibility to conclude that in the studied simplified case, with the supposition of the horizontal trust developed by the aircraft engines, negligibly small rate of the speed of flight change, stability of the flying object center of mass relatively to the aircraft fuselage, also abstract data for simulation, there is a promising potential for the flying apparatus to perform the mentioned optimal flight. Extremal of the optimal flight speed delivers maximal value to the objective functional of the flight endurance. Acceleration assumed insignificant at the problem setting finally proved to be so (see Figures 2, 4, 5, also 8).

The further research according to the proposed approach has some perspectives for the investigations of and optimization with the purpose of the aircraft maximum range horizontal flight performance.

Disclosure statement

The Author declares that he has not any competing financial, professional, or personal interests from other parties.

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Notations

Symbols and abbreviations

In the order of their appearance through the text:
f – the function to be variated, or the free function: the law for the flying apparatus mass variation;
v – the speed of the flying object center of mass;
v_0 – the value of the initial speed of the flying apparatus horizontal flight;
A and B – the constant values determined by expressions of Kosmodemjanskiy (1966);
C_{x_0} – the 'zero' value of the drag;
ρ – the air density at the given altitude;
S – the characterizing square area of the object;
M_0 – the initial mass of the flying apparatus (at the initial moment of time when the aircraft is at the initial point of the rectilinear horizontal trajectory of its flight);
b – a certain constant value determined at the specified speed diapason during blowing tests in the wind tunnels;
g – the acceleration stipulated by the force of gravity;
n – some constant determined analogously to the C_{x_0} and b coefficients;
V_r – the effective velocity of the burning products ejection from the engine nozzle (Kosmodemjanskiy, 1966);
P – the thrust developed by the aircraft engines;
R – the aerodynamic force of the drag;
G – the force of gravity;
Y – the aerodynamic force of lift;
η_H – the coefficient of the proportionality between the engine thrust and aviation fuel consumption;
m – the mass of the flying object;
t – time;
C_x – the coefficient of the ram air flow resistance aerodynamic force;
C_y – the coefficient of the aerodynamic force of lift;
η – the coefficient of the useful action (efficiency) of the flying apparatus propulsion complex;
Q – the calorific value per a unit mass of the burnt aviation fuel;
M_E – the mass of the flying apparatus at the end of the active fragment of the horizontal flight;
F – the under-integral function (integrand) of the objective functional (Eq. (9));
δ – the small variation for the solution (Eq. (10)) (see Figure 2).