ESTIMATION OF WARRANTY PERIOD FOR STRUCTURAL COMPONENTS OF AIRCRAFT

N.A. Nechval*, K.N. Nechval, E.K. Vasermanis***

**Mathematical Statistics Department, University of Latvia, Raina Blvd 19, LV-1050 Riga, Latvia.*
***Computer Science Department, Institute of Transport and Telecommunication, Lomonosov Street 1, LV-1019 Riga, Latvia. E-mail: nechval@junik.lv*
Received 01 10 2004, accepted 07 10 2004

Nicholas A. NECHVAL, Prof
Education: 1969 – Riga Civil Aviation Engineers Institute, PhD.
Affiliations and functions: 1993 – DSc degree in radio engineering at Riga Aviation University; Professor of Applied Mathematics at RAU from 1993 to 1999. At present, he is Professor of Mathematics and Computer Science at the University of Latvia.
Research interests: Mathematics, Stochastic Processes, Pattern Recognition, Multidimensional Statistical Detection and Estimation, Multiresolution Stochastic Signal Analysis, Digital Radar Signal Processing, Operations Research, Statistical Decision Theory, and Adaptive Control.
Professional memberships: Member of Latvian Statistical Association, Institute of Mathematical Statistics, CHAOS asambl (Institute of Mathematics, based in Liege, Belgium), Latvian Association of Professors.
Awards: 1992 – Silver Medal of the Exhibition Committee (Moscow, Russia) in connection with research on the problem of Prevention of Collisions between Aircraft and Birds.
Patents: Several patents in Radio Engineering.
Present position: Professor of Mathematics and Computer Science at the University of Latvia.

Konstantin N. NECHVAL, PhD student
Date and place of birth: 1975 in Riga, Latvia.
Education: 1998 – Aviation University of Riga, MS Degree.
Research interests: Stochastic Processes, Pattern Recognition, Operations Research, Statistical Decision Theory and Adaptive Control.
Present position: PhD student in automatic control and systems engineering at Riga Technical University

Edgars VASERMANIS, Prof
Education: 1974 – Riga Civil Aviation Engineers Institute (RCAEI), PhD degree in automatic control and systems engineering; 1989 – Tallinn University, DSc degree in economics and management.
Professional memberships: A professional member of the Latvian Statistical Association and the Institute of Mathematical Statistics. He is a member of the Latvian Association of Professors.
Research interests: Economics and management, operations research, statistical decision theory, and adaptive control.
Present position: Dean of the Faculty of Economics and Management at the University of Latvia.

Abstract. One of the most important problems in fatigue analysis and design of aircraft structures is the prediction of fatigue crack growth in service. Available in-service inspection data for various types of aircraft indicate that the fatigue crack damage accumulation in service involves considerable statistical variability. In this paper, we consider the problem of estimating the minimum time to crack initiation (or warranty period) for a number of aircraft structural components, before which no cracks (that may be detected) in materials occur, based on the results of previous warranty period tests on the structural components in question. This problem is a special case of a general class of problems concerned with the analysis of fatigue crack damage accumulation in aircraft service. The technique proposed here for solving this problem emphasizes pivotal quantities relevant for obtaining ancillary statistics. Attention is restricted to invariant families of distributions. Numerical examples are given.

Keywords: aircraft structures, fatigue cracks, warranty period, estimation.

Introduction

Aircraft structures have many components. Maintaining high reliability for these structures generally requires that the individual structural components have extremely high reliability, even after long periods of time. Prediction of fatigue crack growth in such components has not been an easy task. This is mainly because the manner in which the various parameters, such as loads, properties of materials and geometries of cracks, affect the propagation of cracks is not clearly understood [2]. This, consequently, has led to a proliferation of hypotheses and laws for describing the propagation fatigue cracks [2, 9, 1]. Most of these models are based on concepts of the continuum theory with the assumption that cracks propagate in an ideal continuum media. Actual metallic materials, however, are composed of random microstructures described by various micro parameters, which can seriously affect the growth of a crack in these materials. As a result, the deterministic theories can only be accepted as an approximation of the actual random fatigue crack propagation process, which, broadly speaking, has five phases:

1) Dormant. There are no cracks in the materials.
2) Nucleation. The crack is initially formed.
3) Micro-crack growth. The crack grows rather haphazardly up to about 1 mm in length.
4) Macro-crack growth. The crack continues to propagate before its growth rate finally increases dramatically.
5) Failure. The component fails; this occurs very quickly relative to the other phases and can be ignored as a factor in determining reliability.

In the Fracture Mechanics approach to fatigue problems it is assumed that failure is caused by the unstable growth of a leading crack, which initiates, propagates, and reaches a critical size due to the fluctuations of the stress field around the crack tip. The transition from the initiation to the propagation stages corresponds to the distinction made between micro- and macro-cracks. Once a crack has attained a certain threshold size, failure occurs very rapidly. Thus, statistical fatigue life of structural components of aircraft may be divided, in general, into three stages, namely, crack nucleation, small crack growth, and large crack growth. Crack nucleation and small crack growth show a wide variation and hence a big spread on a cycles versus crack length graph. Relatively, large crack growth shows a lesser variation. Therefore, different models are fitted to the different stages of the fatigue evolution process, thus treating different stages as different phenomena. With these independent models, it is impossible to predict one phenomenon based on the information available about the other phenomenon. Experimentally, it is easier to carry out crack length measurements of large cracks compared to nucleating cracks and small cracks. Thus, it is easier to collect statistical data for large crack growth compared to the painstaking effort it would take to collect statistical data for crack nucleation and small crack growth.

We consider in this paper the problem of estimating the minimum time to crack initiation (warranty period or time to a first inspection) for a number of aircraft structural components, before which no cracks (that may be detected) in materials occur, based on the results of previous warranty period tests on the structural components in question. If in a fleet of k aircraft there are km of the same individual structural components, operating independently, the length of time until the first crack initially forms in any of these components is of basic interest and provides a measure of assurance concerning the operation of the components in question. This leads to the consideration of the following problem. Suppose we have observations X_1, ... X_n as the result of tests conducted on the components; suppose also that there are km components of the same kind to be put into future use, with times to crack initiation Y_1, .., Y_{km}. Then we want to be able to estimate, on the basis of X_1, ... X_n, the shortest time to crack initiation $Y_{(1,km)}$ among the times to crack initiation Y_1, ... Y_{km}. In other words, it is desirable to construct lower simultaneous prediction limit, L_γ, that is exceeded with probability γ by observations or functions of observations of all k future samples, each consisting of m units. In this paper, the problem of estimating $Y_{(1,km)}$, the smallest of all k future samples of m observations from the underlying distribution, based on an observed sample of n observations from the same distribution, is considered. A solution is proposed for constructing a lower simultaneous prediction limit, L_γ, for $Y_{(1,km)}$. Various properties of these solutions are derived, and illustrations are given for some important special cases.

The results have a direct application in reliability theory, where the time until the first failure in a group of m items in service provides a measure of assurance regarding the operation of the items.

In this paper, attention is restricted to invariant families of distributions. The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics. It is a special case of the method of invariant embedding of sample statistics into a performance index applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space (i.e. in problems where there is a unique best invariant procedure) [4-7]. The analysis of the problem considered here is easily seen to be invariant under changes of location and scale.

1. Equation for constructing lower simultaneous one-sided prediction limits

An equation, which shows how to construct lower simultaneous one-sided prediction limits for the order statistics in all of future samples when a one-sided prediction limit for a single future sample is available, is given by the following theorem.

Theorem 1. Let (X_1, ..., X_n) be a random sample from the cdf F(.), and let $(Y_{1_j},\dots,Y_{m_j})$ be the jth random sample of m_j "future" observations from the same cdf, $j\in\{1, \dots ,k\}$. Assume that (k+1) samples are independent. Let $H=H(X_1, \dots ,X_n)$ be any statistic based on the preliminary sample and let $Y_{(r_j,m_j)}$ denote the r_jth order statistic in the jth sample of size m_j. Then

$$\Pr\left(Y_{(r_1,m_1)}\ge H,\dots,Y_{(r_j,m_j)}\ge H,\dots,Y_{(r_k,m_k)}\ge H\right)$$

$$=\sum_{i_1=0}^{r_1-1}\cdots\sum_{i_j=0}^{r_j-1}\cdots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\cdots\binom{m_j}{i_j}$$

$$\dots\binom{m_k}{i_k}\frac{\Pr\left(Y_{(i_\Sigma+1,m_\Sigma)}\ge H\right)-\Pr\left(Y_{(i_\Sigma,m_\Sigma)}\ge H\right)}{\binom{m_\Sigma}{i_\Sigma}}, \qquad (1)$$

where

$$i_\Sigma=\sum_{j=1}^{k} i_j, \qquad m_\Sigma=\sum_{j=1}^{k} m_j. \qquad (2)$$

Proof.

$$\Pr\left(Y_{(r_1,m_1)}\ge H,\dots,Y_{(r_j,m_j)}\ge H,\dots,Y_{(r_k,m_k)}\ge H\right)$$

$$=\prod_{j=1}^{k}\Pr\left(Y_{(r_j,m_j)}\ge H\right)$$

$$=E\left\{\prod_{j=1}^{k}\sum_{i_j=0}^{r_j-1}\binom{m_j}{i_j}\left[F(H)\right]^{i_j}\left[1-F(H)\right]^{m_j-i_j}\right\}$$

$$=\sum_{i_1=0}^{r_1-1}\dots\sum_{i_j=0}^{r_j-1}\dots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\dots\binom{m_j}{i_j}$$

$$\dots\binom{m_k}{i_k}E\left\{\left[F(H)\right]^{i_\Sigma}\left[1-F(H)\right]^{m_\Sigma-i_\Sigma}\right\}. \qquad (3)$$

Since

$$E\left\{\left[F(H)\right]^{i_\Sigma}\left[1-F(H)\right]^{m_\Sigma-i_\Sigma}\right\}$$

$$=\binom{m_\Sigma}{i_\Sigma}^{-1}\left[E\left\{\begin{aligned}&\sum_{i=0}^{i_\Sigma}\binom{m_\Sigma}{i}\left[F(H)\right]^{i}\left[1-F(H)\right]^{m_\Sigma-i}\\&-\sum_{i=0}^{i_\Sigma-1}\binom{m_\Sigma}{i}\left[F(H)\right]^{i}\left[1-F(H)\right]^{m_\Sigma-i}\end{aligned}\right\}\right]$$

$$=\frac{\Pr\left(Y_{(i_\Sigma+1,m_\Sigma)}\ge H\right)-\Pr\left(Y_{(i_\Sigma,m_\Sigma)}\ge H\right)}{\binom{m_\Sigma}{i_\Sigma}}, \qquad (4)$$

The joint probability can be written as

$$\Pr\left(Y_{(r_1,m_1)}\ge H,\dots,Y_{(r_j,m_j)}\ge H,\dots,Y_{(r_k,m_k)}\ge H\right)$$

$$=\sum_{i_1=0}^{r_1-1}\dots\sum_{i_j=0}^{r_j-1}\dots\sum_{i_k=0}^{r_k-1}\binom{m_1}{i_1}\dots\binom{m_j}{i_j}$$

$$\dots\binom{m_k}{i_k}\frac{\Pr\left(Y_{(i_\Sigma+1,m_\Sigma)}\ge H\right)-\Pr\left(Y_{(i_\Sigma,m_\Sigma)}\ge H\right)}{\binom{m_\Sigma}{i_\Sigma}}. \qquad (5)$$

This ends the proof.

Corollary 1. If $r_j=1$, $\forall j=1(1)\,k$, then

$$\Pr\left(Y_{(1,m_1)}\ge H,\dots,Y_{(1,m_j)}\ge H,\dots,Y_{(1,m_k)}\ge H\right)$$
$$=\Pr\left(Y_{(1,m_\Sigma)}\ge H\right). \qquad (6)$$

2. Invariant embedding technique for obtaining prediction limits

This paper is concerned with the implications of group theoretic structure for invariant performance indexes. We present an invariant embedding technique based on the constructive use of the invariance principle of mathematical statistics. This technique allows one to solve many problems of the theory of statistical inferences in a simple way. The aim of the present paper is to show how the invariance principle may be employed in the particular case of finding prediction limits. The technique used here is a special case of more general considerations applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space.

2.1. Preliminaries

Our underlying structure consists of a class of probability models ($\mathcal{X}$, $\mathcal{A}$, $\mathcal{P}$), a one-one mapping ψ taking $\mathcal{P}$ onto an index set Θ, a measurable space of actions ($\mathcal{U}$, $\mathcal{B}$), and a real-valued function r defined on Θ × $\mathcal{U}$. We assume that a group G of one-one $\mathcal{A}$ - measurable transformations acts on $\mathcal{X}$ and that it leaves the class of models ($\mathcal{X}$, $\mathcal{A}$, $\mathcal{P}$) invariant. We further assume that homomorphic images $\overline{G}$ and $\tilde{G}$ of G act on Θ and $\mathcal{U}$ respectively. ($\overline{G}$ may be induced on Θ through ψ; $\tilde{G}$ may be induced on $\mathcal{U}$ through r). We shall say that r is invariant if for every (θ, u) ∈ Θ × $\mathcal{U}$

$$r(\bar{g}\theta,\tilde{g}u)=r(\theta,u),\quad g\in G. \qquad (7)$$

Given the structure described above there are aesthetic and sometimes admissibility grounds for restricting attention to decision rules φ: $\mathcal{X}\to\mathcal{U}$, which are $(G,\tilde{G})$ equivariant in the sense that

$$\varphi(gx)=\tilde{g}\varphi(x),\quad x\in X,\quad g\in G. \qquad (8)$$

If $\overline{G}$ is trivial and (7), (8) hold, we say φ is G-invariant, or simply invariant [5].

2.2. Invariant Functions

We begin by noting that r is invariant in the sense of (9) if and only if r is a $G^\bullet$-invariant function, where $G^\bullet$ is defined on Θ × $\mathcal{U}$ as follows: to each g∈G, with homomorphic images $\bar{g},\tilde{g}$ in $\overline{G},\tilde{G}$ respectively, let $g^\bullet(\theta, u)=(\bar{g}\theta,\tilde{g}u)$, (θ, u)∈(Θ × $\mathcal{U}$). It is assumed that $\tilde{G}$ is a homomorphic image of $\overline{G}$.

Definition 1 (*Transitivity*). A transformation group $\overline{G}$ acting on a set Θ is called (uniquely) transitive if for every θ, ϑ∈Θ there exists (unique) $\bar{g}\in\overline{G}$ such that $\bar{g}\,\theta=\vartheta$.

When $\overline{G}$ is transitive on Θ we may index $\overline{G}$ by Θ: fix an arbitrary point θ∈Θ and define $\overline{g}_{\theta_1}$ to be the unique $\overline{g}\in\overline{G}$ satisfying $\overline{g}\,\theta=\theta_1$. The identity of $\overline{G}$ clearly corresponds to θ. An immediate consequence is Lemma 1.

Lemma 1 (*Transformation*). Let $\overline{G}$ be transitive on Θ. Fix θ∈Θ and define $\overline{g}_{\theta_1}$ as above. Then $\overline{g}_{\overline{q}\theta_1}=\overline{q}\overline{g}_{\theta_1}$ for θ∈Θ, $\overline{q}\in\overline{G}$.

Proof. The identity $\overline{g}_{\overline{q}\theta_1}\theta=\overline{q}\theta_1=\overline{q}\overline{g}_{\theta_1}\theta$ shows that $\overline{g}_{\overline{q}\theta_1}$ and $\overline{q}\overline{g}_{\theta_1}$ both take θ into $\overline{q}\theta_1$, and the lemma follows by unique transitivity.

Theorem 2 (*Maximal Invariant*). Let $\overline{G}$ be transitive on Θ. Fix a reference point $\theta_0\in\Theta$ and index $\overline{G}$ by Θ. A maximal invariant M with respect to $G^{\bullet}$ acting on Θ × 𝒰 is defined by

$$M(\theta,u)=\tilde{g}_{\theta}^{-1}u,\ \ (\theta,u)\in\Theta\times U\ . \tag{9}$$

Proof. For each (θ,u)∈(Θ × 𝒰) and $\overline{g}\in\overline{G}$

$$M(\overline{g}\theta,\tilde{g}u)=(\tilde{g}_{\overline{g}\theta}^{-1})\tilde{g}u=(\tilde{g}\tilde{g}_{\theta})^{-1}\tilde{g}u$$

$$=\tilde{g}_{\theta}^{-1}\tilde{g}^{-1}\tilde{g}u=\tilde{g}_{\theta}^{-1}u=M(\theta,u) \tag{10}$$

by Lemma 1 and the structure preserving properties of homomorphisms. Thus M is $G^{\bullet}$-invariant. To see that M is maximal, let M (θ_1,u_1)=M(θ_2,u_2). Then $\tilde{g}_{\theta_1}^{-1}u_1=\tilde{g}_{\theta_2}^{-1}u_2$ or $u_1=\tilde{g}\,u_2$, where $\tilde{g}=\tilde{g}_{\theta_1}\tilde{g}_{\theta_2}^{-1}$. Since $\theta_1=\overline{g}_{\theta_1}\theta_0=\overline{g}_{\theta_1}\overline{g}_{\theta_2}^{-1}\theta_2=\overline{g}\theta_2$, $(\theta_1,u_1)=g^{\bullet}(\theta_2,u_2)$ for some $g^{\bullet}\in G^{\bullet}$, and the proof is complete.

Corollary 2.1 (*Invariant Embedding*). An invariant function, r(θ,u), can be transformed as follows:

$$r(\theta,u)=r(\overline{g}_{\hat\theta}^{-1}\theta,\tilde{g}_{\hat\theta}^{-1}u)=\ddot{r}(v,\eta), \tag{11}$$

where $v=v(\theta,\hat\theta)$ is a function (a pivotal quantity) such that the distribution of v does not depend on θ; $\eta=\eta(u,\hat\theta)$ is an ancillary factor; $\hat\theta$ is the maximum likelihood estimator of θ (or the sufficient statistic for θ).

Corollary 2.2 (*Best Invariant Decision Rule*). If r(θ,u) is an invariant loss function, the best invariant decision rule is given by

$$\varphi^*(x)=u^*=\eta^{-1}(\eta^*,\hat\theta), \tag{12}$$

where

$$\eta^*=\arg\inf_{\eta}E_v\{\ddot{r}(v,\eta)\}. \tag{13}$$

Corollary 2.3 (*Risk*). A risk function (performance index)

$$R(\theta,\varphi(x))=E_x\{r(\theta,\varphi(x))\}=E_{v_\circ}\{\ddot{r}(v_\circ,\eta_\circ)\} \tag{14}$$

is constant on orbits when an invariant decision rule φ(x) is used, where $v_\circ=v_\circ(\theta,x)$ is a function whose distribution does not depend on θ; $\eta_\circ=\eta_\circ(u,x)$ is an ancillary factor.

For instance, consider the problem of estimating the location-scale parameter of a distribution belonging to a family generated by a continuous cdf F: 𝒫 ={P_θ: F ((x-μ)/σ), x∈R, θ∈Θ}, Θ={(μ,σ): μ,σ∈R, σ>0}=𝒰. The group G of location and scale changes leaves the class of models invariant. Since $\overline{G}$ induced on Θ by $P_\theta\to\theta$ is uniquely transitive, we may apply Theorem 2 and obtain invariant loss functions of the form

$$r(\theta,\varphi(x))=r[(\varphi_1(x)-\mu)/\sigma,\varphi_2(x)/\sigma], \tag{15}$$

if θ=(μ,σ) and φ(x)=(φ_1(x), φ_2(x)). Let $\hat\theta=(\hat\theta_1,\hat\theta_2)$, u=$(u_1,u_2)$, then

$$r(\theta,u)=\ddot{r}(v,\eta)=\ddot{r}(v_1+\eta_1v_2,\eta_2v_2), \tag{16}$$

where v=(v_1,v_2), $v_1=(\hat\theta_1-\mu)/\sigma$, $v_2=\hat\theta_2/\sigma$; η=(η_1,η_2), $\eta_1=(u_1-\hat\theta_1)/\hat\theta_2$, $\eta_2=u_2/\hat\theta_2$.

The invariant embedding technique, which is used for constructing lower simultaneous tolerance limits, is based on the result of Corollary 2.1.

3. Examples

Example 1. For instance, suppose that X_1, ... X_n and Y_{1j}, ... Y_{mj} (j=1, ... k) denote n+km independent and identically distributed random variables from a left-truncated Weibull distribution with pdf

$$f(x;a,b,\delta)=\frac{\delta}{\sigma}x^{\delta-1}\exp\left[-\left(x^{\delta}-\mu^{\delta}\right)/\sigma\right],\ \ x\geq\mu,\sigma,\delta>0, \tag{17}$$

which is characterized by being three-parameter (μ,σ,δ) where δ is termed the shape parameter, σ is the scale parameter, and μ is the truncation parameter interpreted as the minimum time to crack initiation (warranty period). It is assumed that the parameter δ is known. Let $X_{(1)}$ be the smallest observation in the initial sample of size n and

$$T_n=\sum_{i=1}^{n}(X_i^{\delta}-X_{(1)}^{\delta}). \tag{18}$$

It can be justified by using the factorization theorem that $(X_{(1)},T_n)$ is a sufficient statistic for (μ,σ). Let $Y_{(1,m_j)}$ be the smallest observation in the jth future sample of size m_j=m, ∀j=1(1) k. We wish, on the basis of a sufficient statistic $(X_{(1)},T_n)$ for (μ,σ), to construct simultaneous one-sided lower 100γ% prediction limits for $Y_{(1,m_j)}$, j=1, ...k. It follows from Corollary 1 that this problem reduces to the problem of constructing a lower 100γ% prediction limit, L_γ, for

$$Y_{(1,km)}=\min_{1\leq j\leq k}Y_{(1,m_j)}. \tag{19}$$

By using the above technique of invariant embedding of $(X_{(1)},T_n)$, if $X_{(1)}<Y_{(1,km)}$, or $(Y_{(1,km)},T_n)$, if

$X_{(1)} \ge Y_{(1,km)}$, into a pivotal quantity $(Y^{\delta}_{(1,km)} - \mu^{\delta})/\sigma$ or $(X^{\delta}_{(1)} - \mu^{\delta})/\sigma$, respectively, we obtain an ancillary statistic

$$W = \left(Y^{\delta}_{(1,km)} - X^{\delta}_{(1)}\right)/T_n \tag{20}$$

whose distribution does not depend on any unknown parameter. The pdf of W is given by

$$g(w) = \begin{cases} \dfrac{n(n-1)km}{n+km}\left(\dfrac{1}{1+kmw}\right)^{n}, & \text{if } w > 0, \\ \dfrac{n(n-1)km}{n+km}\left(\dfrac{1}{1-nw}\right)^{n}, & \text{if } w \le 0. \end{cases} \tag{21}$$

Therefore, in this case L_γ can be found explicitly as

$$L_\gamma = \begin{cases} \left(X^{\delta}_{(1)} + \dfrac{T_n}{km}\left[\left(\dfrac{n}{\gamma(n+km)}\right)^{\frac{1}{n-1}} - 1\right]\right)^{1/\delta}, & \text{if } \dfrac{n}{n+km} > \gamma, \\ \left(X^{\delta}_{(1)} + \dfrac{T_n}{n}\left[1 - \left(\dfrac{km}{(1-\gamma)(n+km)}\right)^{\frac{1}{n-1}}\right]\right)^{1/\delta}, & \text{if } \dfrac{n}{n+km} \le \gamma. \end{cases} \tag{22}$$

If, for instance, n=10, δ=8, k=3, m=5, γ=0.95, $X_{(1)}$=5 (in number of 10^4 flight-hours), and T_n=10917240. Then we find from (22) that, with n/(n+km) = 10/(10+15) < γ,

$$L_\gamma = \left(X^{\delta}_{(1)} + \frac{T_n}{n}\left[1 - \left(\frac{km}{(1-\gamma)(n+km)}\right)^{\frac{1}{n-1}}\right]\right)^{1/\delta}$$

$$= \left(5^8 + \frac{10917240}{10}\left[1 - \left(\frac{15}{(0.05)(10+15)}\right)^{\frac{1}{9}}\right]\right)^{1/8} = 3.8 \tag{23}$$

and we have 95% assurance that no cracks will occur in aircraft structural components before L_γ=3.8 (×10^4) flight-hours.

Example 2. Let $X_{(1)} < X_{(2)} < \ldots < X_{(r)}$ be the first r ordered observations of time to crack initiation for identical structural components of aircraft from a sample of size n from a two-parameter Weibull distribution with probability density function

$$f(x;\sigma,\delta) = \begin{cases} \dfrac{\delta}{\sigma}\left(\dfrac{x}{\sigma}\right)^{\delta-1}\exp[-(x/\sigma)^{\delta}], & x \ge 0, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

where the parameters σ and δ (σ>0, δ>0) are unknown. Two types of censoring are generally recognized. In Type I censoring, the time, when censoring occurs, is fixed, and the number of survivors at this time are random variables. In Type II censoring, which is of primary interest here, the number of survivors is fixed and $X_{(r)}$ is a random variable. In Type II censoring, the likelihood may be written as follows:

$$L \propto \left(\frac{\delta}{\sigma}\right)^{r}\left(\prod_{i=1}^{r}\left(\frac{x_{(i)}}{\sigma}\right)^{\delta-1}\right)\exp\left(-\sum_{i=1}^{r}\left(\frac{x_{(i)}}{\sigma}\right)^{\delta}\right)$$

$$\times\left[\int_{x_{(r)}}^{\infty}\frac{\delta}{\sigma}\left(\frac{x}{\sigma}\right)^{\delta-1}\exp[-(x/\sigma)^{\delta}]dx\right]^{n-r}$$

$$\propto \left(\frac{\delta}{\sigma^{\delta}}\right)^{r}\left(\prod_{i=1}^{r}x^{\delta-1}_{(i)}\right)\exp\left(-\frac{1}{\sigma^{\delta}}\left[\sum_{i=1}^{r}x^{\delta}_{(i)} + (n-r)x^{\delta}_{(r)}\right]\right), \tag{25}$$

$$\ln L = \text{constant} + r(\ln\delta - \delta\ln\sigma) + (\delta-1)\sum_{i=1}^{r}\ln x_{(i)} - \frac{\sum_{i=1}^{r}x^{\delta}_{(i)} + (n-r)x^{\delta}_{(r)}}{\sigma^{\delta}}. \tag{26}$$

This leads to the likelihood equations

$$\frac{\partial \ln L}{\partial\sigma} = -\frac{r\delta}{\sigma} + \frac{\sum_{i=1}^{r}x^{\delta}_{(i)} + (n-r)x^{\delta}_{(r)}}{\sigma^{2\delta}}\delta\sigma^{\delta-1} = 0, \tag{27}$$

$$\frac{\partial \ln L}{\partial\delta} = \frac{r}{\delta} - r\ln\sigma + \sum_{i=1}^{r}\ln x_{(i)} - \left(\sum_{i=1}^{r}x^{\delta}_{(i)}\ln x_{(i)} + (n-r)x^{\delta}_{(r)}\ln x_{(r)}\right)\sigma^{-\delta}$$
$$+\left(\sum_{i=1}^{r}x^{\delta}_{(i)} + (n-r)x^{\delta}_{(r)}\right)\sigma^{-\delta}\ln\sigma = 0. \tag{28}$$

Then the MLE's $\hat{\sigma}$ and $\hat{\delta}$ are solutions of

$$\hat{\sigma} = \left(\frac{\sum_{i=1}^{r}x^{\hat{\delta}}_{(i)} + (n-r)x^{\hat{\delta}}_{(r)}}{r}\right)^{1/\hat{\delta}}, \tag{29}$$

$$\hat{\delta} = \left[\left(\sum_{i=1}^{r}x^{\hat{\delta}}_{(i)}\ln x_{(i)} + (n-r)x^{\hat{\delta}}_{(r)}\ln x_{(r)}\right)\times\left(\sum_{i=1}^{r}x^{\hat{\delta}}_{(i)} + (n-r)x^{\hat{\delta}}_{(r)}\right)^{-1} - \frac{1}{r}\sum_{i=1}^{r}\ln x_{(i)}\right]^{-1}. \tag{30}$$

The results given here apply to the Weibull distribution in the form (24). The results are presented more naturally, however, if we consider the variable lnX, which follows the extreme-value distribution,

$$f(\ln x; a, b) = \frac{1}{b}\exp\left(\frac{\ln x - a}{b}\right)\exp\left(-\exp\left(\frac{\ln x - a}{b}\right)\right), \tag{31}$$
$$-\infty < \ln x < \infty,$$

where a = lnσ and b = δ^{-1}. Now (31) is a distribution with location and scale parameters a and b, and it is well known that if $\hat{a}$, $\hat{b}$ are maximum likelihood estimates for a, b from a complete sample of size n, then $(\hat{a}-a)/b$, $(\hat{a}-a)/\hat{b}$ and $\hat{b}/b$ are quantities whose distributions depend only on n.

We are interested in estimating $Y_{(1,km)}$, the smallest order statistic in all k future samples, each consisting of m units from the distribution (24). It is easily shown that

$$W=(\ln Y_{(1)}-\hat{a})/\hat{b}=\hat{\delta}(\ln Y_{(1)}-\ln\hat{\sigma})=\hat{\delta}\ln\left(\frac{Y_{(1)}}{\hat{\sigma}}\right), \quad (32)$$

where $Y_{(1)} \equiv Y_{(1,km)}$, is parameter-free, with distribution depending only on n and km. Hence, probability statements for V lead to confidence interval statements for $Y_{(1)}$.

Let $X_{(1)}$, $X_{(2)}$, ... $X_{(n)}$ and $Y_{(1)}$, $Y_{(2)}$, ... $Y_{(km)}$ represent ordered observations. In particular, let $X_{(1)} < X_{(2)} < \ldots < X_{(r)}$ be the first r ordered observations from a sample of size n from the distribution (24), i.e. we deal with Type II censoring. The joint density of $\ln X_{(1)} \ldots \ln X_{(r)}$ is

$$f(\ln x_{(1)},\ldots,\ln x_{(r)};a,b)$$

$$=\frac{n!}{(n-r)!}\prod_{i=1}^{r} b^{-1}\exp\left(\frac{\ln x_{(i)}-a}{b}-\exp\left(\frac{\ln x_{(i)}-a}{b}\right)\right)$$

$$\times\exp\left(-(n-r)\exp\left(\frac{\ln x_{(r)}-a}{b}\right)\right). \quad (33)$$

Let $\hat{a}$, $\hat{b}$ be the maximum likelihood estimators of a, b, based on $X_{(1)}, \ldots X_{(r)}$, and let $V_1=(\hat{a}-a)/b$, $V_2=\hat{b}/b$, and ${}^{\bullet}W_i=(\ln X_{(i)}-\hat{a})/\hat{b}$ (i=1, …, r). It is easily shown that the distributions of V_1, V_2 are parameter-free, and that any r-2 of the ${}^{\bullet}W_i$'s, say ${}^{\bullet}W_1, \ldots {}^{\bullet}W_{r-2}$, form a set of r-2 functionally independent ancillary statistics. We then find in a straightforward manner that the joint density of V_1, V_2, conditional on fixed ${}^{\bullet}\mathbf{W}=({}^{\bullet}W_1, \ldots {}^{\bullet}W_{r-2})$, is

$$f(v_1,v_2;{}^{\bullet}w_1,\ldots,{}^{\bullet}w_{r-2})$$

$$=\vartheta({}^{\bullet}\mathbf{w})v_2^{r-2}\exp\left(\begin{array}{l} rv_1+v_2\sum_{i=1}^{r}{}^{\bullet}w_i-\sum_{i=1}^{r}\exp(v_1+{}^{\bullet}w_iv_2) \\ -(n-r)\exp(v_1+{}^{\bullet}w_iv_2)\end{array}\right), \quad (34)$$

where $\vartheta({}^{\bullet}\mathbf{w})$ is a normalizing constant. For notational convenience we include all of ${}^{\bullet}w_1, \ldots {}^{\bullet}w_r$ in (34); ${}^{\bullet}w_{r-1}$ and ${}^{\bullet}w_r$ can be expressed as function of ${}^{\bullet}w_1, \ldots {}^{\bullet}w_{r-2}$ only.

Let $Y_{(1)}$ be the smallest observation from an independent second sample of km observations also from the distribution (24). Writing $V=(\ln Y_{(1)}-a)/b$ and noting that exp(V) is the smallest observation in a sample of size km from the standard exponential distribution, we have the density of V as

$$f(v)=km\exp(v)\exp(-km\exp(v). \quad (35)$$

Since V is distributed independently of V_1, V_2 we find the joint density of V, V_1, V_2, conditional on ${}^{\bullet}\mathbf{W}={}^{\bullet}\mathbf{w}$, as the product of (34) and (35). Note that $W=(\ln Y_{(1)}-\hat{a})/\hat{b}$ =(V-V_1)/V_2; making the transformation W=(V-V_1)/V_2, V_1=V_1, V_2=V_2, we find the joint density of W, V_1, V_2, conditional on ${}^{\bullet}\mathbf{W}={}^{\bullet}\mathbf{w}$, as

$$f(w,v_1,v_2;{}^{\bullet}\mathbf{w})$$

$$=km\vartheta({}^{\bullet}\mathbf{w})\exp\left((r+1)v_1+\left(w+\sum_{i=1}^{r}{}^{\bullet}w_i\right)v_2\right)$$

$$\times\exp[-km\exp(v_1+wv_2)]$$

$$\times\exp\left(-\exp(v_1)\left(\sum_{i=1}^{r}\exp({}^{\bullet}w_iv_2)+(n-r)\exp({}^{\bullet}w_iv_2)\right)\right). \quad (36)$$

Now v_1 can be integrated out of (36) in a straightforward way to give

$$f(w,v_2;{}^{\bullet}\mathbf{w})$$

$$=\frac{km\vartheta({}^{\bullet}\mathbf{w})v_2^{r-1}\exp\left(\left(w+\sum_{i=1}^{r}{}^{\bullet}w_i\right)v_2\right)}{\left(km\exp(wv_2)+\sum_{i=1}^{r}\exp({}^{\bullet}w_iv_2)+(n-r)\exp({}^{\bullet}w_rv_2)\right)^{r+1}}. \quad (37)$$

Consider, for fixed w (–∞<w<∞),

$$\Pr\{W>w;{}^{\bullet}\mathbf{w}\}=\int_0^{\infty}\int_w^{\infty}f(w,v_2;{}^{\bullet}\mathbf{w})dwdv_2. \quad (38)$$

A straightforward integration then gives us

$$\Pr\{W>w;{}^{\bullet}\mathbf{w}\}=\Pr\left\{\hat{\delta}\ln\left(\frac{Y_{(1)}}{\hat{\sigma}}\right)>w;{}^{\bullet}\mathbf{w}\right\}$$

$$=\left(\begin{array}{l}\int_0^{\infty}s^{r-2}e^{s\hat{\delta}\sum_{i=1}^{r}\ln(x_{(i)}/\hat{\sigma})} \\ \times\left(kme^{sw}+\sum_{i=1}^{r}e^{s\hat{\delta}\ln(x_{(i)}/\hat{\sigma})}+(n-r)e^{s\hat{\delta}\ln(x_{(r)}/\hat{\sigma})}\right)^{-r}ds\end{array}\right)$$

$$\times\left(\int_0^{\infty}s^{r-2}e^{s\hat{\delta}\sum_{i=1}^{r}\ln(x_{(i)}/\hat{\sigma})}\left(\sum_{i=1}^{r}e^{s\hat{\delta}\ln(x_{(i)}/\hat{\sigma})}+(n-r)e^{s\hat{\delta}\ln(x_{(r)}/\hat{\sigma})}\right)^{-r}ds\right)^{-1}. \quad (39)$$

The above expression holds for 3≤r≤n, with r=n corresponding to complete (uncensored) sampling. In the case r=2, $\hat{a}$, $\hat{b}$ are jointly sufficient for a, b, so that it can be considered the unconditional probability Pr(W>w). It will be noted that in this case, the correct expression is also given by (39), with r=2. Now the probability statement

$$\Pr\left\{\hat{\delta}\ln\left(\frac{Y_{(1)}}{\hat{\sigma}}\right) > w; {}^{\bullet}\mathbf{w}\right\} = \gamma \qquad (40)$$

leads to the warranty period $(0, \hat{\sigma}\exp(w/\hat{\delta}))$ with confidence level γ, i.e. a lower $100\gamma\%$ prediction limit, L_γ, for $Y_{(1)}$ is equal to $\hat{\sigma}\exp(w/\hat{\delta})$.

For instance, consider the data of fatigue tests on a particular type of structural component of the aircraft IL-86. The data are for a complete sample of size r = n = 5, with observations

Table. The Data of Fatigue Tests on a Particular Type of Structural Component of IL-861 Aircraft

Observations	Time to crack initiation (in number of 10^4 flight-hours)
$x_{(1)}$	5
$x_{(2)}$	6.25
$x_{(3)}$	7.5
$x_{(4)}$	7.9
$x_{(5)}$	8.1

The results are being expressed here in number of 10^4 flight-hours. On the basis of these data, the wish is to estimate a lower 0.95 prediction limit on $Y_{(1)}$ in a group of m = 5 identical components (for a fleet of k=1 IL-86 aircraft) that are to be put into service.

Goodness-of-fit testing. We assess the statistical significance of departures from the Weibull model by performing the empirical distribution function goodness-of-fit test. We use the S statistic [3]. For censoring (or complete) data sets, the S statistic is given by

$$S = \frac{\sum_{i=[r/2]+1}^{r-1}\left(\frac{\ln(x_{(i+1)}/x_{(i)})}{M_i}\right)}{\sum_{i=1}^{r-1}\left(\frac{\ln(x_{(i+1)}/x_{(i)})}{M_i}\right)} = \frac{\sum_{i=3}^{4}\left(\frac{\ln(x_{(i+1)}/x_{(i)})}{M_i}\right)}{\sum_{i=1}^{4}\left(\frac{\ln(x_{(i+1)}/x_{(i)})}{M_i}\right)} = 0.184, \qquad (41)$$

where [r/2] is a largest integer $\le r/2$, the values of M_i are given in Table 13 [3]. The reject region for the α level of significance is $\{S > S_{n;1-\alpha}\}$. The percentage points for $S_{n;1-\alpha}$ were given by Kapur and Lamberson [3]. For this example,

$$S = 0.184 < S_{n=5;\,1-\alpha=0.95} = 0.86. \qquad (42)$$

Thus, there is no evidence to rule out the Weibull model.

The maximum likelihood estimates are $\hat{\sigma} = 7.42603$ and $\hat{\delta} = 7.9081$. It follows from (39) that

$$\Pr\left\{\hat{\delta}\ln\left(\frac{Y_{(1)}}{\hat{\sigma}}\right) > -4.55761\right\} = \frac{0.000016192}{0.0000170442} = 0.95 \qquad (43)$$

and a lower 0.95 prediction limit for $Y_{(1)}$ is 4.1730 ($\times 10^4$) flight-hours, i.e. we have obtained the warranty period equal to 41730 flight-hours with confidence level γ=0.95.

Conclusions

In this paper we consider the important situation in which it can be assumed that the structural components of the aircraft in question have the time to crack initiation following the Weibull distribution. It will be noted that the general problem considered here, that of predicting on the basis of an ordered sample the smallest observation $Y_{(1,km)}$ from k future samples, each consisting of m units, has application in reliability theory other than described above. For example, if one has a series system consisting of m identical components, with lifetimes $Y_1, \ldots Y_m$, then $Y_{(1)} \equiv Y_{(1,m)}$ represents the life of the system; it is often wished to estimate $Y_{(1)}$ for a given system, on the basis of previous life test data on the components.

Acknowledgments

The authors wish to acknowledge the partial support of this research by the Latvian Council of Science and the National Institute of Mathematics and Informatics of Latvia under Grant No. 02.0918 and Grant No. 01.0031.

References

1. **Chand S., Garg S.B.L.** Propagation Under Constant Amplitude Loading // Engineering Fracture Mechanics. – 1985. – Vol. 21. – P. 1–30.
2. **Hoeppner D.W., Krupp W.E.** Prediction of Component Life by Application of Fatigue Crack Growth Knowledge // Engineering Fracture Mechanics. – 1974. – Vol. 6. – P. 47–70.
3. **Kapur K.C., Lamberson L.R.** Reliability in Engineering Design. – John Wiley & Sons, 1977.
4. **Nechval N.A.** Modern Statistical Methods of Operations Research. – Riga: RCAEI, 1982.
5. **Nechval N.A.** Theory and Methods of Adaptive Control of Stochastic Processes. – Riga: RCAEI, 1984.
6. **Nechval N.A., Nechval K.N.** State Estimation of Stochastic Systems via Invariant Embedding
7. Technique // Cybernetics and Systems'2000 / Edited by R. Trappl. – Vienna: Austrian Society for Cybernetic Studies, 2000. – Vol. 1. – P. 96–101.
8. **Nechval N.A., Nechval K.N., Vasermanis E.K.** Invariant Embedding Technique and its Applications to Statistical Decision-Making // Proceedings of the Second World Congress of Latvian Scientists. – Riga: LZP, 2001. – P. 578.
9. **Nechval N.A., Nechval K.N., Vasermanis E.K.** Optimization of Interval Estimators via Invariant Embedding Technique // IJCAS (International Journal of Computing Anticipatory Systems). – 2001. – Vol 9. – P. 241–255.
10. **Schwalbe K-H.** Comparison of Several Fatigue Crack Propagation Laws With Experimental Results // Engineering Fracture Mechanics. –1974. – Vol 6. – P. 325–341.