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ADJOINT APPROACH SENSITIVITY ANALYSIS OF THIN-WALLED BEAMS AND FRAMES

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Abstract. Sensitivity analysis of beams and frames assembled of thin-walled members is presented within the adjoint approach. Static loads and structures composed of thin-walled members with the bisymmetrical open cross-section are considered. The analysed structure is represented by the one-dimensional model consisting of thin-walled beam elements based on the classical assumptions of the theory of thin-walled beams of non-deformable cross-section together with superelements applied in place of location of structure nodes, restraints and stiffeners. The results of sensitivity analysis, obtained for the structure model described above, are compared with the results of the detailed FEM model, where the whole structure is discretised with the use of QUAD4 shell elements of the system MSC/NASTRAN.

Keywords: thin-walled structures, beams, frames, stiffeners, sensitivity analysis.

1. Introduction

Behaviour of beams and frames assembled of thin-walled members depends on their geometrical and material characteristics, as well as on parameters of additional elements like restraints and stiffeners. In designing, especially optimal designing and parametric identification, it is very useful to have the first variation of the response variable expressed by the design variable variations. This relation can be obtained with an aid of the sensitivity theory [1, 2]. The scope of the present research is limited to structures composed of thin-walled members with the bisymmetrical open cross-section. The sensitivity analysis of arbitrary displacements, internal forces or reactions due to certain variations of the design variables can be performed by an analytical or numerical method. The former can be used only in simple cases of one-dimensional beams. To analyse more complicated structures like frames or beams with warping stiffeners one should use the numerical method. The computational model of the structure employed in the numerical method [3] consists of thin-walled beam elements [4] based on the classical assumptions of the theory of thin-walled beams of non-deformable cross-section [5] combined with superelements applied in place of location of structure nodes, restraints and stiffeners [6, 7]. Such a model captures the effect of deformable cross-section and the stress transfer mechanism within the regions of frame nodes and in the areas near stiffeners.

To verify the proposed strategy the results of the sensitivity analysis are compared with the results of the static analysis for changed values of the design variables. To obtain an additional reference solution the detailed FEM model has been employed in which the whole frame is represented as an assembly of QUAD4 shell finite elements of the MSC/NASTRAN [8].

2. Numerical method in the sensitivity analysis

It is assumed that the sensitivity analysis can be done in the same discrete manner as the static FEM analysis of thin-walled beams and frames carried out by using the superelements, as proposed in [7]. The fundamental matrix equation of equilibrium of the structure is]

$$\mathbf{K}(\mathbf{x})\mathbf{s} = \mathbf{P}, \tag{1}$$

where x denotes the design variable vector, s is the state variable vector, K stands for the stiffness matrix and P represents the load vector. In the sequel it is assumed that dimensions of vectors x and s are n and m respectively.

We are searching for the variation of a function $f(\mathrm{x, s})$ due to arbitrary variation of the design variable. The first variation of this function due to a component x_i of the design variables vector x, can be written as

$$\delta f = \frac{\partial f}{\partial x_i}\delta x_i + \frac{\partial f}{\partial \mathbf{s}}\frac{d\mathbf{s}}{dx_i}\delta x_i \,. \tag{2}$$

In order to obtain the unknown derivative ds/dxi one can differentiate both sides of (1):

$$\mathbf{K}\frac{d\mathbf{s}}{dx_i}+\frac{d\mathbf{K}}{dx_i}\mathbf{s}=\frac{d\mathbf{P}}{dx_i} \qquad (3)$$

and hence arrive at

$$\frac{d\mathbf{s}}{dx_i}=\mathbf{K}^{-1}\left(\frac{d\mathbf{P}}{dx_i}-\frac{d\mathbf{K}}{dx_i}\mathbf{s}\right) \qquad (4)$$

Substituting relation (4) into (2), one can derive the variation of function under consideration linearly related to the design variable variation

$$\delta f=\left[\frac{\partial f}{\partial x_i}+\frac{\partial f}{\partial \mathbf{s}}\mathbf{K}^{-1}\left(\frac{d\mathbf{P}}{dx_i}-\frac{d\mathbf{K}}{dx_i}\mathbf{s}\right)\right]\delta x_i. \qquad (5)$$

Equation (5) is a basis of a direct method of sensitivity analysis. Unfortunately, in case of a large dimension of the stiffness matrix K, it is a very burdensome task to find its inverse. Evidently, instead of applying (5) it is possible to solve equation (3) *n* times with respect to derivative ds/dx_i, and substitute it directly into (2) to obtain the desired variation of the function *f*. Derivatives dP/dx_i and dK/dx_i can be estimated from the following difference relations

$$\frac{d\mathbf{P}}{dx_i}=\frac{\mathbf{P}(x_i+\Delta x_i)-\mathbf{P}(x_i)}{\Delta x_i},\quad \frac{d\mathbf{K}}{dx_i}=\frac{\mathbf{K}(x_i+\Delta x_i)-\mathbf{K}(x_i)}{\Delta x_i} \qquad (6)$$

or by central differences

$$\frac{d\mathbf{P}}{dx_i}=\frac{\mathbf{P}(x_i+\Delta x_i)-\mathbf{P}(x_i-\Delta x_i)}{2\Delta x_i},\quad \frac{d\mathbf{K}}{dx_i}=\frac{\mathbf{K}(x_i+\Delta x_i)-\mathbf{K}(x_i-\Delta x_i)}{2\Delta x_i}. \qquad (7)$$

Sometimes it is more convenient to apply the method of adjoint system [1, 2]. The adjoint variables vector l is introduced with the following relation

$$\mathbf{K}\bullet=-\left(\frac{\partial f}{\partial \mathbf{s}}\right)^{\mathrm{T}}, \qquad (8)$$

where superscript T denotes the transposition. It should be noted that (8) has the same structure as equation (1), thus the adjoint system is the same as the structure under consideration, except it is subjected to different adjoint loads defined by right side of (8). Having calculated the adjoint variable vector l from (8) and taking the advantage of the symmetry of the stiffness matrix K one can find that

$$\left(\frac{\partial f}{\partial \mathbf{s}}\right)\mathbf{K}^{-1}=-\bullet^{\mathrm{T}} \qquad (9)$$

what can be used in equation (5) to obtain

$$\delta f=\left[\frac{\partial f}{\partial x_i}+\bullet^{\mathrm{T}}\left(\frac{d\mathbf{P}}{dx_i}-\frac{d\mathbf{K}}{dx_i}\mathbf{s}\right)\right]\delta x_i. \qquad (10)$$

It is observed that in many cases of engineering practice the load vector P and the function *f* under consideration are independent of the design variables, therefore the derivatives df/dx_i and dP/dx_i vanish and (10) takes a simplified for

$$\delta f=\lambda^{\mathrm{T}}\frac{d\mathbf{K}}{dx_i}\mathbf{s}\,\delta x_i. \qquad (11)$$

If one investigates the discrete structure, where in each element *k* one design variable x_k is assumed, then, using (10), it is possible to determine the sensitivity coefficients S_k for all members

$$S_k=\frac{\partial f}{\partial x_k}+\bullet^{\mathrm{T}}\left(\frac{d\mathbf{P}}{dx_k}-\frac{d\mathbf{K}}{dx_k}\mathbf{s}\right) \qquad (12)$$

This means one can obtain the influence lines of the first variation of the function *f*, due to a unit "point" change of the design variable. Moreover, it should be emphasised that the matrix operation in the above equations is restricted to one element with the design variable under investigation.

3. Numerical examples

3.1. Sensitivity analysis of a simply supported thin-walled I-beam with battens

In the first example we examine the change of the torsion angle in the middle of the simply supported I-beam under a unit torque at the central cross-section of the beam, due to the inclusion of a pair of battens at a varying position of the beam (Fig 1). The analysed I-beam is built of 10 mm thick steel panels (E = 205 GPa, υ = 0,3) with the height of the web equal to 30 cm and flanges 20 cm wide. The width of the batten, *b*, is assumed as the design variable.

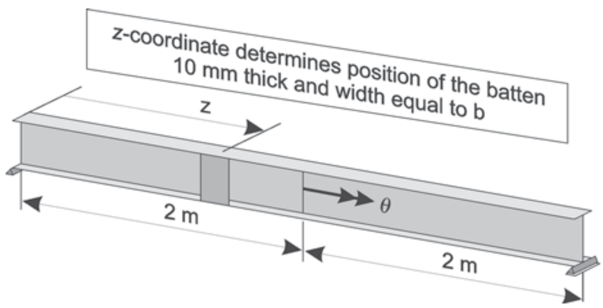

Fig 1. Simply supported I-beam with a pair of battens at a varying position

The analysed beam is divided into 10 segments of an equal length. Each segment is represented by an appropriate superelement. The variation of the torsion angle is estimated from the relation (11). The incremental approximation (6) is applied to evaluate the derivative of the stiffness matrix with respect to the width of the batten:

$$\left.\frac{d\mathbf{K}}{db}\right|_{b=0} \approx \frac{\mathbf{K}(b=0{,}05\,\mathrm{m}) - \mathbf{K}(b=0)}{0{,}05}, \qquad (13)$$

where K(b=0,05 m) is the stiffness matrix of the superelement with the batten 5 cm wide, K(b=0) represents the stiffness matrix of the superelement without any stiffener. Discrete values of the underintegral function have been calculated at the points corresponding to the subsequent locations of the batten:

$$z_i = 0{,}2\,\mathrm{m} + (i-1)\times 0{,}4\,\mathrm{m}, \qquad i = 1,2,\ldots,10\,. \quad (14)$$

The obtained representation of the influence line of the torsion angle at the central cross-section of the beam due to the inclusion of the batten is presented in Fig 2.

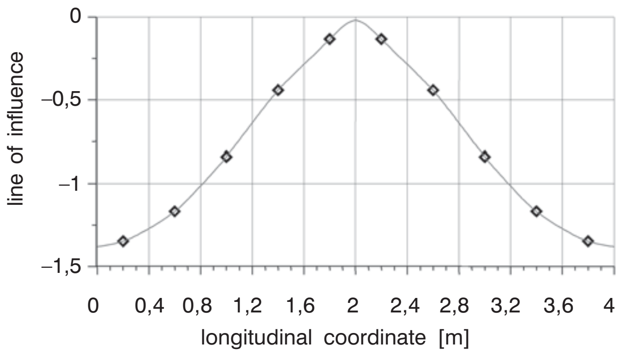

Fig 2. Relative increase of torsion angle due to inclusion of unit width batten placed at the coordinate z

Looking at the line of influence presented in Fig 2 one can find that the most significant reduction of the torsion angle results from the inclusion of battens at the supports of the beam, whereas battens located in the mid-span of the beam have almost no effect on the value of the torsion angle of the mid-span cross-section.

To verify the obtained results a simply validation procedure has been applied as presented schematically in Fig 3.

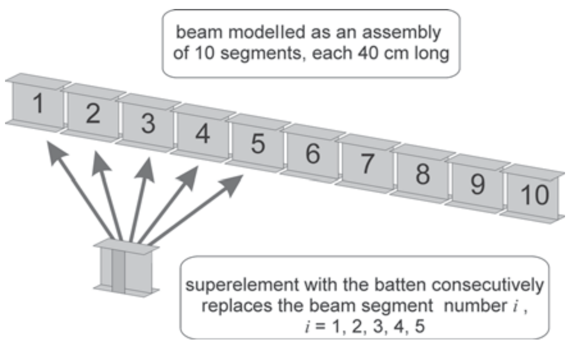

Fig 3. Scheme of the validation procedure

Because of the beam symmetry only the first 5 positions of the battens are considered. Three values of the width of the batten have been applied in the study: 5 cm, 10 cm and 20 cm. The corresponding graphs of the relative increase of the torsion angle at the mid-span section of the beam are given in Fig 4 ÷ 6.

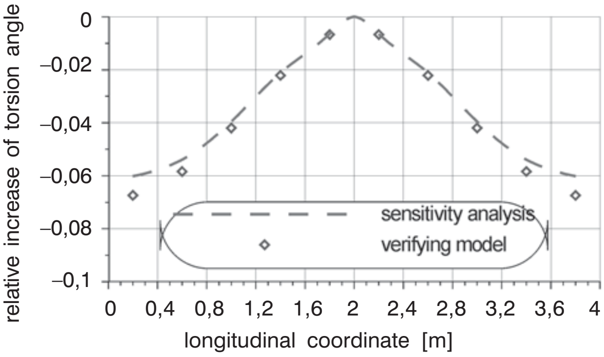

Fig 4. Relative increase of torsion angle resulted from application of battens 5 cm wide

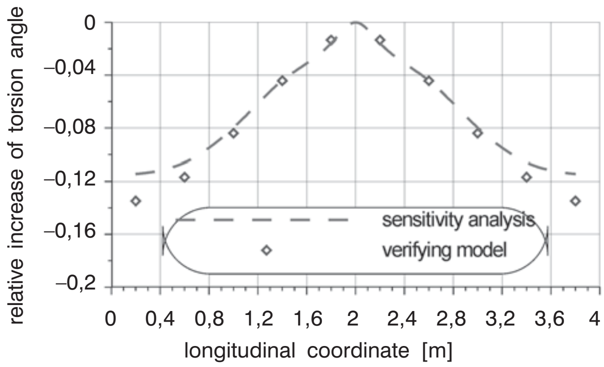

Fig 5. Relative increase of torsion angle resulted from application of battens 10 cm wide

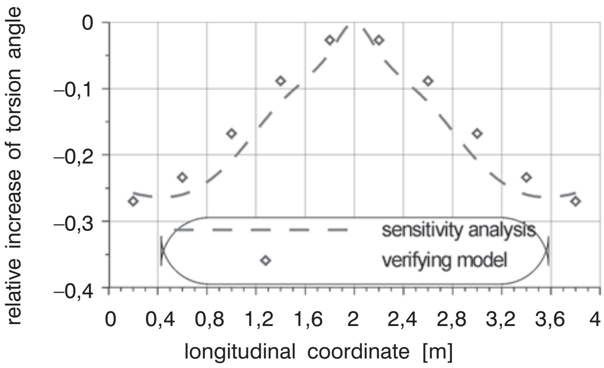

Fig 6. Relative increase of torsion angle resulted from application of battens 20 cm wide

The results of the validation procedure seem to prove that the numerical model can provide a good estimation of the change of the torsion angle due to the inclusion of battens. However, with the increase of the width of the battens the difference between the results of the numerical sensitivity analysis and the validation procedure becomes larger. This is a consequence of the introduced linearisation approximation. In Fig 7 the relative increase of the torsion angle as the function of the width of the batten is presented for the varying location of the batten. Looking at these graphs, one can notice that all the presented relations are non-linear.

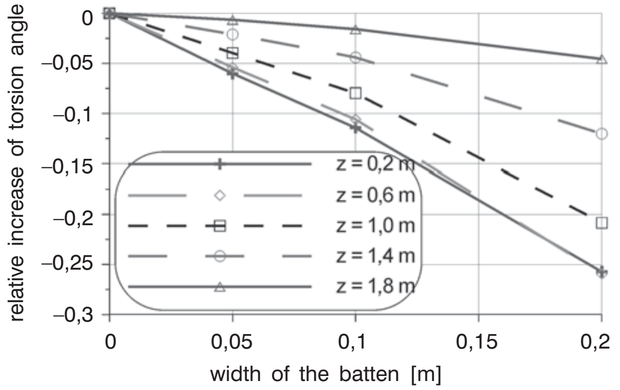

Fig 7. Relative increase of torsion angle resulted from application of battens at positions described by *z*-coordinate

In the present example the possibilities of the sensitivity analysis have been used to determine the alterations in the performance of the thin-walled beams resulted from the changes in the properties of the stiffeners. In particular, employing the sensitivity analysis with the numerical model one can estimate how the performance of the I-beam changes after adding the battens of a varying width at the arbitrary position. This can be obtained without the necessity of performing additional static analysis. The results of the validation procedure fully confirmed that such a technique offers an accuracy which is fully sufficient for the practical engineering applications.

3.2. Sensitivity analysis of frame with lateral restraints at top flange level

A simple frame constructed of I-beams as shown in Fig 8 is subjected to the unit torque M = 1 kNm acting on the mid-span cross section (point 2) of the horizontal beam. The frame can be stiffened by lateral restraints on the upper flange level at any cross-section along the frame beam axis (Fig 8).

It is assumed that suitable stiffeners are applied to eliminate the possibility of the crippling buckling of the flanges.

Two different numerical models of the frame are applied in the comparative study:

1. thin-walled beam elements with node superelement having 14 degrees of freedom each (warping effects included);
2. detailed FEM model – discretisation with QUAD4 shell elements available in computer system MSC/NASTRAN.

The state variable is the torsion angle θ_2 at point 2.

The method of adjoint system is applied [2, 3]. In this case the adjoint load is the unit torque acting at point 2 in the opposite direction to torsion angle θ_2.

The sensitivity analysis method can be applied to determine the torsion angle variation $\delta\theta_2$ cased by variation of the restraint stiffness δk

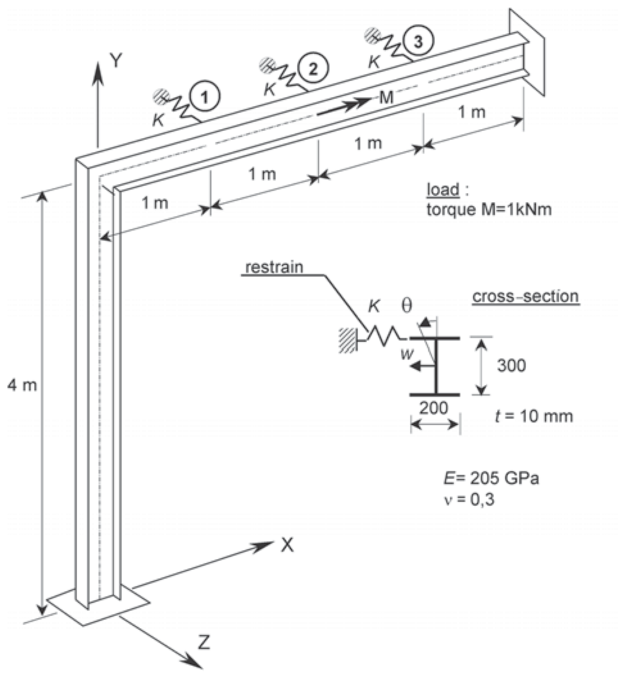

Fig 8. Frame with lateral restraints

$$\delta\theta_2 = \int_L (w_h \overline{w}_h)\delta k\, ds = \int_L F_{\theta k}(s)\delta k\, ds\,, \qquad (15)$$

where w_h and $\overline{w}_h$ are displacements at the connection point of restraint calculated for initial solution and for adjoint solution, respectively. The underintegral function $F_{\theta k}(s)$ is an influence line of the torsion angle variation dq_2 due to the restraint stiffness variations δk.

The relative changes of the state variables $\delta\theta_2/\theta_2$ can be calculated as

$$\frac{\delta\theta_2}{\theta_2} = \int_L \frac{F_{\theta k}(s)}{\theta_2}\delta k\, ds = \int_L \overline{F}_{\theta k}(s)\delta k\, ds\,. \qquad (16)$$

The graphs of the underintegral function $\overline{F}_{\theta k}$ are illustrated in Fig 9. A very good agreement can be observed between results of the sensitivity analysis obtained with both considered models.

To verify the results of the sensitivity analysis a static analysis of the frame shown in Fig 8 has been performed assuming three identical restraints added at points 1, 2 and 3. Four values of the restraint stiffness K = 50, 100, 200, 500 kN/m were applied, respectively, in the computations.

The relative variation of the torsion angle can be estimated according to (16). For the discrete values of the restraint stiffness the adequate relation is:

$$\frac{\delta\theta_2}{\theta_2} = \sum_{i=1}^{3}(\overline{F}_{\theta k} K_i) = K\sum_{i=1}^{3}\overline{F}_{\theta k}\,, \qquad (16a)$$

where $\overline{F}_{\theta k}$ is an underintegral sensitivity function value at cross section i with K being the stiffness of restraint.

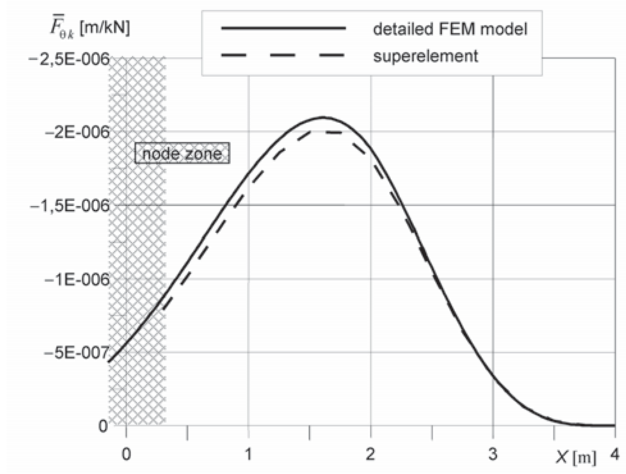

Fig 9. Distribution of underintegral sensitivity function along the horizontal beam

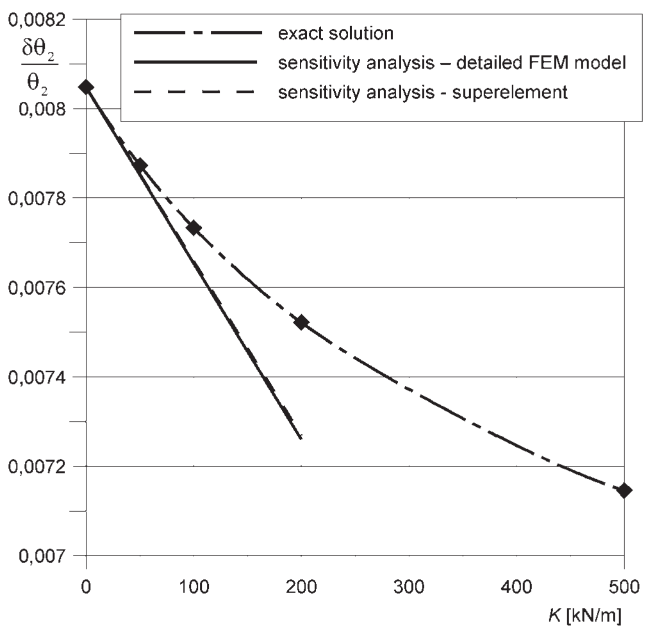

Fig 10. Comparison of sensitivity with exact solution (three identical restraints at points 1, 2 and 3)

Results of the sensitivity analysis for both considered models are compared in Fig 10 with the exact solution obtained by the parametric analysis of the frame with restraints.

One can observe that the sensitivity analysis provides a good estimation of the relative change of the torsion angle due to application of the lateral restraints of a moderate stiffness (up to 150 kN/m). However, the results of the first order sensitivity analysis for higher values of the stiffness differ from the exact solution due to the non-linear dependence of the results with respect to the stiffness of the restraint.

3.3. Change of torsional angle and bimoment due to variations of thickness of I-beam flange

A simple frame built of I-beams as considered in the previous example is examined now in the sensitivity analysis with the thickness t of the I-beam flanges taken as the design variable as shown in Fig 11. The frame is subjected to the unit torque M = 1 kNm acting in the mid-span of the horizontal beam (point *0*). Two various stage variables are considered:

1. torsion angel at point $0 - \theta_0$;
2. bimoment at point $B - B_B$ – upper support.

A discrete numerical model is applied consisting of the thin-walled one-dimensional beam elements along the frame column and the beam combined with the node superelement. The discretisation assumed in the computations is shown in Fig 11.

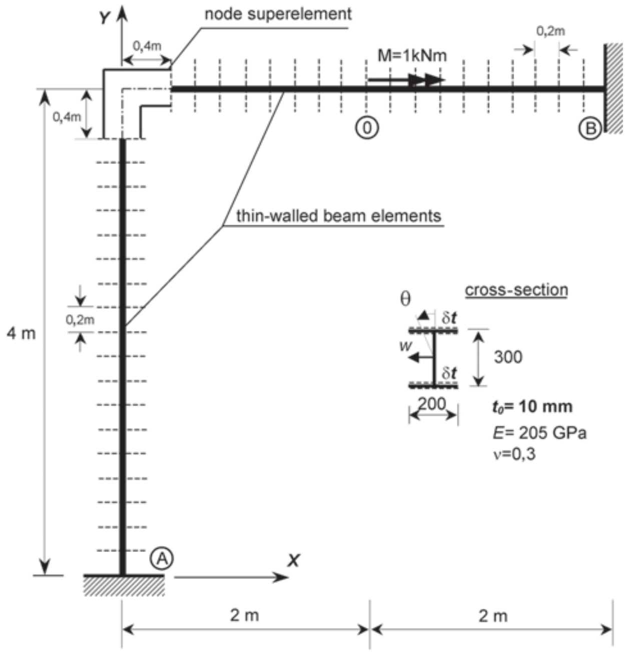

Fig 11. Sensitivity analysis of thin-walled frame – flange thickness t as a design variation

In this example, the load vector P and the state variable function f under consideration are independent of the design variable t_i, therefore the simplified equation (11) should be applied in the form

$$\delta f = \bullet^{\mathrm{T}} \frac{d\mathbf{K}}{dt_i} \mathbf{s}\, \delta t_i, \tag{17}$$

where the design variable t_i is the thickness of an I-beam flange for the *i*-th element and the state variable function f is taken $f=\theta_0$ or $f=B_B$. The remaining nomenclature is analogous as used in (11).

As the adjoint system load corresponding to the state variable θ_0 is taken a unit torque acting at point *0* in the opposite direction to the torsion angle θ_0. For the bimoment B_B being the state variable the adjoint system load is a negative unit warping enforced at point *B*.

The derivatives of the stiffness matrix of thin-walled beam elements [4] can be calculated in the analytical manner. In the case of the node superelement a numerical evaluation of derivatives is necessary – here central difference relations (7) are used.

Two cases of the flange thickness changes Δt are considered for the node superelement as presented in Fig 12:

- case A – change of the flange thickness along the frame column;
- case B – thickness change along the horizontal beam.

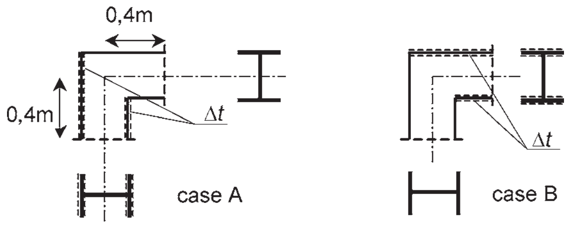

Fig 12. Two cases of flange thickness variations

The underintegral sensitivity function of the torsion angle relative changes $\delta\theta_0/\theta_0$ due to relative I-beam flange thickness variations $\delta t/t_0$ along the frame column and the horizontal beam is shown in Figs 13 and 14, respectively.

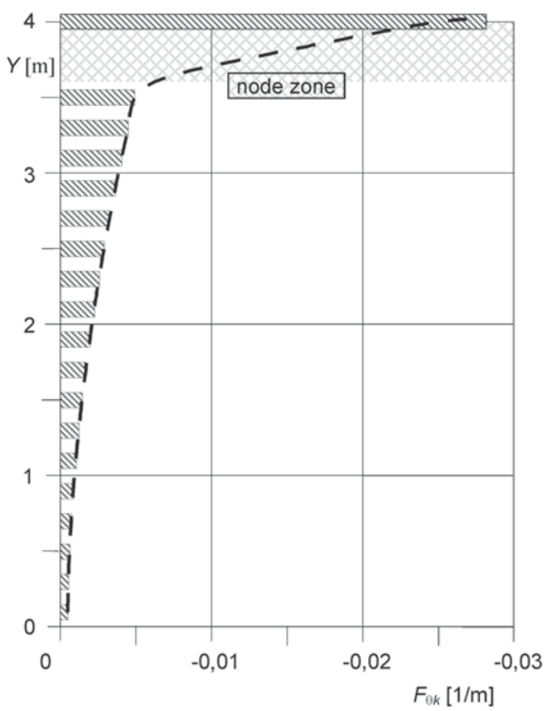

Fig 13. Underintegral sensitivity function $\delta\theta_0/\theta_0$ – distribution along frame column

The accuracy of the sensitivity analysis of the torsional angle q_0 is examined in Fig 15 by comparing its results with the exact solution obtained in a parametric analysis of the same structure performed for a varying thickness of the I-beam flange along the horizontal beam.

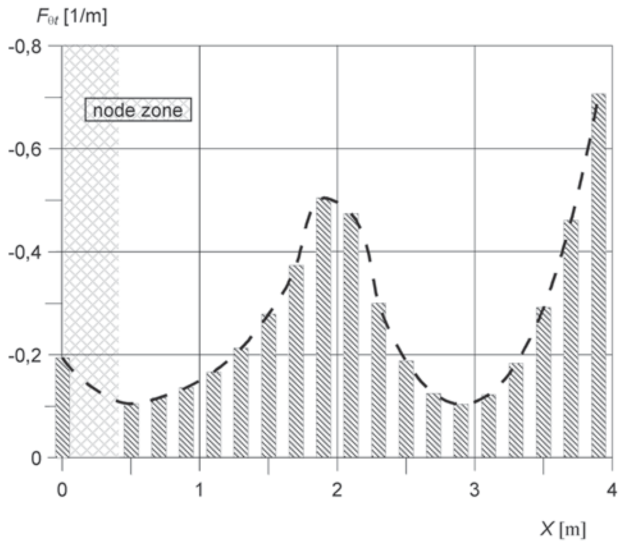

Fig 14. Underintegral sensitivity function $\delta\theta_0/\theta_0$ – distribution along horizontal beam

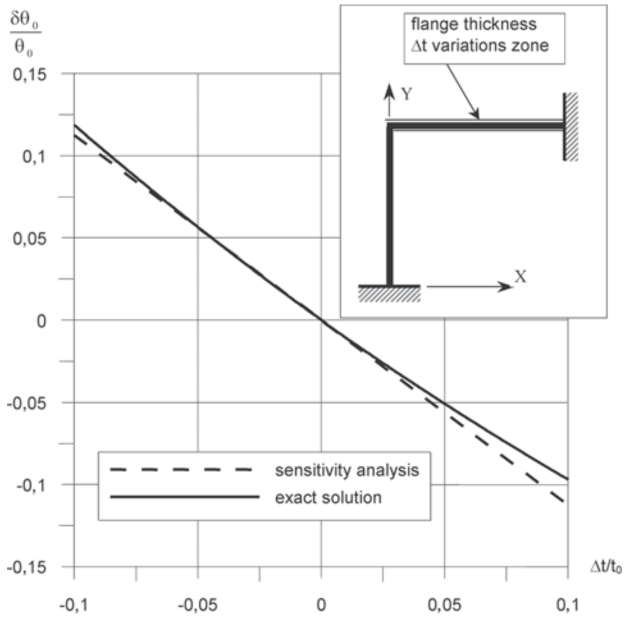

Fig 15. Accuracy of the sensitivity analysis

The underintegral sensitivity function of the second state variable – bimoment relative changes $\delta B_B/B_B$ due to relative I-beam flange thickness variations $\delta t/t_0$ is presented in Fig 16 (along the frame column) and in Fig 17 (along the horizontal beam).

The results of the sensitivity analysis for the bimoment B_B agree quite well with the exact solution as shown in Fig 18. In this case the variations of the I-beam flange thickness are restricted to the 1m span of the horizontal beam near the support B, where the underintegral sensitivity function is positive (Fig 16).

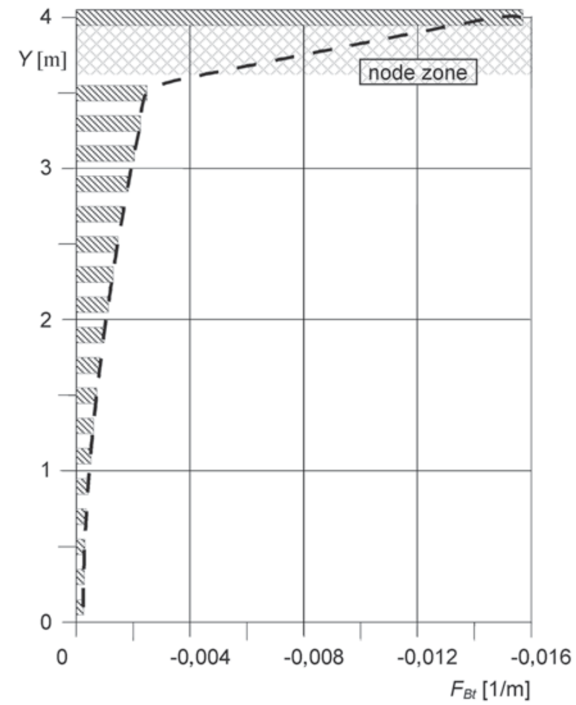

Fig 16. Underintegral sensitivity function $\delta B_B/B_B$ – distribution along frame column

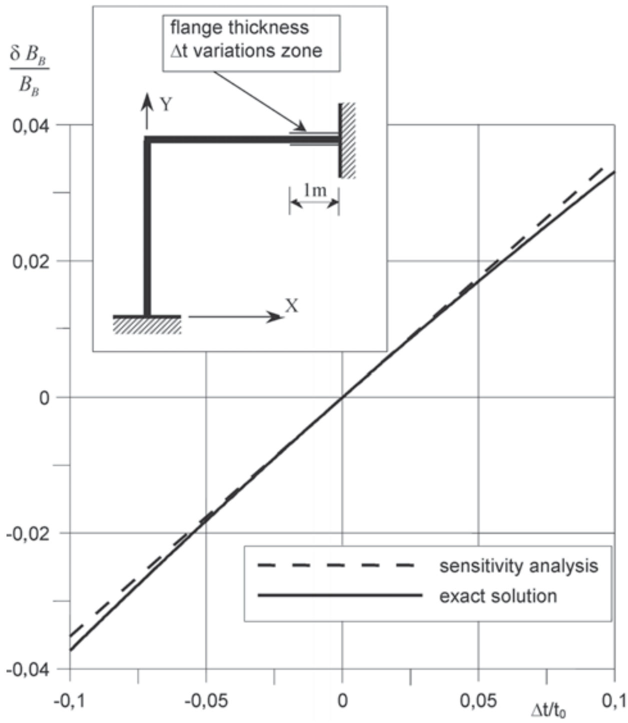

Fig 18. Accuracy of the sensitivity analysis

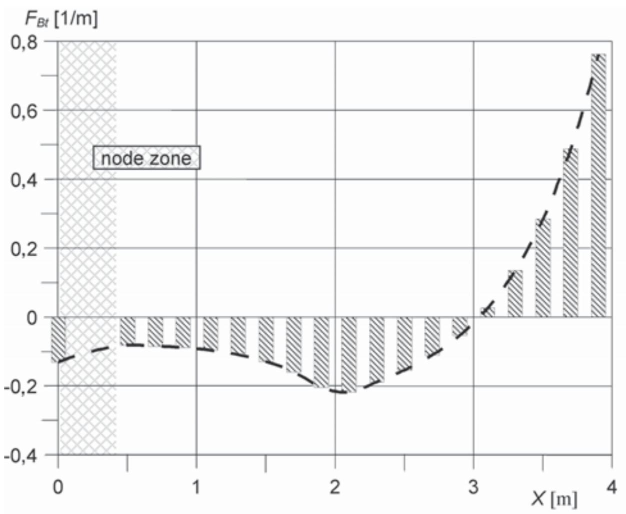

Fig 17. Underintegral sensitivity function $\delta B_B/B_B$ – distribution along horizontal beam

4. Conclusions

Sensitivity analysis of displacements and internal forces has been performed for beams and frames made of thin-walled members. The concept of superelement has been applied for frame nodes and for thin-walled segments containing warping stiffeners.

In particular, the proposed strategy proved to be the right tool for the sensitivity analysis of thin-walled beams with different type of stiffeners. Applying the sensitivity analysis, one can estimate the changes in the performance of the I-beam resulting from addition the stiffeners of varying characteristic at the arbitrary position without the necessity of performing additional static analysis.

The concept of the superelement seems to be especially attractive for applications in the sensitivity analysis of frames. The other one-dimensional models based on the classical theory of bars do not take into consideration the essential effect of bimoment transfer through the nodes. The only acceptable alternative approach offering comparable accuracy is the detailed FEM model which however requires much more computational "power" since the numerical size of the detailed model is much larger than any model composed of beam elements and the superelement.

In the sensitivity analysis it is required to determine the derivatives of the stiffness matrix with respect to the given design variables. The necessary derivatives for the superelement can be estimated only numerically. If the equivalent difference relationships are employed to evaluate the derivatives one should carefully select appropriate increment of the design variable to avoid singularities which can appear when both, the numerator and the denominator, are close to zero.

The accuracy of the sensitivity analysis in comparison with the "exact" solution obtained by parametric analysis depends primarily on the type of the adopted state variable, the assumed design variables, and on the type of the analysis. In the present paper attention was confined to the static problems. The design variable variations ranging from 10 % to 20 % provide a solution of the sensitivity analysis with an accuracy of 1 % – 2 % which is quite sufficient for the practical engineering applications.

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PLONASIENIŲ SIJŲ IR RĖMŲ JAUTRUMO ANALIZĖ TAIKANT GRETINIMO METODĄ

I. Kreja, T. Mikulski, C. Szymczak

Santrauka

Pateikiama sijų ir rėmų, sukomponuotų iš plonasienių elementų, jautrumo analizė taikant gretinimo metodą. Nagrinėjamos statinės apkrovos ir konstrukcijos, sumodeliuotos plonasieniais dviejų ašių atžvilgiu simetriniais atviro kontūro elementais. Nagrinėjama konstrukcija yra modeliuojama taikant vienmatį modelį. Šį modelį sudaro plonasieniai sijiniai elementai, kuriems aprašyti taikomos klasikinės nedeformuojamo skerspjūvio prielaidos bei superelementai, naudojami konstrukcijų mazguose standumo briaunų ir kitų suvaržymų vietose. Aprašytu būdu sumodeliuotai konstrukcijai gauti jautrumo analizės rezultatai palyginti su rezultatais, gautais naudojant detalų BEM modelį, taikant programos *MSC/NASTRAN* kevalinius *QUAD*4 tipo elementus.

Raktažodžiai: plonasienės konstrukcijos, sijos, rėmai, standumo briaunos, jautrumo analizė.

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