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R. Baušys

1. Introduction

The numerical simulation of acoustic problems is usually performed using standard computational techniques such as boundary element method (BEM), finite difference or finite element method (FEM). The finite element method can be preferred to the boundary element technique due to the fact that FEM does not encounter the difficulties of numerical implementation which are common for BEM approach. However, even after more than 30 years of development of the finite element method, the question of estimating and controlling discretization errors remains a major topic of concern even for the commonest case of modelling linear elasticity problems. In practice, the user has to discretize the domain of certain problems according to his earlier experience with similar applications. Such discretization process, being unable to predict proper resolution and the proper order of the approximation at each location, usually produces a mesh with too may elements. An alternative is to find some means to identify critical regions, which have to be refined. That is, starting solving problems on a crude mesh, one has to estimate truncation errors in different locations. The posteriori error estimation is the most important ingredient of this adaptive mesh design strategy.

In acoustic problems, steady state sound response is governed by Helmholtz equation, which can be characterised by a potential loss of ellipticity with increasing wave number in the propagation region. The Galerkin method provides good phase and amplitude accuracy as long as the mesh is fine enough with respect to wave number.

Acousticians often use so-called "rule of the thumb" which prescribes the minimal discretization of a wavelength. Such a discretization process, which is based on "rule of thumb", is unable to predict a proper resolution and the proper order of the approximation at each location and usually produces a mesh with too many elements and still we do not have a direct measure of the error.

The most important ingredient in finite element adaptive strategies is the estimation of the error of the approximate solutions. Constructing a new solution of a higher quality since the exact solution for complex-engineering problems is generally unknown usually performs the error estimation. Typically, this new improved solution is obtained by *a posteriori* procedure, which utilise finite element solution itself. By now a considerable success have been achieved mainly on problems of linear elliptic type, such as linear elastostatics and stationary heat conduction problems [1] - [5].

Bouillard *et al* [6] implemented the original superconvergent patch recovery (SPR) technique for acoustic finite element analysis. The original concepts are extended to complex variables and the reliability of the error estimation is studied. Tetambe and Rajakumar [7] presented the error estimation strategy for acoustic analysis based on nodal averaging technique. Residual-based a *posteriori* error estimator for Helmholtz equation presented by Harari *et al* [8].

In the present study we have implemented the superconvergent patch recovery for prime variables (SPRD) technique to estimate the discretization error of the solution. The SPRD technique is based on a higher order prime variable field fitted to superconvergent values in a least squares sense over local element patches.

2. Governing equations

The sound field in the enclosure is treated as a compressible, inviscid, non-flowing medium [9]. Consider acoustic medium of uniform density ρ vibrating in the enclosure with volume V having surface S. The acoustical pressure $p(x,y,z,t)$ is then governed by the linearized wave equation

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{1}$$

where ∇^2 is the Laplacian operator and c is the sound speed.

This equation also assumes an adiabatic process, local changes which are small, and small amplitude displacement and velocity of the fluid particles. Since the viscous dissipation has been neglected, Eq (1) is referred to as the lossless wave equation for propagation of sound in fluids. Noise sources interior to an enclosed cavity can be concluded as forcing terms in the wave equation. For the case of monopole source the time-varying mass flow rate is $\dot{m}(x,y,z,t) = \rho\, Q(x,y,z,t)$, so that

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\rho}{V}\frac{\partial Q}{\partial t}. \tag{2}$$

Other noise sources can be represented as combinations of monopole sources or else can be included directly in the wave equation in a similar way.

The boundary conditions for p determine the reflection, absorption and transmission of sound waves at the enclosure's surfaces and are derived from fluid mechanical considerations. The boundary conditions of interest for the present paper are of the following types.

Dirichlet type condition

$$p = p_e \quad on\ S_1. \tag{3}$$

Neumann type boundary condition requires that the air particle velocity v normal to boundary surface to be related to pressure p through

$$\frac{1}{\rho}\nabla p \cdot \boldsymbol{n} = -\frac{\partial v}{\partial t} \quad on\ S_2 \tag{4}$$

where $\boldsymbol{n}$ is a unit normal to S_2. It is assumed that energy is lost by overcoming damping forces at the boundaries. The corresponding boundary condition is usually expressed as a Cauchy type condition

$$\frac{1}{\rho}\nabla p \cdot \boldsymbol{n} = -\frac{1}{Z_a}\frac{\partial p}{\partial t} \quad on\, S_3 \tag{5}$$

where Z_a is the normal acoustic impedance.

In the case time-harmonic excitation, a steady solution is sought of the form

$$p(x,y,z,t) = Re\left\{p_0(x,y,z)\,e^{j\omega t}\right\}, \tag{6}$$

$$Q(x,y,z,t) = Re\left\{Q_0(x,y,z)\,e^{j\omega t}\right\} \tag{7}$$

where $j = \sqrt{-1}$.

In frequency domain the Eq (2-4) then become

$$\nabla^2 p_0 + \left(\frac{\omega}{c}\right)^2 p_0 = -\frac{j\,\omega\,\rho\,Q_0}{V} \tag{8}$$

with

$$p_0 = \overline{p}_e \quad on\ S_1, \tag{9}$$

$$\nabla p_0 \cdot \boldsymbol{n} = -j\rho\omega\,\overline{v}_n \quad on\ S_2, \tag{10}$$

$$\nabla p_0 \cdot \boldsymbol{n} = -\frac{j\rho\omega}{Z_a}\overline{p}_e \quad on\ S_3. \tag{11}$$

In the finite element context, using standard Galerkin procedure, we have Eq (8-11) in the form

$$\left([K] + j\rho\omega\,[C] - \omega^2[M]\right)q^h = -j\rho\omega \boldsymbol{F} \tag{12}$$

where *[K]* is the stiffness matrix

$$[K] = \int_{\Omega} (\nabla[N])^T(\nabla[N])\,d\Omega \tag{13}$$

and *[N]* denotes standard shape functions. Matrix *[C]* stands for the damping matrix, representing mixed boundary conditions (11)

$$[C] = \int_{S_3} [N]^T[N]\frac{1}{Z_a}dS \tag{14}$$

and *[M]* is the mass matrix

$$[M] = \int_{\Omega} [N]^T[N]\,d\Omega. \tag{15}$$

The forcing term in the right hand side of eq.(12) has the following form

$$\boldsymbol{F} = \int_{\Omega} [N]^T\frac{Q}{\Omega}d\Omega + \int_{S_2} [N]^T\overline{v}_n dS. \tag{16}$$

The vector $\boldsymbol{q}^h$ contains nodal pressure values, which satisfy Dirichlet boundary conditions, Eq (9).

With the finite element solution of the problem, pressure and velocity approximations $\boldsymbol{q}^h$ and $v^h = -\frac{1}{j\rho\omega}\nabla \boldsymbol{q}^h$, respectively, are obtained.

3. A postprocessed error estimation

The purpose of the postprocessed error estimation is to provide a local estimate of the solution error in some norm. All acoustic variables are complex so the error of the finite element solution can be defined as

$$\|p\|^2_{H^1(\Omega)} = \|p\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)}, \qquad (17)$$

where H^1 is complex Sobolev space and L^2 is scalar norm. Considering the fact that $\|p\|^2_{L^2(\Omega)}$ is asymptotically negligible behind $\|\nabla p\|^2_{L^2(\Omega)}$, the error of the finite element solution can be expressed as

$$\|e\|^2_{L^2(\Omega)} = \int_\Omega (v^T - v_h^T)(\tilde{v} - \tilde{v}_h) d\Omega, \qquad (18)$$

where $\tilde{\bullet}$ denotes the complex conjugate.

A relative error of the finite element approximation can be expressed as follows

$$\eta = \frac{|e|_{L^2(\Omega)}}{|q|_{L^2(\Omega)}}, \ with\ |q|^2_{L^2(\Omega)} = \int_\Omega v^T \tilde{v} dx. \qquad (19)$$

From a priori error estimation results [10], we can express that the global relative error is bounded as follows

$$\eta \le C_1(p)\theta^p + C_2(p)k\theta^{2p}, \qquad (20)$$

with $\theta = \frac{kh}{2p}$, wave number $k = \frac{\omega}{c}$, where p is the order of the FE basis functions. For linear elements $p=1$, the equation (9) will be of the form

$$\eta \le C_1 kh + C_2 k^3 h^2. \qquad (21)$$

The first term in the error bound reflects the classical best approximation error while the second term indicates pollution of optimal error for high wave number.

In general case, exact solution is not available, and error is estimated using a postprocessed velocity field v^* of the higher order accuracy

$$\|\bar{e}\|^2_{L^2(\Omega)} = \int_\Omega (v^{*T} - v_h^T)(\tilde{v}^* - \tilde{v}_h) d\Omega. \qquad (22)$$

In practice, we calculate this norm by summing over all elements in the domain Ω:

$$\|e\|^2 = \sum_{i=1}^{nel} (\|e\|^2_{L^2(\Omega_i)})^{1/2}. \qquad (23)$$

Where Ω_i is an element domain and *nel* is the total number of elements.

The quality of the error estimator is measured by its effectivity index, defined as the ratio between the estimated and the exact errors

$$\theta = \frac{\|\bar{e}\|}{\|e\|}. \qquad (24)$$

An error estimator is named asymptotically exact if θ approaches to unity as the characteristic size of the finite element h tends to zero.

4. An improved solution by patch recovery

A new improved finite element solution can be constructed by the SPRD technique. The SPRD technique is essentially a least square fit of the prime variables at superconvergent points. This approach provides superconvergent pressure field q^* over local element patches. Since the SPRD technique recovers superconvergent pressure field that is at least one order higher than finite element solution, the method can successfully be implemented in error estimation of the finite element solution. This approach is a local updating method, so no global system of equations has to be constructed and solved. The recovered pressure field over an element $\tau \in T_h$ is constructed as

$$\mathbf{q}^*(\mathbf{x}) = \sum_r [\mathbf{N}]_r^*(\mathbf{x})(\mathbf{q}_r^*) + \sum_s [\mathbf{N}]_s^*(\mathbf{x})(\mathbf{q}_s^*). \qquad (25)$$

Where r is used to denote finite element τ nodes, s denotes additional nodes of the element of the recovered displacement field, $N_r^*(x)$ and $N_s^*(x)$ are local basis functions of the order $p+1$ associated with the original element nodes and the additional ones, respectively.

The nodal values of the original finite element displacements are of the superior accuracy and are assumed fixed $(q_r^*) \equiv (q_r^h)$. The recovered displacement values (q_s^*) at the additional nodes are obtained by solving least squares problem in the reduced element patch Ω_τ which represents the union of the element under consideration and the part of the surrounding elements:

Find $q^* \in P_{p+1}$ such that

$$J_{\Omega_\tau}(\mathbf{q}^*) = \min_{(q^{f*}) \in P_{p+1}} J_{\Omega_\tau}(\mathbf{q}^{f*}). \quad (26)$$

Where

$$J_{\Omega_\tau}(\mathbf{q}^{f*}) = \sum_{j=1}^{ns} w_j^2 \mathbf{R}_q^T(\mathbf{x}_j)\mathbf{R}_q(x_j). \quad (27)$$

Where the residual $R_q(x_j)$ is defined by expression as

$$R_q = (q^{f*}) - (q_r^h) \quad (28)$$

and

$$(q^{f*}) = [Q(x)]b \quad (29)$$

Here x_j is the location of j-th sampling point in the element patch Ω_τ, w_j is the weight assigned to the *j*-th sampling point and ns *is* the total number of the sampling (nodal) points in the element patch Ω_τ and b are complex unknown coefficients. Details of this approach are available in [11-14].

When the recovered pressure are determined over all elements $\tau \in T_h$, we obtain the pressure field of the higher accuracy and a postprocessed error velocities can be determined and error estimation can be performed.

5. Numerical example

We consider a tube of length $L=1.0\ m$ and width $H=0.1\ m$ which is shown in Fig 1. The surface of the excitation is at left end of the tube. On the other boundaries normal velocity set be zero. Regular meshes for both elements: linear quadrilateral and triangular, are considered. Typical meshes for quadrilateral and triangular elements are presented in Figs 1 and 2.

A sequence of three regular meshes with 20x2, 40x4 and 80x8 elements is used to study the rate of convergence and the accuracy of the results for both quadrilateral and triangular elements.

For this problem an analytical solution is available and can be expressed by

$$v(x) = \frac{v_0}{sin(kL)} sin[k(L-x)]. \quad (30)$$

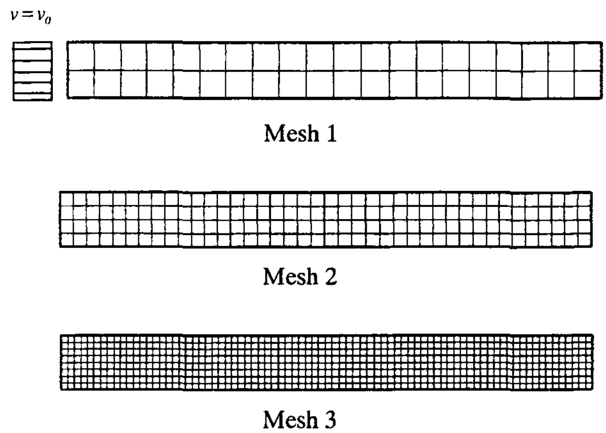

Fig 1. Geometry, excitation and typical quadrilateral meshes

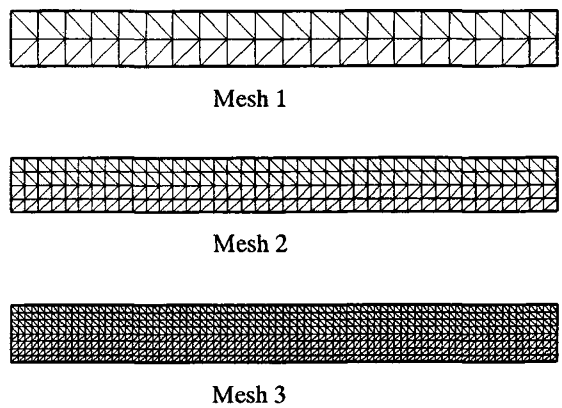

Fig 2. Typical triangular meshes

The error estimation was performed for two frequencies of the excitation, *50 Hz* and *750 Hz.* In the first case the non-dimensional wave number *k'* is then equal to *0.92* and for the coarsest mesh: $kh=0.046<<1$ and $k^2h=0.0432<<1$ and we will study behaviour of postprocessed error estimation in the asymptotic range since both assumptions concerning *kh* and k^2h are hold. In the second case *k'* is equal to *13.85* and for the coarsest mesh: $kh=0.691<1$ and $k^2h=9.59>1$ and we will study behaviour of postprocessed error estimation in preasymptotic range since only assumption concerning *kh* is hold. Notice that only in the first case non-dimensional wave number respects the criterion of low wave number. The numerical results of the convergence rate for linear triangular elements are plotted in Figs 3-4. The error in energy norm of the original finite element solution, of the post-processed solution and of the estimated error of the finite element solution is presented in these figures.

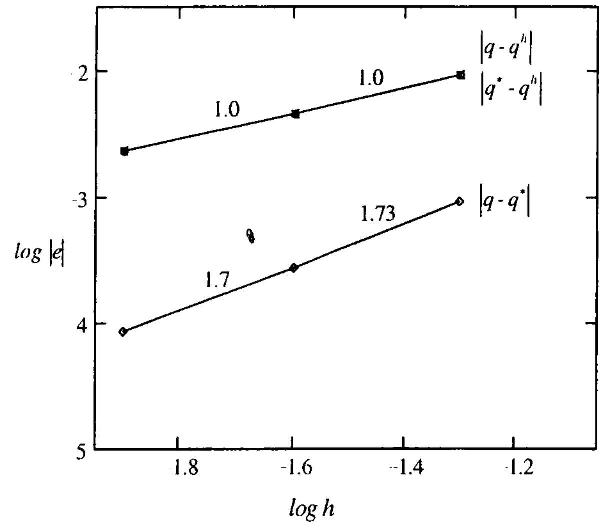

Fig 3. Convergence rate of triangular elements at 50 Hz

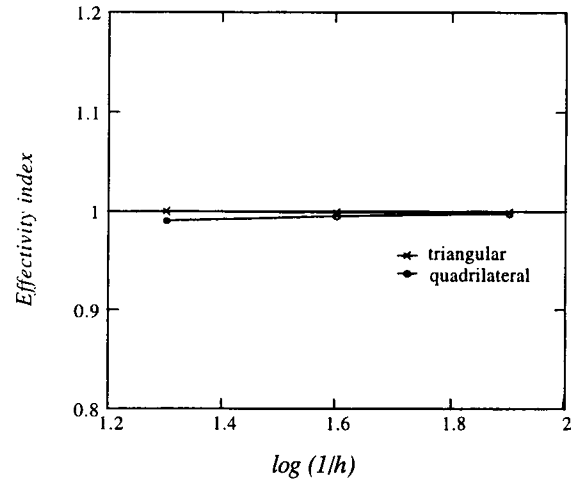

Fig 5. Effectivity indices of triangular and quadrilateral elements at 50 Hz

With the results of the numerical experiments at hand, we can make the following observations:

1. The original finite element solution exhibit order of accuracy $O(h)$ as predicted by *a priori* error estimation.
2. The recovered solution obtained by SPRD technique demonstrates superior accuracy with respect to original finite element solution.

The convergences of the effectivity indices are plotted in Figs 5 and 6.

We observe that the effectivity indices converge to one rapidly for both quadrilateral and triangular elements tested when the finite element mesh is refined. The numerical results show an asymptotic exactness of the proposed error estimator based on the SPRD technique.

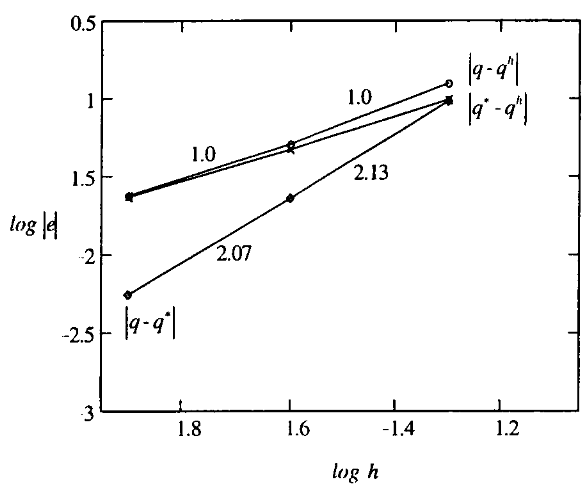

Fig 4. Convergence rate of triangular elements at 750 Hz

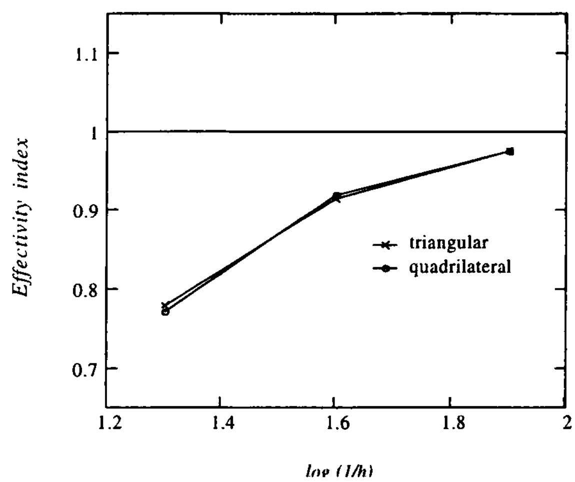

Fig 6. Effectivity indices of triangular and quadrilateral elements at 750 Hz

3. The superconvergent properties of the improved solution are demonstrated for both cases in asymptotic and preasymptotic ranges.
4. The proposed SPRD technique slightly underestimates the exact error of the finite element solution.

6. Conclusions

For the error estimation we have implemented the Superconvergent Patch Recovery technique for the prime variables (SPRD). SPRD technique is extended to the region of complex variables. A reduced element patch is implemented which has an extension of the size 2h (where h is a characteristic element size in the local patch) in order to maintain locality of the least squares fit. This enables us to reduce the cost of computation and at the same time to increase the accuracy of recovered pressure field. Only boundary patches, which have not enough number of elements for reduced patch, are constructed in the usual way. The described approach is a local updating method, so no global system of the equations has to be solved. The number of equations to be solved is small and the cost of the recovery procedure is proportional to the number of the mesh nodes. Numerical experiments show reliability of the proposed SPRD technique due to the fact that recovered gradient field exhibits superconvergence properties. In acoustic analysis distribution of the pressure field is a function of the frequency of the excitation. This case is illustrated by performed numerical experiments.

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AKUSTIKOS UŽDAVINIŲ PAKLAIDŲ ĮVERTINIMAS

R. Baušys

S a n t r a u k a

Straipsnyje pateikiamas originalus metodas, skirtas akustikos uždavinių diskretizacijos paklaidoms nustatyti. Diskretizacijos paklaidos randamos poprocesoriniu superkonvergenciniu lopinio atkūrimo būdu. Darbe pateikta superkonvergencinio lopinio atkūrimo metodo versija, pritaikyta tiesioginiams aproksimacijos kintamiesiams (slėgiams). Pradžioje šis būdas buvo taikytas siekiant įvertinti laisvųjų svyravimų uždavinių diskretizacijos paklaidas. Darbe superkonvergencinis lopinio atkūrimo metodas pritaikytas kompleksinių skaičių sričiai. Šioje procedūroje taikytas sumažintas baigtinių elementų lopinys, kurio dydis yra 2h, kur h yra charakteringas baigtinio elemento dydis. Tai leidžia lokalizuoti sritį, kurioje atliekamas atkūrimas mažiausių kvadratų metodu, ir kartu padidinti atkurto slėgio lauko tikslumą bei sumažinti skaičiavimo sąnaudas. Tiktai kraštiniams lopiniams, kuriuose yra nepakankamas elementų skaičius, taikomas tradicinis lopinys. Pasiūlytas metodas yra lokalus, globali algebrinių lygčių sistema nėra sprendžiama. Atlikti skaitiniai eksperimentai atskleidė pasiūlyto metodo patikimumą, nes gautas aukštesnės tikslumo klasės sprendinys turi superkonvergencines savybes. Taikant pasiūlytą metodą, gaunamas patikimas paklaidų įvertinimas tiek asimptotinėje, tiek priešasimptotinėje srityse. Gauti rezultatai atveria galimybes rasti kokybiškai įvertintus akustikos uždavinių sprendinius.

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