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SOLUTIONS OF A NONLINEAR DIRICHLET PROBLEM IN WHICH THE NONLINEAR PART IS BOUNDED FROM ABOVE AND BELOW BY POLYNOMIALS

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ABSTRACT

In this paper we study the existence and multiplicity solutions of nonlinear elliptic problem of the form

$$\begin{aligned} \Delta u + \lambda \cdot u - f(u) \cdot u = g \qquad & in\ \Omega \\ u = 0 \qquad & on\ \delta\Omega \end{aligned}$$

Here, Ω is a smooth and bounded domain in R^N, $N \geq 2$, $\lambda \in R$ and $f : R \longrightarrow R$ is a continuous, even function satisfying the following condition

$$\begin{aligned} c_1 \cdot |t|^{\alpha} \leq f(t) \leq c_2 \cdot |t|^{p-1} \qquad & for\ |t| > 1 \\ 0 \leq f(t) \leq c_3 \cdot |t| \qquad & for\ |t| \leq 1 \end{aligned}$$

for some $c_1, c_2, c_3, p, \alpha \in R$, $c_1, c_2, c_3, \alpha > 0$ and $p > 1 + \alpha$.
We shall show that, for $\lambda \in R$, $g \in L_r(\Omega)$ if $N = 2$, $r > 1$, $p > 1 + \alpha$ or $N \geq 3$, $r \geq \frac{2 \cdot N}{N+2}$, $1 + \alpha < p < \frac{N+2}{N-2}$, the above problem has solutions.
Assuming additionally that, $\lambda \leq \lambda_1$ and f is decreasing for $t \leq 0$, we shall show that, this problem have exctly one solution.
We take advantage of the fact, that a continuous, proper and odd (injective) map of the form I + C (where C is compact) is suriective (a homeomorphism).

1. THE MAIN RESULTS

We will consider the following nonlinear Dirichlet problem

$$\begin{aligned} \Delta u + \lambda \cdot u - f(u) \cdot u = g \qquad & in\ \Omega \\ u = 0 \qquad & on\ \delta\Omega \end{aligned} \tag{1}$$

Here, Ω is a smooth and bounded domain in R^N, $N \geq 2$, $\lambda \in R$ and $f : R \longrightarrow R$ is a continuous, even function satisfying the following condition

$$\begin{array}{ll} c_1 \cdot |t|^{\alpha} \leq f(t) \leq c_2 \cdot |t|^{p-1} & for\ |t| > 1 \\ 0 \leq f(t) \leq c_3 \cdot |t| & for\ |t| \leq 1 \end{array} \tag{2}$$

for some $c_1, c_2, c_3, p, \alpha \in R$, $c_1, c_2, c_3, \alpha > 0$ and $p > 1 + \alpha$.
We assume that, $p > 1 + \alpha$ for $N = 2$ and $1 + \alpha < p < \frac{N+2}{N-2}$ for $N \geq 3$.

For each $u \in \overset{\circ}{W}_{12}(\Omega)$, let us define the elements Lu, Tu as follows:

$$\begin{array}{l} (Lu,\ \phi)_{12} = \int_{\Omega} u \cdot \phi dx \\ (Tu,\ \phi)_{12} = \int_{\Omega} f(u) \cdot u \cdot \phi dx \end{array} \tag{3}$$

for every $\phi \in \overset{\circ}{W}_{12}(\Omega)$.

Lemma 1. *If* $u \in \overset{\circ}{W}_{12}(\Omega)$, *then* $Lu, Tu \in \overset{\circ}{W}_{12}(\Omega)$.

P r o o f. Let $u \in \overset{\circ}{W}_{12}(\Omega)$. Then, for every $\phi \in \overset{\circ}{W}_{12}(\Omega)$, we have

$$|q_1(\phi)| = |\int_{\Omega} u \cdot \phi dx| \leq \int_{\Omega} |u \cdot \phi| dx \leq \ \| u \|_{02} \cdot \| \phi \|_{02} \ \leq \ c \cdot \| \phi \|_{12}$$

and

$$\begin{array}{lcl} |q_2(\phi)| & = & |\int_{\Omega} f(u) \cdot u \cdot \phi dx| \ \leq \ \int_{\Omega} |f(u) \cdot u \cdot \phi| dx \\ & \leq & \| f(u) \cdot u \|_{0\frac{2 \cdot N}{N+2}} \cdot \| \phi \|_{0\frac{2 \cdot N}{N+2}} \ \leq c \cdot \| \phi \|_{12} \end{array}$$

for $N \geq 3$ or

$$|q_2(\phi)| \ \leq \ \| f(u) \cdot u \|_{02} \cdot \| \phi \|_{02} \ \leq c \cdot \| \phi \|_{12}$$

when N = 2.

This means that, functions q_1 and q_2 are linear continuous functionals defined on the Hilbert space $\overset{\circ}{W}_{12}(\Omega)$. The Riesz theorem implies that, $Lu, Tu \in \overset{\circ}{W}_{12}(\Omega)$.□

Thus we can define the operators $L, T : \overset{\circ}{W}_{12}(\Omega) \longrightarrow \overset{\circ}{W}_{12}(\Omega)$. Now we shall prove some properties of L and T.

Lemma 2. *The operator L is linear and compact.*

P r o o f. It is obvious that, L is linear. Let us consider a bounded sequence $\{u_n\}$ in $\overset{\circ}{W}_{12}(\Omega)$. Since the imbedding $\overset{\circ}{W}_{12}(\Omega) \subset L_2(\Omega)$ is compact, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \longrightarrow u$ in $L_2(\Omega)$, $k \longrightarrow \infty$. Hence

$$\| Lu_{n_k} - Lu \|_{12} \leq c \cdot \| u_{n_k} - u \|_{02} \longrightarrow 0.$$

This means that, $Lu_{n_k} \longrightarrow Lu$ in $\overset{\circ}{W}_{12}(\Omega)$ *as* $k \longrightarrow \infty$ and the operator L is compact.□

LEMMA 3. *The map T is continuous and compact.*

P r o o f. Let $u_n \longrightarrow u$ *in* $\overset{\circ}{W}_{12}(\Omega)$, $n \longrightarrow \infty$. The continuity of the imbedding $\overset{\circ}{W}_{12}(\Omega) \subset L_{\frac{2\cdot N}{N-2}}(\Omega)$ implies that $u_n \longrightarrow u$ in $L_{\frac{2\cdot N}{N-2}}(\Omega)$.

It follows from inequality (2) and the Vajnberg theorem that, $f(u_n) \cdot u \longrightarrow f(u) \cdot u$ in $L_{\frac{2\cdot N}{(N-2)\cdot p}}(\Omega)$. Hence

$$\begin{aligned} \| Tu_n - Tu \|_{12} &= \sup_{\|\phi\|_{12}=1} |(T(u_n - Tu, \phi)_{12}| \\ &= \sup_{\|\phi\|_{12}=1} |\int_\Omega (f(u_n) \cdot u_n - f(u) \cdot u) \cdot \phi dx \\ &\leq \sup_{\|\phi\|_{12}=1} \| f(u_n) \cdot u_n - f(u) \cdot u \|_{0\frac{2\cdot N}{N+2}} \cdot \| \phi \|_{0\frac{2\cdot N}{N-2}} \\ &\leq c \cdot \sup_{\|\phi\|_{12}=1} \| f(u_n) \cdot u_n - f(u) \cdot u \|_{0\frac{2\cdot N}{N+2}} \cdot \| \phi \|_{12} \\ &\leq c \cdot \| f(u_n) \cdot u_n - f(u) \cdot u \|_{0\frac{2\cdot N}{N+2}} \longrightarrow 0 \end{aligned}$$

for $n \longrightarrow \infty$, $N > 2$ and

$$\begin{aligned} \| Tu_n - Tu \|_{12} &\leq \sup_{\|\phi\|_{12}=1} \| f(u_n) \cdot u_n - f(u) \cdot u \|_{02} \cdot \| \phi \|_{02} \\ &\leq c \| f(u_n) \cdot u_n - f(u) \cdot u \|_{02} \longrightarrow 0 \end{aligned}$$

for $N = 2$.

This proves continuity of T.

Now let $\{u_n\}$ be a bounded sequence in $\overset{\circ}{W}_{12}(\Omega)$. From the compactness of the imbedding $\overset{\circ}{W}_{12}(\Omega) \subset L_{\frac{2\cdot N\cdot p}{N+2}}(\Omega)$, we conclude that, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \longrightarrow w$ *in* $L_{\frac{2\cdot N\cdot p}{N+2}}(\Omega)$, $k \longrightarrow \infty$. It follows from inequality (2) and the Vajnberg theorem that, $f(u_{n_k}) \cdot u_{n_k} \longrightarrow f(w) \cdot w$ *in* $L_{\frac{2\cdot N}{N+2}}(\Omega)$. The continuity of T implies that T is compact.□

Let $A : \overset{\circ}{W}_{12}(\Omega) \longrightarrow \overset{\circ}{W}_{12}(\Omega)$ be a map defined by

$$A = I - \lambda \cdot L + T.$$

We shall prove some properties of A.

Lemma 4. *The map is odd.*

P r o o f. $A(-u) = (-u) - \lambda \cdot L(-u) + T(-u) = -u + \lambda \cdot Lu - Tu = -(u - \lambda \cdot Lu + Tu) = -Au.\Box$

Lemma 5. *The map A is proper.*

P r o o f. We shall show that, $\| Au \|_{12} \longrightarrow \infty$ if $\| u \|_{12} \longrightarrow \infty$. Let $\{u_n\} \subset \overset{\circ}{W}_{12}(\Omega)$ and $\| u_n \|_{12} \longrightarrow \infty$ as $n \longrightarrow \infty$. Then

$$\begin{aligned}
(Au_n, u_n)_{12} &= (u_n, u_n)_{12} - \lambda \cdot (Lu_n, u_n)_{12} + (Tu_n, u_n)_{12} \\
&= \| u \|_{12}^2 - \lambda \cdot \| u \|_{02}^2 + \int_\Omega f(u_n) \cdot u_n^2 dx \\
&\geq \| u_n \|_{12}^2 - \lambda \cdot \| u_n \|_{02}^2 + \int_\Omega |u_n|^{2+\alpha} dx \\
&= \| u_n \|_{12}^2 - \lambda \cdot \| u_n \|_{02}^2 + \| u_n \|_{02+\alpha}^{2+\alpha} \\
&\geq \| u_n \|_{12}^2 - \lambda \cdot \| u_n \|_{02}^2 + c \cdot \| u_n \|_{02}^{2+\alpha} \\
&= \| u_n \|_{12}^2 + \| u_n \|_{02}^2 (c \cdot \| u_n \|_{02}^{\alpha} - \lambda).
\end{aligned}$$

But $(Au_n, u_n)_{12} \leq \| Au_n \|_{12} \cdot \| u_n \|_{12}$, therefore

$$\| Au_n \|_{12} \geq \| u_n \|_{12} + \| u_n \|_{02}^2 (c \cdot \| u_n \|_{02}^{\alpha} - \lambda) / \| u_n \|_{12} \longrightarrow \infty \, as \, n \longrightarrow \infty.$$

$\Box$

Theorem 1. *The map A is suriective.*

P r o o f. It follows from fact that, a continuous, proper and odd map of the form I + C (where C is a compact) is suriective.$\Box$

Theorem 2. *If $\lambda \leq \lambda_1$ and f is decreasing for $t \leq 0$, then A is a homeomorphism.*

P r o o f. We shall prove that A is injective. Let us assume that, there exist $u, v \in \overset{\circ}{W}_{12}(\Omega)$, such that $Au = Av$. Then

$$(Au, u - v)_{12} = (Av, u - v)_{12}$$

$$(u - v, u)_{12} - \lambda \cdot (Lu, u - v)_{12} + (Tu, u - v)_{12} -$$

$$- (v, u - v)_{12} + \lambda \cdot (Lv, u - v)_{12} - (Tv, u - v)_{12} = 0$$

$$(u - v, u - v)_{12} - \lambda \cdot (Lu - Lv, u - v)_{12} + (Tu - Tv, u - v)_{12} =$$

$$\| u - v \|_{12}^2 - \lambda \cdot \| u - v \|_{02}^2 + \int_\Omega (f(u) \cdot u - f(v) \cdot v) \cdot (u - v) dx = 0.$$

Hence

$$\| u - v \|_{12}^2 - \lambda \cdot \| u - v \|_{02}^2 \quad = - \int_\Omega (f(u) \cdot u - f(v) \cdot v) \cdot (u - v) dx.$$

Since $\lambda \leq \lambda_1$, the left-hand side of the equation is nonnegative, the right-hand side is nonpositive, so $u = v$ and therefore A is injective. From theorem 1 we conclude that, A is homeomorphism.□

An element $u \in \overset{\circ}{W}_{12}(\Omega)$ is called a weak solution of problem (1), if the following condition is satisfied

$$\int_\Omega \nabla u \cdot \nabla \phi dx - \lambda \cdot \int_\Omega u \cdot \phi dx + \int_\Omega f(u) \cdot u \cdot \phi dx = - \int_\Omega g \cdot \phi dx \tag{4}$$

for every $\phi \in \overset{\circ}{W}_{12}(\Omega)$.

Let $g \in W_{-12}(\Omega)$, then there exists $h \in \overset{\circ}{W}_{12}(\Omega)$ such that $- \int_\Omega g \cdot \phi dx = (h, \phi)_{12}$ and (4) can be written in the form

$$(u - \lambda \cdot Lu + Tu, \phi)_{12} = (h, \phi)_{12} \tag{5}$$

for every $\phi \in \overset{\circ}{W}_{12}(\Omega)$.
This equation is equivalent to

$$u - \lambda \cdot Lu + Tu = h \tag{6}$$

or

$$Au = h.$$

Theorem 3. *If $g \in L_r(\Omega)$ where, $r > 1$, $N = 2$, $p > 1 + \alpha$ or $N \geq 3$, $r > \frac{2 \cdot N}{N+2}$, $1 + \alpha < p < \frac{N+2}{N-2}$, then there exists a weak solution of problem (1). If additionally $\lambda \leq \lambda_1$ and function f is decreasing for $t \leq 0$, the problem (1) has exactly one weak solution.*

P r o o f. Let g satisfies the assumptions of theorem. Then

$$\left| \int_\Omega (-g) \cdot \phi dx \right| \;\leq\; \| g \|_{02} \cdot \| \phi \|_{02} \;\leq\; c \cdot \| \phi \|_{12} \qquad in\ case\ N = 2$$

or

$$\left| \int_\Omega (-g) \cdot \phi dx \right| \;\leq\; \| g \|_{0 \frac{2 \cdot N}{N+2}} \cdot \| \phi \|_{0 \frac{2 \cdot N}{N-2}} \;\leq\; c \cdot \| \phi \|_{12} \qquad for\ N \geq 3.$$

It denotes that $g \in W_{-12}(\Omega)$. The thesis of theorem follows from theorems 1 and 2.□

REFERENCES

[1] A. Ambrosetti and G. Prodi. On the inversion of some differentiable mappings with singularities between Banach spaces, *Ann. Math. Pure Appl.*, 93, 1972, P. 213–246.

[2] A. Bahri and B. Berestycki. A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.*, 267, 1981, P. 1–30.

[3] M. S. Berger. Nonlinearity and functional analysis, *Academic Press*, New York, 1977.

[4] M. S. Berger and E. Podolak. On the solutions of a nonlinear Dirichlet problems, *Indiana Univ. Math. J.*, 24, 1975, P. 837–846.

[5] J. Bęczek. On solutions of some nonlinear Dirichlet problems, *Zeszyty naukowe UG*, 1987, P. 7-15.

[6] J. Bęczek. On the existence of solutions of some nonlinear Dirichlet problems, *Annales Polonici Math.*, XLIX, 1988, P. 45-51.

[7] E. N. Dancer. On the Dirichlet Problem for Weakly Nonlinear Elliptic Partial Differential Equations, *Proc. Royal. Soc.*, Edinburgh 76 A , 1977, P. 283-300.

[8] J. Dugundi and A. Granas. Fixed point theory, PWN Warszawa, 1982.

[9] O. Ladyzenskaya and N. Uralceva. Linear and Quasilinear Elliptic Partial Differential Equations, *Academic Press*, New York , 1968.

[10] L. Nirenberg. Topics in Nonlinear Functional Analysis, *Courant Inst. of Math. Sciences*, New York, 1974.