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THE NONLINEAR CONVOLUTIONAL EQUATIONS SOLVABLE IN CLOSED FORM

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1. SECTION 1

Let C: $H \leftrightarrow W$ is bijective linear operator, operating from algebra W in linear space H. We define convolution with weight $\gamma \in W$ by equation [1,2]
$\varphi * \psi = C^{-1}\left[\gamma \cdot (C\varphi) \cdot (C\psi)\right], \varphi, \psi$ and $\varphi * \psi \in H$
Then
$\varphi * \psi = \psi * \varphi$ and $(\varphi * \psi) * \chi = \varphi * (\psi * \chi), \forall \varphi, \psi, \chi \in H$

$$(\varphi * \psi) * \chi = C^{-1}\left\{\gamma \cdot \left[C(\varphi * \psi)\right] \cdot (C\chi)\right\} = C^{-1}\left[\gamma^2 \cdot C(\varphi) \cdot C(\psi) \cdot (C\chi)\right] = \varphi * (\psi * \chi),\ \varphi, \psi,\ \chi \in H$$

Then from here the possibility of definition of "power" of convolution follows [3,4]

$$(\varphi)^n = \underbrace{\varphi * \varphi * \ldots * \varphi}_{n} = C^{-1}\left[\gamma^{n-1} \cdot C(\varphi)^n\right],\ n \in N.$$

Particular cases are:

$$(\varphi)^1 = \varphi,\ (\varphi)^n * (\varphi)^m = (\varphi)^{n+m}\ \forall n, m \in N.$$

We shall suppose further, that the algebra H is commutative normalized ring [5] concerning of norm $\|\cdot\|$ such, that

$$\|\varphi * \psi\| \le \|\varphi\| \cdot \|\psi\|, \forall \varphi, \psi \in H \text{ and, particularly, } \|(\varphi)^n\| \le \|\varphi\|^n \tag{1}$$

At last, we shall suppose, that for element $\varphi \in H$ exists finite or infinite limit

$$\lim_{n \to \infty}\left[\gamma^{n-1} \cdot (C\varphi)^n\right] = \varphi_\infty \tag{2}$$

where from $\lim\limits_{n \to \infty} (\varphi)^n = C^{-1}(\varphi_\infty)$ or belong to H, or this limit is infinite.

2. SECTION 2

Assuming, that $\gamma^{-1} \in W$, power series has form (3)

$$y = a(x) = \sum_{n=1}^{\infty} a_n x^n, \ a_1 = 1, \ |x| < r_a. \tag{3}$$

and we compare “convolutional” series (compare [4]) to power series (3)

$$\sum_{n=1}^{\infty} a_n (\varphi)^n = \sum_{n=1}^{\infty} a_n C^{-1} \left[\gamma^{n-1} \cdot C(\varphi)^n\right] = C^{-1} \left[\sum_{n=1}^{\infty} a_n \gamma^{n-1} \cdot C(\varphi)^n\right] =$$

$$C^{-1} \left\{\gamma^{-1} \cdot \sum_{n=1}^{\infty} a_n \left[\gamma^n \cdot C(\varphi)^n\right]\right\} = C^{-1} \left\{\gamma^{-1} \cdot a\left[\gamma \cdot C(\varphi)^n\right]\right\}, \tag{4}$$

which, obviously, make sence, if $\|\varphi\| \leq r_a$.

Let the series

$$x = \sum_{k=1}^{\infty} b_k y^k, \ b_1 = 1, \ |y| < r_b \tag{5}$$

convert series (3):

$$b(a(x)) = x, \ |x| < s_a < r_a \quad a(b(y)) = y, \ |y| < s_b < r_b, \tag{6}$$

then the factors b_k of series (5) determining (see p.755 in [6]) by formulae:

$$b_m = \sum_{\alpha_2 + 2\alpha_3 + \cdots + (m-1)\alpha_m = m-1} \frac{(m-1+\alpha_2+\cdots+\alpha_m)!}{m!\alpha_2! \cdots \alpha_m!} \times (-a_2)^{\alpha_2} \cdot \ldots \cdot (-a_m)^{\alpha_m}, \tag{7}$$

where $\alpha_j \in N$.

In particular, if series (3) is polinomial of power p, then instead (7) we have:

$$b_m = \sum_{\alpha_2 + 2\alpha_3 + \cdots + (p-1)\alpha_p = m-1} \frac{(m-1+\alpha_2+\cdots+\alpha_p)!}{m!\alpha_2! \ldots \alpha_p!} \cdot (-a_2)^{\alpha_2} \cdot \ldots \cdot (-a_p)^{\alpha_p}. \tag{8}$$

3. SECTION 3

We consider convolutional equation

$$a(\varphi) = \sum_{n=1}^{\infty} a_n (\varphi)^n = \psi, \tag{9}$$

in which ψ - is known, and φ - is required element from H. In general case this equation have infinite order, but it may be polynomial equation of power $p \geq 2$.

We allows, that function $a(x)$ is invertible with help series (5) and, hence, the equality (6) and (7) (or (8)) take place. We notice, that function $a(x)$, is strictly monotonic on interval $\{|x| < s_a < r_b\}$, is obviously invertible.

Takinig into account (4) we find that $b(\gamma \cdot (C\psi)) = \gamma \cdot (C\varphi)$ or

$$\varphi = C^{-1} \left[\gamma^{-1} \cdot b(\gamma \cdot (C\psi))\right] = \sum_{n=1}^{\infty} b_n (\psi)^n . \tag{10}$$

From here, if $\|\psi\| < s_b$, then equality (9) has the solution $\varphi \in H$, determined by formula (10) $\|\varphi\| < s_a$ and by virtue of (1).

It is easy to see, that convolutional equation

$$\sum_{n=1}^{\infty} b_n (\varphi)^n = \psi, \ \psi \in H, \ \|\psi\| < s_b$$

has the solution

$$\varphi = \sum_{n=1}^{\infty} a_n (\psi)^n \in H, \ \|\varphi\| < s_a.$$

4. SECTION 4

We made two remarks:

a) Departing from decomposition of kind [7]

$$\sin \lambda x = 2 \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(\lambda) T_{2n-1}(x) =$$
$$\frac{2}{\lambda} \sum_{n=1}^{\infty} (-1)^n J_{2n}(\lambda) U_{2n-1}(x), \ |x| < 1, \ 0 < \lambda < \pi$$

naturally series (3) to replace on more general series

$$\sum_{n=1}^{\infty} b_n Q_{2n-1}(x), \ Q_{2n-1}(x) = \sum_{s=1}^{n} q_{n,s} x^{2s-1}.$$

b) Probably, for the first time particular nonlinear equation in convolutions with p=2 was investigated in article [8]. One of variants of research of polynomial equations with convolutions is offered in [4].

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