MMCA–97 Conference, pp. 105–116
R. Čiegis (Ed)

USING VARIATIONAL FORMULAE FOR SOLVING EXTREMAL PROBLEMS IN LINEARLY-INVARIANT CLASSES

E.G.KIRJACKIJ

Vilnius Gediminas Technical University
Saulėtekio ave. 11, LT–2054 Vilnius

1. INTRODUCTION

Let for every function $f(z)$ belonging to some class V one can separate a family of functions $f(z,\varepsilon)$ uniformly differentiable with respect to ε at $\varepsilon = 0$ inside the domain D of the class, then the expansion of the form

$$f(z,\varepsilon) = f(z) + \varepsilon Q(z) + o(|\varepsilon|, D)$$

is called to be a variational formula in class V, written for $f(z)$.

Constructing of the variational formula is usually a complicated separate problem.

Variational method belongs to a group of main methods of geometrical theory of functions of complex variable. By this method one can solve a whole set of extremal problems, especially in a theory of univalent functions. Applying variational-geometrical M.A. Lavrentjev had obtained outstanding results in applicatory problems. The method proposed by American mathematician Schiffer for extremal problems in the class of univalent functions leads to differential equations for extremal function. G.M. Goluzin proposed method of variations of his own, due to which he obtained the same differential equations Schiffer had obtained. The shortcoming of this method is the fact that solving of the arising differential equation with respect to extremal function not always comes to an end.

At the same time the class of univalent functions was either shrinked to some subclasses (the class of convex functions) or, in contrary, extended and submerged into wider classes (class of many-sheeted functions).

Such an extension was proposed by German mathematician Ch. Pommerenke [4], which was familiar to the author of this paper, as well (see [2]). Let $\tilde{A}_1(E)$ be a class of analytical functions $f(z)$ in a unit circle E, i.e. in the circle $|z| < 1$, and let functions $f(z)$ be normalized by conditions $f(0) = 0, f'(0) = 1$, and $f'(z) \neq 0$ in E. Let also Λ be a set of all analytical automorphisms of unit circle (Omega-transformations) having the shape

$$\omega = \omega(z) = \frac{e^{i\Theta} z + \xi}{1 + \bar{\xi} e^{i\Theta} z}, \quad \xi \in E, \quad \Theta \in (-\infty, \infty).$$

We introduce an operator Pommerenke on the class $\tilde{A}_1(E)$

$$\Omega^{\omega}\big[f(z)\big] = \frac{f\big(\omega(z)\big) - f(\xi)}{e^{i\Theta}\big(1 - |\xi|^2\big) f'(\xi)}.$$

This operator transforms any function from the class $\tilde{A}_1(E)$ into a function belonging to the same class. *We denote as $\tilde{\mathcal{I}}(E)$ the class of analytical functions $f(z)$ from $\tilde{A}_1(E)$, having the following property: If function $f(z) \in \tilde{\mathcal{I}}_1(E)$, then the function $f(z; \omega) = \Omega_1^{\omega}\big[f(z)\big] \in \tilde{\mathcal{I}}_1(E)$ for any $\omega \in \Lambda$. as well. The class $\tilde{\mathcal{I}}_1(E)$ was named by Pommerenke as linearly-invariant.* For example, the class $\tilde{A}_1(E)$ as well as the class $\tilde{K}_1(E)$ of univalent and normalized functions in E can be concidered as linearly-invariant classes.

We note, that the idea to use intensely the analytical automorphism of unit circle in combination of the operator of Pommerenke to analyse the properties of functions from the class $\tilde{K}_1(E)$ belongs also to French mathematician Marti [3], who obtained a lot of outstanding results in this class.

The told above yields that *in order to build any linearly-invariant class of analytical functions in the unit circle, one need to introduce a particular operator defined on that class which depends both on the analytical automorphism and a corresponding normalization of functions belonging to that class.* For the examples which clearly show the way how combining of introduced set of normalized functions, automorphism of the circle, and operator can lead to interesting ideas in the analysis of properties of analytical functions, see the book of the author of this paper [1].

The study of the properties of functions belonging to some class is build according to classical scheme usually used in theory of analytical functions. We establish the various criteria of dependency to a given class, state the problem of compactness of some families of functions from this class, find the range of functionals, give various estimates of modulus of functions and derivatives, analyse the behaviour of the coefficients of an expansion, search for the fixed points of operators, solve the extremal problems, and introduce variational formulae.

Here we study the linearly-invariant classes of analytical functions in E introduced by the author of this paper. Considering the lack of space, the possible reader is introduced to some variational formulae, as well as to application for solving of extremal problems.

2. VARIATIONAL FORMULAE

We denote as $\tilde{A}_0(E)$ a class of functions $f(z)$ analytical in E normalized by the condition $f(0)=1$ having the property $f(z)\neq 0$ in E. We introduce an operator Ω^ω on the class $\tilde{A}_0(E)$ as follows:

$$\Omega^\omega[f(z)] = \frac{f(\omega(z))}{f\big(\omega(0)\big)}.$$

This operator transforms the function of the class $\tilde{A}_0(E)$ into a function of the same class $\tilde{A}_0(E)$. *We will call the set $\tilde{\mathcal{I}}(E)$ of functions $f(z)$ of $\tilde{A}_0(E)$ as linearly-invariant class, if $f(z)\in\tilde{\mathcal{I}}(E)$ implies $f(z;\omega)=\Omega^\omega[f(z)]\in\tilde{\mathcal{I}}(E)$ for any $\in\Lambda$.* Obviously, the class $\tilde{A}_0(E)$ is the widest linearly-invariant class.

We get one more example of a linearly-invariant class, if we fix in an arbitrary way the function $f(z)\in\tilde{A}_0(E)$ and then adjoin to it all of the functions $f(z;\omega)$, where ω takes all the values from the set of omega-transformations of Λ. We note, that a single function $f(z)\equiv 1$ makes the linearly-invariant class.

We denote as Λ^* the set of omega-transformations of the shape

$$\omega=\omega(z)=\frac{z+\xi}{1+\bar{\xi}z},\quad \xi\in E.$$

Obviously, $\Lambda^*\subset\Lambda$. Let $\tilde{\mathcal{I}}(E)$ any linearly-independent class. The definitions of such a class implies that operator Ω^ω, where $\omega\in\Lambda^*$ transforms the function

$$f(z)=1+\sum_{k=1}^{\infty}a_k z^k \tag{1}$$

of the class $\tilde{\mathcal{I}}(E)$ to the function

$$f(z;\xi)=1+\sum_{k=1}^{\infty}a_k(\xi)z^k. \tag{2}$$

belonging to the class $\tilde{\mathcal{I}}(E)$ too. It is not difficult to prove the following lemma.

LEMMA 1. *For the k-th coefficient $a_k(\xi)$ of the function $f(z;\xi)$ the following formula holds:*

$$a_k(\xi)=\sum_{m=1}^{k-1}\frac{(k-1)!(-1)^m}{m!(k-1-m)!}\big(1-|\xi|^2\big)\bar{\xi}^m\frac{f^{(k-m)}(\xi)}{(k-m)!f(\xi)}. \tag{3}$$

Obviously,

$$f(z;0)\equiv f(z),\quad a_k(0)=a_k. \tag{4}$$

THEOREM 1. *If function*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \in \tilde{\mathcal{I}}(E),$$

then the function

$$f(z;\xi) = 1 + \sum_{k=1}^{\infty} a_k(\xi) z^k \in \tilde{\mathcal{I}}(E),$$

for any $\xi \in E$ and the representation

$$f(z;\xi) = f(z) + \big(f'(z) - a_1 f(z)\big)\xi - z^2 f'(z)\bar{\xi} + o(|\xi|), \tag{5}$$

holds for any sufficiently small values of the modulus of the parameter ξ.

P r o o f. Dependency of the function $f(z;\xi)$ to the class $\tilde{\mathcal{I}}(E)$ for any $\xi \in E$ follows from the definition of the linearly-invariant class. Next, according to the lemma 1, the formula (3) for the k-th coefficient of the function $f(z;\xi)$ holds. Let $\xi = xe^{i\gamma}$, $-1 < x < 1$, $0 \le \gamma < 2\pi$. The function $f(z; xe^{i\gamma})$for fixed values of z and γ is analytical on $x = 0$. Consequently, considering (4), we may write

$$f\big(z; xe^{i\gamma}\big) = f(z) + G(z)x + o(|x|), \tag{6}$$

where

$$G(z) = \frac{\partial f\big(z; xe^{i\gamma}\big)}{\partial x}\Big|_{x=0}. \tag{7}$$

Next, the k-th coefficient $a_k(xe^{i\gamma})$ for fixed $\gamma \in [0, 2\pi]$ is an analytical function at the point $x = 0$. Applying (3) it is not difficult to get the formula

$$\frac{\partial a_k(xe^{i\gamma})}{\partial x}\Big|_{x=0} = (k+1)a_{k+1}e^{i\gamma} - a_k a_1 e^{i\gamma} - (k-1)a_{k-1}e^{-i\gamma}. \tag{8}$$

Let us evaluate $G(z)$. Basing upon (2),(7), and (8), we get

$$\begin{aligned} G(z) &= \frac{\partial f(z; xe^{i\gamma})}{\partial x}\Big|_{x=0} = \sum_{k=1}^{\infty} \frac{\partial a_k(xe^{i\gamma})}{\partial x}\Big|_{x=0} \\ &= \sum_{k=1}^{\infty} \big[(k+1)a_{k+1}e^{i\gamma} - a_k a_1 e^{i\gamma} - (k-1)a_{k-1}e^{-i\gamma}\big] z^k \\ &= e^{i\gamma}\sum_{k=1}^{\infty}(k+1)a_{k+1}z^k - a_1 e^{i\gamma}\sum_{k=1}^{\infty} a_k z^k - e^{-i\gamma}\sum_{k=1}^{\infty}(k-1)a_{k-1}z^k \\ &= e^{i\gamma}\big(f'(z) - a_1\big) - a_1 e^{i\gamma}\big(f(z) - 1\big) - e^{-i\gamma}z^2 f'(z) \\ &= \big(f'(z) - a_1 f(z)\big)e^{i\gamma} - z^2 f'(z)e^{-i\gamma}. \end{aligned}$$

Multiplying G(z) by x and substituting into (6) one can see, that the function $f(z;\xi)$ has the representation (5) for sufficiently small values of the modulus of ξ.

Formula (5) is called a variational formula for the function $f(z)$ in the class $\tilde{\mathcal{I}}(E)$. Altogether with this formula we introduce a variational formula for the k-th coefficient of the function $f(z)$.

THEOREM 2. *Let function*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \in \tilde{\mathcal{I}}(E).$$

Then for the k-th coefficient $a_k(\xi)$ of the function

$$f(z;\xi) = 1 + \sum_{k=1}^{\infty} a_k(\xi) z^k \in \tilde{\mathcal{I}}(E)$$

the representation

$$a_k(\xi) = a_k + \big((k+1)a_{k+1} - a_k a_1\big)\xi - (k-1)a_{k-1}\bar{\xi} + o(|\xi|) \tag{9}$$

holds for any sufficiently small values of the modulus of the parameter ξ.

P r o o f. Let $\xi = xe^{i\gamma}, -1 < x < 1, 0 \le \gamma < 2\pi$. According to (5) for fixed $\gamma \in [0, 2\pi)$ the function $a_k(xe^{i\gamma})$ is analytical in the point $x = 0$. Therefore

$$a_k(xe^{i\gamma}) = a_k + \frac{\partial a_k(xe^{i\gamma})}{\partial x}\Big|_{x=0} + o(|x|)$$

for any $|x| < p$, where p is sufficiently small. Basing upon (5) we get (9).

The following theorem produces one more variational formula gor the function $f(z)$ in the class $\tilde{\mathcal{I}}(E)$. The proof is of no difficulty.

THEOREM 3. *If function $f(z) \in \tilde{\mathcal{I}}(E)$, then the function $f\big(e^{i\gamma}\big) \in \tilde{\mathcal{I}}(E)$ for any real γ. Moreover, for any real γ the following representation holds:*

$$f\big(e^{i\gamma}z\big) = f(z) + if'(z)z\gamma + o(|\gamma|). \tag{10}$$

3. EXTREMAL PROBLEMS

Using the variational formulae (5),(9), and (10) we will solve a few extremal problems.

THEOREM 4. *Let in a linearly-invariant class $\tilde{\mathcal{I}}(E)$ there exists function $f_1(z) = 1 + a_1 z + a_2 z^2 + \cdots$, having the following property: for the k-th* *coefficient of it the equality*

$$|a_k| = \max_{f(z)\in\tilde{\mathcal{I}}(E)} \frac{1}{k!}\left|f^{(k)}(0)\right| \tag{11}$$

holds for fixed value of $k \geq 1$. *Then*

$$\bar{a}_k((k+1)a_{k+1} - a_k a_1) - (k-1)a_k\bar{a}_{k-1} = 0. \tag{12}$$

P r o o f. Since $f_1(z) \in \tilde{\mathcal{I}}(E)$, the according to Theorem 2, the function $f_1(z;\xi)$ for any $\xi \in E$, and there exists a positive number p such that for the k-th coefficient $a_k(\xi)$ of the function $f_1(z;\xi)$ the representation

$$a_k(\xi) = a_k + \big((k+1)a_{k+1} - a_k a_1\big)\xi - (k-1)a_{k-1}\bar{\xi} + o(|\xi|)$$

holds if $|\xi| < p$. The assumption (11) of Theorem 4 implies

$$|a_k(\xi)| \leq |a_k|, \quad \forall|\xi| < p. \tag{13}$$

Next, for any $|\xi| < p$ one can get the following:

$$\begin{aligned} |a_k(\xi)|^2 &= |a_k + \big((k+1)a_{k+1} - a_k a_1\big)\xi - (k-1)a_{k-1}\bar{\xi} + o(|\xi|)|^2 \\ &= \left|a_k\right|^2 + 2Re\big\{\big[\bar{a}_k\big((k+1)a_{k+1} - a_k a_1\big) - (k-1)a_k\bar{a}_{k-1}\big]\xi\big\} \\ &\quad + Re\big\{o(\left|\xi\right|)\big\}. \end{aligned}$$

This and (13) implies

$$Re\big\{\big[\bar{a}_k + \big((k+1)a_{k+1} - a_k a_1\big) - (k-1)a_k\bar{a}_{k-1}\big]\xi\big\} + Re\big\{o(|\xi|)\big\} \leq 0,\ \forall|\xi| < p.$$

Taking into account the arbitrarity of the value of the argument of the complex number ξ, we conclude that (12) is valid.

THEOREM 5. *Let in a linearly-invariant class* $\tilde{\mathcal{I}}(E)$ *there exists the function* $f_2(z) = 1 + a_1 z + a_2 z^2 + \cdots$, *the* k*-th coefficient of which has either the property*

$$Re\{a_k\} = \max_{f(z)\in\tilde{\mathcal{I}}(E)} Re\big\{\frac{1}{k!}f^{(k)}(0)\big\} \tag{14}$$

or

$$Re\{a_k\} = \min_{f(z)\in\tilde{\mathcal{I}}(E)} Re\big\{\frac{1}{k!}f^{(k)}(0)\big\} \tag{15}$$

which holds for fixed $k \geq 1$. *Then the equality*

$$(k+1)a_{k+1} - a_k a_1 - (k-1)\bar{a}_{k-1} = 0. \tag{16}$$

holds.

P r o o f. Let for the function $f_2(z) \in \tilde{\mathcal{I}}(E)$ condition (14) holds. According to Theorem 2, the function $f_2(z;\xi) \in \tilde{\mathcal{I}}(E)$ for any $\xi \in E$ and there exists positive number p such that for the k-th coefficient $a_k(\xi)$ of this function the representation

$$a_k(\xi) = a_k + \big((k+1)a_{k+1} - a_k a_1\big)\xi - (k-1)a_{k-1}\bar{\xi} + o(|\xi|),$$

is valid if $|\xi| < p$. Taking into account condition (14) in Theorem 5 it implies

$$Re\big\{a_k(\xi)\big\} \le Re\big\{a_k\big\}, \quad \forall|\xi| < p.$$

Consequently,

$$Re\big\{\big((k+1)a_{k+1} - a_k a_1\big)\xi - (k-1)a_{k-1}\bar{\xi} + o(|\xi|)\big\} \le 0, \quad \forall|\xi| < p.$$

Hence

$$Re\big\{\big((k+1)a_{k+1} - a_k a_1 - (k-1)\bar{a}_{k-1}\xi\big\} + Re\big\{o(|\xi|)\big\} \le 0, \quad \forall|\xi| < p.$$

Since the argument of the complex number ξ one can take in an arbitrary way, then it is easy to see the validity of (16). In case of condition (15) the theorem is proved analogously.

Corollary 1. Let in a linearly-invariant class $\tilde{\mathcal{I}}(E)$ there exists a function $f_3(z) = 1 + a_1 z + a_2 z^2 + \cdots$, the k-th coefficient of which a_1 has the following property:

$$|a_1| = \max_{f(z)\in\tilde{\mathcal{I}}(E)} \big|f'(0)\big|.$$

Then

$$a_2 = \frac{1}{2}a_1^2. \tag{17}$$

To prove (17) it is enough to take $k = 1$ in Theorem 4.

Corollary 2. Let in a linearly-invariant class $\tilde{\mathcal{I}}(E)$ there exists a function $\varphi(z) = 1 + a_1 z + a_2 z^2 + \cdots$ all the coefficients of an expansion of which are real numbers and the k-th coefficient a_k has the following property:

$$\big|a_k\big| = \max_{f(z)\in\tilde{\mathcal{I}}(E)} \big|\frac{1}{k!}f^{(k)}(0)\big|.$$

Then the equality

$$(k+1)a_{k+1} - a_k a_1 - (k-1)\bar{a}_{k-1} = 0.$$

holds.

THEOREM 6. *Let in a linearly-invariant class $\tilde{\mathcal{I}}(E)$ there exists a function $f_0(z) = 1 + a_1 z + a_2 z^2 + \cdots$, having the following property: at the point $z_0 \in E$ either the equality*

$$\big|f_0(z_0)\big| = \max_{f(z)\in\tilde{\mathcal{I}}(E)} \big|f(z_0)\big|. \tag{18}$$

or

$$\big|f_0(z_0)\big| = \min_{f(z)\in\tilde{\mathcal{I}}(E)} \big|f(z_0)\big|. \tag{19}$$

holds. Then

$$a_1 = \Big(1 - |z_0|^2\Big)\frac{f_0'(z_0)}{f_0(z_0)}. \tag{20}$$

P r o o f. Let the condition (18) for the function $f_0(z) \in \tilde{\mathcal{I}}(E)$ holds. According to Theorem 1 the function $f_0(z;\xi) \in \tilde{\mathcal{I}}(E))$ for any $\xi \in E$ and there exists a positive number p such that

$$f_0(z;\xi) = f_0(z) + \big(f_0'(z) - a_1 f_0(z)\big)\xi - z^2 f_0'(z)\bar{\xi} + o\big(|\xi|\big).$$

holds. Taking into account

$$|f_0(z_0;\xi)| \le |f_0(z_0)|, \forall|\xi| < p$$

this implies the inequality

$$\big|f_0(z_0) + (f_0'(z_0) - a_1 f_0(z_0))\xi - z_0^2 f_0'(z_0)\bar{\xi} + o(|\xi|)\big| \le \big|f_0(z_0)\big|, \quad \forall|\xi| < p,$$

which can be easily substituted by the inequality

$$Re\Big\{\Big[\bar{f}_0(z_0)\big(f_0'(z_0) - a_1 f(z_0)\big) - f_0(z_0)\bar{z}_0^2\bar{f}_0'(z_0)\Big]\xi\Big\} + Re\big\{o(|\xi|)\big\} \le 0, \quad \forall|\xi| < p.$$

According to arbitrarity of the values of the argument of the complex number ξ, we get either

$$\bar{f}_0(z_0)\big(f_0'(z_0) - a_1 f_0(z_0)\big) - f_0(z_0)\bar{z}_0^2\bar{f}_0'(z_0) = 0$$

or

$$\bar{f}_0(z_0)f_0'(z_0) - f_0(z_0)\bar{z}_0^2\bar{f}_0'(z_0) - a_1\bar{f}_0(z_0)f_0(z_0) = 0. \tag{21}$$

Next, the function $\varphi_\gamma(z) = f_0(e^{i\gamma}z) \in \tilde{\mathcal{I}}(E)$ for any real γ therefore according to (18) we get

$$\big|\varphi_\gamma(z_0)\big| = \big|f_0\big(e^{i\gamma}z_0\big)\big| \le \big|f_0(z_0)\big|, \quad \forall\gamma \in (-\infty, \infty).$$

Hence applying the variational formulae (10) we come to an inequality

$$\gamma Re\{iz_0\bar{f}_0(z_0)f_0'(z_0)\} + Re\{o(|\gamma|)\} \le 0, \quad \forall\gamma \in (-\infty,\infty).$$

But it is not difficult to notice, that

$$Re\{iz_0 f_0'(z_0)\bar{f}_0(z_0)\} = 0.$$

The latter equality can be substituted by the equality

$$Im\{z_0 f_0'(z_0)\bar{f}_0(z_0)\} = 0.$$

therefore

$$\bar{z}_0^2 f_0(z_0)\bar{f}_0'(z_0) = |z_0|^2 \bar{f}_0(z_0)f_0'(z_0). \tag{22}$$

Taking into account both (22) and $\bar{f}_0(z_0) \ne 0$, the equality (21) we substitute by the equality

$$f_0'(z_0) - a_1 f_0(z_0) - |z_0|^2 f_0'(z_0) = 0,$$

which yields (20). Theorem 6 in case the function mentioned in the assumption satisfies the condition (19) is proved analogously.

THEOREM 7. *Let in a linearly-invariant class $\tilde{\mathcal{I}}(E)$ there exists a function $f_4(z) = 1+a_1z+a_2z^2+\cdots$, having the following property: at the point $z_0 \in E$ either the equality*

$$Re\{f_4(z_0)\} = \max_{f(z)\in\tilde{\mathcal{I}}(E)} Re\{f(z_0)\} \tag{23}$$

or

$$Re\{f_4(z_0)\} = \min_{f(z)\in\tilde{\mathcal{I}}(E)} Re\{f(z_0)\} \tag{24}$$

holds. Then

$$a_1 = \left(1 - |z_0|^2\right)\frac{f_4'(z_0)}{f_4(z_0)}. \tag{25}$$

P r o o f. Let for the function $f_4(z) \in \tilde{\mathcal{I}}(E)$ (23) holds. According to Theorem 1, the function $f_4(z;\xi) \in \tilde{\mathcal{I}}(E)$ for any $\xi \in E$ and there exists a positive number p such that

$$f_4(z_0;\xi) = f_4(z_0) + \left(f_4'(z_0) - a_1 f_4(z_0)\right)\xi - z_0^2 f_4'(z_0)\bar{\xi} + o(|\xi|), \quad \forall|\xi| < p.$$

Taking into account (23) this implies

$$\begin{aligned} &Re\{\left(f_4'(z_0) - a_1 f_4(z_0)\right)\xi - z_0^2 f_4'(z_0)\bar{\xi}\} + Re\{o(|\xi|)\} \\ &\quad = Re\{\left(f_4'(z_0) - a_1 f_4(z_0) - \bar{z}_0^2 \bar{f}_4'(z_0)\right)\xi\} + Re\{o(|\xi|)\} \le 0, \quad \forall|\xi| < p. \end{aligned}$$

Taking into account the arbitrarity of the argument of the complex number ξ, we easily come to an equality

$$f_4'(z_0) - a_1 f_4(z_0) - \bar{z}_0^2 \bar{f}_4'(z_0) = 0. \tag{26}$$

Next, the function $\Psi_\gamma(z) = f_4\big(e^{i\gamma} z\big) \in \mathcal{I}(E)$ for any real γ hence according to condition (23), we get

$$Re\big\{\Psi_\gamma(z_0)\big\} = Re\big\{f_4(e^{i\gamma} z_0)\big\} \leq Re\big\{f_4(z_0)\big\}, \quad \forall \gamma \in (-\infty, \infty).$$

Applying the variational formula (10) we come to an inequality

$$\gamma Re\big\{i z_0 f_4'(z_0)\big\} + Re\big\{o(|\gamma|)\big\} \leq 0, \quad \forall \gamma \in (-\infty, \infty).$$

But there is of no difficulty to notice, that

$$Re\big\{i z_0 f_4'(z_0)\big\} = 0.$$

Therefore,

$$Im\big\{z_0 f_4'(z_0)\big\} = 0.$$

Consequently,

$$\bar{z}_0^2 \bar{f}_4'(z_0) = |z_0|^2 f_4'(z_0).$$

Substituting it into (26) yields (25). Theorem 7 in case of the function mentioned in the assumption satisfies the condition (24) is proved analogously.

REMARK. Let us consider the following analytical functions in E :

$$\Phi_{t,a,b}(z) = \Big(\frac{1-\bar{a}z}{1-\bar{b}z}\Big)^{\dfrac{t}{\bar{b}-\bar{a}}}, \quad \Phi_{t,a,b}(z) = \exp\Big\{\frac{tz}{1-\bar{a}z}\Big\},$$

where $|a| < 1, |b| < 1$, and t is complex number. Moreover, let $\Phi_{t,a,b}(0) = 1$, if $a \neq b$, and $\Phi_{t,a,a}(0) = 1$. We will call *those functions as the basic functions in linearly-invariant classes.* There are few of the basic functions:

$$\Phi_{t,-1,0}(z) = (1+z)^t, \quad \Phi_{t,0,1}(z) = (1-z)^t, \quad \Phi_{t,0,0}(z) = e^{tz}.$$

Let us expand the function $\Phi_{t,a,b}(z)$ into the series

$$\Phi_{t,a,b}(z) = 1 + \sum_{k=1}^{\infty} g_{k,a,b}(t) z^k.$$

Then for the k-th coefficient of the expansion the recursion formula

$$g_{k,a,b}(t) = \frac{1}{k}\big(tg_{k-1,a,b}(t)+(k-1)(\bar{a}+\bar{b})g_{k-1,a,b}(t)-(k-2)\bar{a}\bar{b}g_{k-2,a,b}(t)\big), \quad (27)$$

holds, where the assumption of $g_{-1,a,b}(t) \equiv 0$ and $g_{0,a,b}(t) \equiv 1$ is taken. As a separate case,

$$g_{l,a,b}(t) \equiv t.$$

We notice, that $\Phi_{0,a,b} = 1$ and, consequently, $g_{k,a,b}(0) = 0, \forall k \geq 1$. If $a = -1$ and $b = 1$, then we assume for the sake of brevity,

$$\Phi_{t,-1,1}(z) \equiv \Phi_t(z), \quad g_{k,-1,1}(t) = g_k(t).$$

In this case

$$\Phi_t(z) = 1 + \sum_{k=1}^{\infty} g_k(t) z^k,$$

where

$$g_k(t) = \frac{1}{k}\big(tg_{k-1}(t) + (k-2)g_{k-2}(t)\big).$$

We notice, that $\Phi_0(z) \equiv 1$ and, consequently, $g_k(0) = 0, \forall k \geq 1$. It is not difficult to prove, that the basic function $\Phi_{t,a,b}(z)$ is the single solution of the 1st order linear homogeneous differential equation

$$(1-\bar{a}z)(1-\bar{b}z)Z'(z) - tZ(z) = 0$$

subject to initial condition $Z(0) = 1$.

The basic functions are widely used in the analysis of the properties of the linearly-invariant classes, i.e. when solving the various extremal problems they frequently appear to be extremal ones.

Let $z_0 \in E$ and $z_0 \neq 0$. Let us consider the basic function

$$\Phi_{t,a,b}(z) = 1 + \sum_{k=1}^{\infty} g_{k,a,b}(t) z^k, \quad a = -e^{i\gamma_0}, \quad b = e^{i\gamma_0}, \ \gamma_0 = \arg z_0.$$

It is easy to prove that

$$\big(1-|z_0|^2\big)\Phi'_{t,a,b}(z_0) - t\Phi_{t,a,b}(z_0) = 0$$

holds for any t. Since $g_{l,a,b}(t) \equiv t$, then for any t

$$\big(1-|z_0^2|\big)\frac{\Phi'_{t,a,b}(z_0)}{\Phi_{t,a,b}(z_0)} = g_{l,a,b}(t).$$

holds. Next, for any t according to (27) for the coefficients of the function $\Phi_{t,a,b}(z), a = -e^{i\gamma_0}, b = e^{i\gamma_0}, \gamma_0 = \arg z_0$, the equality

$$(k+1)g_{k+1,a,b}(t) - g_{k,a,b}(t)g_{l,a,b}(t) + (k-1)\bar{a}\bar{b}g_{k-1,a,b}(t) = 0$$

holds. Moreover, for any integer m the coefficient $g_{m,a,b}(t)$ is an analytical function on t and $g_{k,a,b} = 0$. Therefore, there exists such a t_0, that $\bar{b}g_{k-1,a,b}(t_0)$ is a real number. In this case $\bar{a}\bar{b}g_{k-1,a,b}(t_0) = -\bar{g}_{k-1,a,b}(t_0)$ and we can write

$$(k+1)g_{k+1,a,b}(t_0) - g_{k,a,b}(t_0)g_{l,a,b}(t_0) - (k-1)\bar{g}_{k-1,a,b}(t_0) = 0$$

It is clear, that there exists such a function $\Phi_{t_0,a,b}(z)$, for which the equalities mentioned in the theorems 5 and 6 hold.

REFERENCES

[**1**] Kirjac E.G. Kirjackij, *Multivalent Functions and Divided Differences.* Vilnius, Technika, 1995, p. 394.

[**2**] Kirjackij E.G. Kirjackij, *Uber funktionen, die keine nullgleiche dividierte differenz besitzen.* Lith. Math. J., **1**(1) 1963, p. 157 – 168.

[**3**] Marti F. Marti, *Sur le module des koefficients de Mac Laurin d'une function univalent.* C.R. Acad. Sci., Paris, 198, 1934, 1569 – 1571.

[**4**] Pomm Ch. Pommerenke, *Linear-invariante familien analitischer funktionen.* Math. An., 155, 1964, p. 108 – 154.