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WEIERSTRASS TRANSFORMATION IN ALGEBRA OF MNEMOFUNCTIONS

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Algebras of mnemofunctions permits to solve the problem of multiplication on distributions. But in those algebras it's not always possible to determine the different integral transformations, which are necessary for the research of number of theoretical physics problems. Integral transformation are widely used in the theory of differential equations, in which distributions and their multiplication play an important role. Therefore it is necessary to determine integral transformations for algebras of mnemofunctions. On the basis of the general method [1] algebras of mnemofunctions with differentiation and Fourier, Laplace and Mellin transformations respectively have been already built [3-5]. This note is dedicated to the building of algebra of mnemofunctions with differentiation and Weierstrass transformation.

Regular Weierstrass transformation is determined by the formula

$$F(z) = \frac{1}{\sqrt{4\pi}} \int\limits_{\mathbf{R}} f(t) e^{-(z-t)^2/4} dt, \tag{1}$$

where $f(t)$ is some regular function of t. Therefore this transformation takes $f(t)$ to the function $F(z)$ of complex argument z.

The basis of the following research will be the variant of determination of generalized Weierstrass transformation suggested by A.H.Zemanian [2].

Let $a,\ b \in R$ and $a < b$. We take two strictly monotonic sequences $a_n \in \mathbf{R}$ and $b_n \in \mathbf{R}$ such that $a_n \to a + 0$ and $b_n \to b - 0$. Then $\mathbf{R}(a, b)$ is space of infinitely differentiable functions with the topology determined by the system of seminorms

$$\mathbf{r}_{s,n}(f) = \max_{k \le s} \sup_{t \in R} |\xi_{-a_n, -b_n}(t) f^{(k)}(t)|, \quad s, n \ge 0, \quad s, n \in Z,$$

where

$$\xi_{\alpha,\beta}(t) = \begin{cases} e^{\alpha t}, & t \geq 0, \\ e^{\beta t}, & t < 0. \end{cases}$$

Therefore, let real part of z be in (a,b). $\mathbf{H}(a,b)$ is the space of analytic functions of z satisfying the condition $\sup_{Re\, z \in \mathbf{A}} |z^k F^{(l)}(z)| < \infty$, where $\mathbf{A}$ is any compact subset of interval (a,b), and k, l are nonnegative integers. On $\mathbf{H}(a,b)$ we shall determine the topology by the system of seminorms

$$\mathbf{h}_{s,\,v,\mathbf{A}}(F) = \max_{\substack{l \leq s \\ i \leq v}} \sup_{Re\, z \in \mathbf{A}} |z^l F^{(i)}(z)|,$$

where s, v, i, l are nonnegative integers. And $\mathbf{A}$ is any compact subset of interval (a,b), $z \in \mathbf{C}$.

$\mathbf{R}(a,b)$ is topological algebra with the multiplication, determined by the formula

$$f(t) \circ g(t) = f(t)g(t)e^{-\alpha t}, \tag{2}$$

where α is a constant in (a,b) [4]. Besides $\mathbf{R}(a,b)$ doesn't depend on selection of sequences (a_n) and (b_n) and Laplace transformation is the isomorphism of $\mathbf{R}(a,b)$ onto $\mathbf{H}(a,b)$. $\mathbf{H}(a,b)$ is the topological algebra [4].

We shall define two new spaces $\mathbf{R}_W(a,b)$ and $\mathbf{H}_W(a,b)$. Further we shall prove that $\mathbf{R}_W(a,b)$ and $\mathbf{H}_W(a,b)$ are the topological algebras and Weierstrass transformation is the isomorpfism of $\mathbf{R}_W(a,b)$ onto $\mathbf{H}_W(a,b)$.

Now $\mathbf{R}_W(a,b)$ is space of infinitely differentiable functions with the topology determined by the system of seminorms

$$\tilde{\mathbf{r}}_{s,\,n}(f) = \max_{k \leq s} \sup_{t \in R} |e^{-t^2/4} \rho_{-a_n,\,-b_n}(t) f^{(k)}(t)|, \quad s,n \geq 0, \quad s,n \in Z,$$

where

$$\rho_{\alpha,\beta}(t) = \begin{cases} e^{-\alpha t/2}, & t < 0, \\ e^{-\beta t/2}, & t \geq 0. \end{cases}$$

Then, $\mathbf{H}_W(a,b)$ is the space of analytic functions of z (for $a < Re\, z < b$) satisfying the condition $\sup_{Re\, z \in \mathbf{A}} |e^{z^2/4} z^k F^{(l)}(z)| < \infty$, where $\mathbf{A}$ is any compact subset of interval (a,b), and k, l are nonnegative integers. On $\mathbf{H}_W(a,b)$ we shall determine the topology by the system of seminorms

$$\tilde{\mathbf{h}}_{s,\,v,\mathbf{A}}(F) = \max_{\substack{l \leq s \\ i \leq v}} \sup_{Re\, z \in \mathbf{A}} |e^{z^2/4} z^l F^{(i)}(z)|,$$

where s, v, i, l are nonnegative integers. And $\mathbf{A}$ is any compact subset of interval (a, b), $z \in \mathbf{C}$.

It's easy to notice that

$$\rho_{\alpha,\,\beta}(t) = \xi_{-\beta/2,\,-\alpha/2}(t). \tag{3}$$

THEOREM 1. *The mapping* $f(t) \to e^{-t^2/4} f(t)$ *is the isomorphism of* $\mathbf{R}_W(a, b)$ *onto* $\mathbf{R}(-b/2, -a/2)$.

P r o o f. First we shall prove that mapping $f(t) \to e^{-t^2/4} f(t)$ is continuous. Let $f(t) \in \mathbf{R}_W(a, b)$, then for all s and n it's possible to determine the topology on $\mathbf{R}(-b/2, -a/2)$ by the following collection of seminorms

$$\mathbf{r}_{s,\,n}(g) = \max_{k \le s} \sup_{t \in R} |\xi_{b_n/2,\,a_n/2}(t) f^{(k)}(t)|.$$

Using the formula (3) we're getting

$$\mathbf{r}_{s,\,n}\left(e^{-t^2/4} f(t)\right) = \max_{k \le s} \sup_{t \in R} \left|\xi_{b_n/2,\,a_n/2}(t)\left(e^{-t^2/4} f(t)\right)^{(k)}\right| \le$$

$$\le c_1 \max_{k \le s} \sup_{t \in R} \left|\rho_{-a_n,\,-b_n}(t) e^{-t^2/4} f(t)^{(k)} P_k(t)\right| =$$

$$= c_1 \max_{k \le s} \sup_{t \in R} \left|e^{-t^2/4} \rho_{-a_{n+1},\,-b_{n+1}}(t) f(t)^{(k)} \rho_{a_{n+1} - a_n,\; b_{n+1} - b_n}(t) P_k(t)\right| \le$$

$$\le c_1 \tilde{\mathbf{r}}_{s,\,n+1}(f) \sup_{t \in R} |\rho_{a_{n+1} - a_n,\; b_{n+1} - b_n}(t) P_k(t)| \le c\, \tilde{\mathbf{r}}_{s,\,n+1}(f),$$

because $P_k(t)$ is a polynomial of degree no more than k. Now if $f(t) \in \mathbf{R}_W(a, b)$, then $e^{-t^2/4} f(t) \in \mathbf{R}(-b/2, -a/2)$ and $\mathbf{r}_{s,\,n}\left(e^{-t^2/4} f(t)\right) \le c\, \tilde{\mathbf{r}}_{s,\,n+1}(f)$.

Continuity of mapping $f(t) \to e^{-t^2/4} f(t)$ has been proven. By analogy we can show that if $g(t) \in \mathbf{R}(-b/2, -a/2)$, then $e^{t^2/4} g(t) \in \mathbf{R}_W(a, b)$ and

$$\tilde{\mathbf{r}}_{s,\,n}\left(e^{t^2/4} g(t)\right) \le c\, \mathbf{r}_{s,\,n+1}(g).$$

It means that mapping inverse to $f(t) \to e^{-t^2/4} f(t)$ is also continuous. This completes the proof because $f(t) \to e^{-t^2/4} f(t)$ is obviously one-to-one mapping.

Straightly from the definition of spaces $\mathbf{H}(-b/2, -a/2)$ and $\mathbf{H}_W(a, b)$ follows

Theorem 2. *The mapping* $F(z) \to e^{-z^2/4}F(-z/2)$ *is the isomorphism* $\mathbf{H}(-b/2,-a/2)$ *onto* $\mathbf{H}_W(a,b)$.

Let $\mathcal{L}'(-b/2,-a/2)$ and $\mathcal{W}'(a,b)$ be the spaces of generalized functions transformed by Laplace and Weierstrass respectively. It's known that mapping $f(t) \to e^{-t^2/4}f(t)$ is the isomorphism of $\mathcal{W}'(a,b)$ onto $\mathcal{L}'(-b/2,-a/2)$ [2]. Besides $\mathbf{R}(-b/2,-a/2)$ is continuously and densely imbedded in $\mathcal{L}'(-b/2,-a/2)$ and Laplace transformation is the isomorphism of $\mathbf{R}(-b/2,-a/2)$ onto $\mathbf{H}(-b/2,-a/2)$ [4]. Therefore using theorem 1 and 2 it can be shown by simple calculations that for any function $f(t)$ in $\mathbf{R}_W(a,b)$ formula

$$W(f)(z) = F(z) = \frac{e^{-z^2/4}}{\sqrt{4\pi}} L\Big(e^{-t^2/4}f(t)\Big)\Big(-\frac{z}{2}\Big), \quad a < Re\, z < b, \tag{4}$$

is true, where L and W are Laplace and Weierstrass transformations respectively. Now on the basis of [4], formula (4) and theorems 1 and 2 the following two theorems are true.

Theorem 3. *The space* $\mathbf{R}_W(a,b)$ *is continuously and densely imbedded in* $\mathcal{W}'(a,b)$.

Theorem 4. *Weierstrass transformation is the isomorphism of* $\mathbf{R}_W(a,b)$ *onto* $\mathbf{H}_W(a,b)$.

From the formula (2) and theorem 1 it follows that $\mathbf{R}_W(a,b)$ is the topological algebra with multiplication, determined by the formula

$$f(t) \diamond g(t) = f(t)g(t)e^{-t^2/4+\alpha t/2},$$

where α is a fixed number in (a,b).

Now $\mathbf{R}_W(a,b)$ is topological algebra continuously and densely imbedded in $\mathcal{W}'(a,b)$. More then that, it follows from the definition of $\mathbf{R}_W(a,b)$ that differentiation on $\mathbf{R}_W(a,b)$ is the continuous operator. $\mathbf{H}_W(a,b)$ is also topological algebra and differentiation on $\mathbf{H}_W(a,b)$ is the continuous operator. It follows from the theorem 2 and respective results for algebra $\mathbf{H}(-b/2,-a/2)$ [4]. Therefore in accordance with general theory [1,3,4] we can build the algebras of mnemofunctions $\mathcal{G}(\mathbf{R}_W(a,b))$ and $\mathcal{G}(\mathbf{H}_W(a,b))$ with differentiation. Taking into account the respective results for algebras of mnemofunctions $\mathcal{G}(\mathbf{R}(-b/2,-a/2))$ and $\mathcal{G}(\mathbf{H}(-b/2,-a/2))$ [4], we obtain that the following theorems are true.

THEOREM 5. *Weierstrass transformation is one-to-one mapping of* $\mathcal{G}(\mathbf{R}_W(a,b))$ *onto* $\mathcal{G}(\mathbf{H}_W(a,b))$.

THEOREM 6. $\mathcal{W}'(a,b)$ *is imbedded in* $\mathcal{G}(\mathbf{R}_W(a,b))$ *and this imbedding is injective.*

Summing up it remains to say that Weierstrass transformations in $\mathcal{W}'(a,b)$ and on $\mathcal{G}(\mathbf{R}_W(a,b))$ are coordinated. Taking into account the formula (4), the character at this relation is the same as for Laplace transformation in algebra of mnemofunctions. Besides the characteristics of Weierstrass transformation remain the same also for algebra $\mathcal{G}(\mathbf{R}_W(a,b))$. For exemple, in $\mathcal{G}(\mathbf{R}_W(a,b))$ the formula of operation's transformation generating the operational calculus for Weierstrass transformation

$$W\big((t-2D_t)f(t)\big)(z) = zF(z), \quad a < Re\, z < b$$

is true.

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