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On the Inverse Problems for a Family of Integro-Differential Equations

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Abstract. An integro-differential equation involving arbitrary kernel in time variable with a family of non-local boundary condition has been considered. Two inverse source problems for integro-differential equations are formulated and the unique-existence results for the solution of inverse source problems are presented. Some particular examples in support of our analysis are discussed.

Keywords: inverse problems, generalized diffusion equation, Bi-orthogonal system of functions, multinomial Mittag-Leffler type functions.

AMS Subject Classification: 26A33; 35R30; 35P10; 44A10; 33E12.

1 Introduction

Partial differential equations have been used to describe problems in various fields. However, there are some fields, such as heat transfer, diffusion concentration and nuclear reactor dynamics, scientist need to consider the effect of the past on present. Hence, a partial integro-differential equation is needed to represent the problem. In this article, we considered the following integro-differential equation

$$ {}^{(g)}D^{\eta}_{0_+,t} v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + \rho \int_0^t v(x,\tau) d\tau + F(x,t), \quad (x,t) \in \Omega, \tag{1.1} $$

where

$$ {}^{(g)}D^{\eta}_{0_+,t}v(x,t);=\int_0^t \eta(t-\tau)v(x,\tau)d\tau, \quad \Omega:=(0,1)\times(0,T), $$

$\eta(t)$ stands for the arbitrary memory kernel and ρ is a positive real number, subject to the following Dirichlet and dynamic boundary conditions

$$ v(0,t)=0, \qquad v_x(0,t)+\alpha v_{xx}(1,t)=0, \qquad t\in(0,T], \qquad \alpha>0. \tag{1.2} $$

The equation with fractional order integro-differential are important as they have been used in modeling several phenomena of engineering, nuclear reactor dynamics and epidemic in biology. The Equation (1.1) is used in the modeling of heat conduction in materials with memory [21, 27], the compression of poro-viscoelastic media [6], the analysis of space-time dependent nuclear reactor dynamics [22], epidemic phenomena in biology [28]. Heat equation with memory has been studied by many researchers for various aspects. In [3], Coleman and Gurti studied heat equation in which they discussed the regular fading memory effects. Some researchers studied it for the aspects of controllability [5, 24, 29] and some studied it for the numerical approximation [13, 14, 28], etc.
There are two inverse source problems, related to (1.1)–(1.2), to discuss.

Inverse Source Problem-I (ISP-I): In ISP-I, we will investigate the source term in (1.1) as space dependent source term, i.e, $F(x,t)=f(x)$. An over-specified condition

$$ v(x,T)=\psi(x), \qquad x\in[0,1], \tag{1.3} $$

is given which is used to determine $f(x)$ and $v(x,t)$ from (1.1)–(1.2). A pair of functions $\{v(x,t),f(x)\}$ is said to be a regular solution of the ISP-I which satisfies the system (1.1)–(1.2) with over-specified condition (1.3) such that $v(x,t)\in C(\overline{\Omega})$; $\overline{\Omega}=[0,1]\times[0,T]$, ${}^{(g)}D^{\eta}_{0_+,t}v(x,\cdot)\in C((0,T])$, $v(\cdot,t)\in C^2([0,1])$ and $f(x)\in C([0,1])$.

Inverse Source Problem-II (ISP-II): In ISP-II, the source term is like $F(x,t)=s(t)f(x,t)$, where $f(x,t)$ is given. We have to determine the time dependent source term $s(t)$ and temperature distribution $v(x,t)$ for the system (1.1)–(1.2). For the determination of the time dependent source term, we need some additional data, i.e., known as over-specified condition and is given by

$$ \int_0^1 v(x,t)dx=E(t), \quad t\in[0,T]. \tag{1.4} $$

The solution of the ISP-II $\{v(x,t),s(t)\}$ is said to be regular solution if $s(t)\in C([0,T])$, $v(x,t)\in C(\overline{\Omega})$, ${}^{(g)}D^{\eta}_{0_+,t}v(x,\cdot)\in C((0,T])$ and $v(\cdot,t)\in C^2([0,1])$.

Motivated by [16], the following transformation is used to deal with non-local term $\int_0^t v(x,\tau)d\tau$,

$$ u(x,t)=\int_0^t v(x,\tau)d\tau. $$

One can easily see that $u(x,0)=0$ and $\forall t\in[0,T]$,

$$\frac{\partial u(x,t)}{\partial t}=v(x,t).$$

Thus the problem (1.1)–(1.2) can be written as

$$\begin{aligned}
{}^{(g)}D^{\eta}_{0_+,t}u(x,t) &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial u(x,t)}{\partial t}\right)+\rho u(x,t)+F(x,t),\ (x,t)\in\Omega, && (1.5)\\
u_t(0,t) &= 0,\ u_{tx}(0,t)+\alpha u_{txx}(1,t)=0,\ t\in(0,T],\ \alpha>0, && (1.6)\\
u(x,0) &= 0. && (1.7)
\end{aligned}$$

After transformation, ${}^{(g)}D^{\eta}_{0_+,t}u(x,t)$ is given by

$${}^{(g)}D^{\eta}_{0_+,t}u(x,t)=\int_0^t \eta(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau.$$

The direct and inverse problems with dynamic boundary conditions for heat equation have been considered by many authors, for readers convenience, we refer [4, 7, 9, 10, 11, 20, 23]. In last few years, the diffusion equations involving integrals and derivatives of fractional order are considered extensively in literature. Chechkin et al. [2] studied Time Fractional Diffusion Equation (TFDE) with varying in space fractional order of time derivative. Wei et al. [26] surveyed the temporal effects in the modeling of anomalous diffusion process using fractional order operators.

Let us mention some works of Inverse Problems (IPs) for Fractional Differential Equations (FDEs) which become important tool in modeling of many real-life problems. Wei et al. [25] investigated the inverse source problem of spatial fractional anomalous diffusion equation by using the so-called coupled method. Ismailov et al. Liao et al. [15] studied an IP of recovering a fractional order and a space dependent source term in a multi-dimensional time fractional diffusion wave equation by the final time measurement data. Malik et al. [18] considered an IP of the determination of the source term and diffusion concentration for a multi-term FDE with integral type over-specified condition. Kinash et al. [12] presented two IPs for a generalized subdiffusion equation with final over-determination condition. Asim et al. [8] studied two IPs for a multi-term time-fractional evolution equation with an involution term, interpolating the heat and wave equations.

The rest of the paper is organized as follows: further, in Section 2, we define the multinomial Mittag-Leffler function and represent its several properties. In Section 3, we give the spectral problem corresponding to system (1.5)–(1.6) and its properties. In Section 4, we construct the solution of ISP-I and prove to be unique existence for the solution of the ISP-I. In Section 5, we formulate the solution of the ISP-II and investigate the existence and uniqueness results of the ISP-II. In Section 6, we give some numerical examples and present the conclusions in the last section.

2 Mittag-Leffler type functions

In this section, we will present Mittag-Leffler type functions and some results related to its estimates.

DEFINITION 1. [17] For $\zeta_i, \eta > 0,\ w_i \in \mathbb{C}, i = 1, 2, ..., m,\ m \in \mathbb{N}$, the multinomial Mittag-Leffler function is defined as

$$E_{(\zeta_1,\zeta_2,\ldots,\zeta_m),\eta}(w_1, w_2, \ldots, w_m) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+\ldots+l_n=k \\ l_1\geq 0,\ldots,l_m\geq 0}} (k; l_1, \ldots, l_n) \frac{\Pi_{i=1}^{m} z_i^{l_i}}{\Gamma\left(\eta + \sum_{i=1}^{m} \zeta_i l_i\right)},$$

where $(k; l_1, \ldots, l_m) = \frac{k!}{l_1! \times \ldots \times l_m!}$.

Remark 1. For $w_1 \neq 0,\ w_2 \neq 0,\ w_3 = ... = w_m = 0,\ m \in \mathbb{N}$ the multinomial Mittag-Leffler function takes the following form

$$\begin{aligned} E_{(\zeta_1,\zeta_2)\eta}(w_1, w_2) &= \sum_{k=0}^{\infty} \sum_{l_1+l_2=k l_1\geq 0, l_2\geq 0} \frac{k!}{l_1! l_2!} \frac{w_1^{l_1} w_2^{l_2}}{\Gamma(\eta + \zeta_1 l_1 + \zeta_2 l_2)}, \\ &= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \frac{w_1^{i} w_2^{k-i}}{\Gamma(\eta + \zeta_1 i + \zeta_2 (k-i))}. \end{aligned}$$

The multinomial Mittag-Leffler function $E_{(\zeta_1,\zeta_2)\eta}(w_1, w_2)$, will be used in Case II.

Remark 2. For $w_1 \neq 0$, $w_2 = 0$, the multinomial Mittag-Leffler function reduces to the well known two parameter Mittag-Leffler function given by

$$E_{(\zeta_1,\zeta_2)\eta}(w_1, 0) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\zeta_1 k + \eta)} := E_{\zeta_1,\eta}(w_1).$$

Lemma 1. *[18]For $\zeta_i, \eta, \tau, \sigma_i > 0,\ i = 1, 2, \ldots, m,\ m \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by*

$$\mathcal{L}\{\tau^{\eta-1} E_{(\zeta_1,\zeta_2,\ldots,\zeta_m),\eta}(-\sigma_1 \tau^{\xi_1}, ..., -\sigma_m \tau^{\xi_m}); s\} = \frac{s^{-\eta}}{1 + \sum_{i=1}^{m} \sigma_i s^{-\zeta_i}},$$

for $\left|\sum_{i=1}^{m} \sigma_i s^{-\zeta_i}\right| < 1$.

3 The spectral problem

The spectral problem of the system (1.5)–(1.6) is given by

$$X^{''}(x) + \lambda X(x) = 0, \tag{3.1}$$

$$X(0) = 0, \quad X^{'}(0) + \alpha X^{''}(1) = 0. \tag{3.2}$$

Suppose that $\alpha \neq \frac{1}{x_i \sin x_i}$ for all i, where x_i satisfy the equation $\sin x + x\cos x = 0$ on $(0, \infty]$. In this case problem (3.1) along with the boundary conditions (3.2) has eigenfunctions

$$X_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x), \qquad n = 0, 1, 2, ..., \tag{3.3}$$

where the eigenvalues $\lambda_n,\ n = 0, 1, 2, ...,$ satisfy the equation $\alpha\sqrt{\lambda}\sin\sqrt{\lambda} = 1$, $Re\sqrt{\lambda} = 0$.

The asymptotic estimate for the eigenvalues

$$\sqrt{\lambda_n} = \pi n + \frac{(-1)^n}{\pi\alpha n} + O\big(1/n^3\big),$$

is valid for large n. It is shown in [19] that the system of eigenfunctions $X_n(x),\ n = 0, 1, 2, ...,$ that is, the system of eigenfunctions of spectral problem with one deleted, is a Riesz basis $L_2([0, 1])$ and the system

$$Y_n(x) = \sqrt{2}\frac{\overline{\sqrt{\lambda_0}}\sin\overline{\sqrt{\lambda_0}}(1-x) - \overline{\sqrt{\lambda_n}}\sin\overline{\sqrt{\lambda_n}}(1-x)}{\overline{\sqrt{\lambda_n}}\cos\overline{\sqrt{\lambda_n}} + \sin\overline{\sqrt{\lambda_n}}} \tag{3.4}$$

is a bi-orthogonal to the system $X_n(x), n = 1, 2, 3, ...,$ i.e.,

$$\langle X_n, Y_m\rangle = \int_0^1 X_n(x)Y_m(x)dx = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Lemma 2. *(Theorem 1, [19]) Suppose that a function* $h \in C([0, 1])$ *has a uniformly convergent Fourier series expansion in the system* $\sqrt{2}\sin(\pi n x)$, $n = 1, 2, ...,$ *on the interval* $[0, 1]$*. Then this function can be expanded in a Fourier series in the system* $X_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n = 1, 2, ...,$ *and this expansion is uniformly convergent on every interval* $[0, b]$, $0 < b < 1$*. If* $C_{\lambda_0} \equiv \sqrt{2\lambda_0}\langle h, \sin\lambda_0(1-x)\rangle = 0$*, then the Fourier series of* h *in the system* $X_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x)$, $n = 1, 2, ...,$ *is uniformly convergent on* $[0, 1]$*.*

We considered that $u(x, t)$ is real valued, but we have some eigenvalues which are complex. To get rid of these terms, method of annihilation of the complex term of Fourier series expansion is used. The class of function which also contains the conditions of the Lemma 2 will be denoted by

$$C^3_{n_\alpha}([0,1]) = \begin{cases} h(x) \in C^3([0, 1]) : \ h(0){=}h'(0){=}h''(0) = 0,\ h(1){=}h''(1){=}0, \\ \int_0^1 h(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0, \quad n = 0, 1, 2, ..., n_\alpha. \end{cases}$$

Lemma 3. *[10] If* $h(x) \in C^3_{n_\alpha}([0, 1])$*, then the inequality*

$$\sum_{n=n_\alpha+1}^{\infty} |\lambda_n h_n| \le c\|h'''\|^2_{L_2([0,1])},\ c = const > 0 \text{ holds, where } h_n = \langle h, Y_n\rangle.$$

From this discussion, by using the Bessel and Schwarz inequalities, one can obtain

$$\sum_{n=n_\beta+1}^{\infty} |\lambda_n h_n| \le c\|h'''\|^2_{L_2([0,1])} \le c\|h\|_{C^3([0,1])}.$$

4 Inverse Source Problem-I

In this section, we want to determine $F(x,t) = f(x)$ and $u(x,t)$, i.e., the solution of the system (1.5)–(1.7).

4.1 Construction of the Solution ISP-I

By using the generalized Fourier method, the solution of the system (1.5)–(1.7) can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x), \qquad f(x) = \sum_{n=1}^{\infty} f_n X_n(x),$$

where $T_n(t)$, and f_n satisfy the following differential equation

$$\eta(t) * T^{'}(t) - \rho\, T_n(t) = \quad -\lambda_n T^{'}(t) + f_n, \tag{4.1}$$

and $f_n = \int_0^1 f(x)Y_n(x)dx,\ n = 1,2,3,....$
Taking Laplace transform of (4.1) and due to initial condition (1.7), we have

$$\mathcal{L}\{T_n(t);s\} = \frac{f_n}{s(s\mathcal{L}\{\eta(t);s\} + s\lambda_n - \rho)}, \implies T_n(t) = f_n A_n(t), \tag{4.2}$$

where $A_n(t) := \int_0^t B_n(\tau)d\tau$, and $B_n(t) = \mathcal{L}^{-1}\left(\frac{1}{s\mathcal{L}\{\eta(t);s\}+s\lambda_n-\rho}\right)$ and " $*$ " represents integral convolution given by

$$s_1(t) * s_2(t) = \int_0^t s_1(t-\tau)s_2(\tau)d\tau, \qquad 0 \le \tau \le t.$$

The expression of f_n is obtained from the over-specified condition (1.3)

$$f_n = \psi_n / A_n^{'}(T), \tag{4.3}$$

where $\psi_n = \int_0^1 \psi(x)Y_k(x)dx,\ n = 1,2,3,....$

By substituting the coefficients of f_n in Equation (4.2), we obtain

$$T_n(t) = \frac{\psi_n}{A_n^{'}(T)} A_n(t). \tag{4.4}$$

Under the conditions that, $\psi(x) \in C^3_{n_\alpha}([0,1])\ \forall\, t \in [0,T]$, we have

$$\int_0^1 \psi(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0,\ n = 0,1,2,...,n_\alpha,$$

implies that the Fourier coefficient $\psi_n = 0$, for $n = 0,1,2,...,n_\alpha$ and

$$\psi_n = \frac{\sqrt{2\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}} \int_0^1 \psi(x)\sin(\sqrt{\lambda_n}(1-x))dx,\ n > n_\alpha,$$

are real constants. Therefore, the formal solution of the system (1.5)–(1.7) is the series

$$u(x,t)=\sum_{n=n_\alpha+1}^{\infty} T_n(t)X_n(x), \quad f(x)=\sum_{n=n_\alpha+1}^{\infty} f_n X_n(x). \tag{4.5}$$

Now, we will discuss some special cases of the ISP-I by taking the several choices of the memory kernel $\eta(t)$.

Case I: Letting the memory kernel $\eta(t)=\delta(t)$ in Equation (1.5), we have

$$A_n(t)=\frac{1}{1+\lambda_n}\int_0^t e^{\frac{\rho}{1+\lambda_n}\tau}d\tau, \quad f_n=\psi_n(1+\lambda_n)^2/\int_0^T e^{\frac{\rho}{1+\lambda_n}\tau}d\tau.$$

Case II: Taking the memory kernel $\eta(t)=\frac{t^{-\beta}}{\Gamma(1-\beta)}$, where $0<\beta<1$ in Equation (1.5), we get

$$A_n(t)=\int_0^t \tau^{\beta-1}E_{(\beta-1,\beta),\beta}(-\lambda_n\tau^{\beta-1},\rho\tau^{\beta})d\tau,$$

$$f_n=\psi_n/\int_0^T \tau^{\beta-2}E_{(\beta-1,\beta),\beta-1}(-\lambda_n\tau^{\beta-1},\rho\tau^{\beta})d\tau.$$

Case III: Introducing the memory kernel $\eta(t)=\sum_{j=1}^{n}\frac{U_j t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $0<\beta_n<...<\beta_2<\beta_1<1,\ \ U_j>0,\ j=1,2,...,n,$ in Equation (1.5), i.e., yields

$$A_n(t)=\frac{1}{U_1}\int_0^t \tau^{\beta_1-1}E_{(\beta_1-\beta_n,...,\beta_1-\beta_2,\beta_1-1,\beta_1),\beta_1}(\vartheta)d\tau,$$

$$f_n=\psi_n U_1/\int_0^T \tau^{\beta_1-1}E_{(\beta_1-\beta_n,...,\beta_1-\beta_2,\beta_1-1,\beta_1),\beta_1}(\vartheta)d\tau,$$

where

$$(\vartheta):=\left(-\frac{U_n}{U_1}\tau^{\beta_1-\beta_n},\ldots,-\frac{U_2}{U_1}\tau^{\beta_1-\beta_2},\ldots,-\frac{\lambda_n}{U_1}\tau^{\beta_1-1},\frac{\rho}{U_1}\tau^{\beta_1}\right).$$

Case IV: Let an exponential cut-off of the power-law memory kernel of the form $\eta(t)=e^{-dt}t^{-\beta}/\Gamma(1-\beta),\ \ 0<\beta<1$, where $d>0$ is the truncation parameter and substitute in Equation (1.5), we obtain

$$A_n(t)=e^{-dt}t^{\beta}E_{(\beta-1,\beta,1,\beta,\beta+1),\beta+1}(-\lambda_n t^{\beta-1},2\lambda_n dt^{\beta},dt,\rho t^{\beta},-(d^2\lambda_n+d\rho)t^{\beta+1}),$$

and f_n is given by the Equation (4.3) with

$$A_n'(T)=-\,de^{-dT}T^{\beta}E_{(\beta-1,\beta,1,\beta,\beta+1),\beta+1}(\omega)+e^{-dT}T^{\beta-1}E_{(\beta-1,\beta,1,\beta,\beta+1),\beta}(\omega),$$

where

$$(\omega):=\left(-\lambda_n T^{\beta-1},2\lambda_n dT^{\beta},dT,\rho T^{\beta},-(d^2\lambda_n+d\rho)T^{\beta+1}\right).$$

4.2 Existence of the solution of the ISP-I

Lemma 4. *[1] We see that $B_n(t)$ is a completely monotone functions, furthermore, for some constant $C_0 = C_0(T) > 0$ independent of λ_n we have*

$$0 < C_0 \leq \lambda_n \int_0^T B_n(\tau) d\tau < 1.$$

Therefore, following the strategy presented in [1], we obtain the estimate of $A_n(t)$ and $A_n(T)$. Let the functions $A_n(t)$ and $A_n(T)$ for $n \in \mathbb{N}$ are continuous on $[0, \infty)$ vanish at $t = 0$, positive and nondecreasing on $\mathbb{R}^+$. The following estimates for $t, T > 0$ are satisfied:

$$A_n(t) \leq \frac{1}{\lambda_n}, \quad \frac{1}{A_n(T)} \leq C_0 \lambda_n,$$

where $A_n(t) = \int_0^T B_n(\tau) d\tau$.

Theorem 1. *Assume*

$$\psi(x) \in C^3_{n_\alpha}([0,1]); \quad \psi_{n_\alpha+1} > 0, \quad \psi_{n_\alpha+k} \geq 0, \quad k = 2, 3, \ldots.$$

Then, there exists a regular solution of the ISP-I for the system (1.5)–(1.7) *and* (1.3)*.*

Proof. For the proof of the existence of the solution of the ISP-I, we need to show the uniform convergence of series corresponding to $f(x), u_t(x,t), u_{xxt}(x,t)$ and ${}^{(g)}D^{\eta}_{0+,t} u(x,t)$. First, we are going to discuss the continuity of $f(x)$. Due to Lemma 4 and Equation 4.3, one gets

$$|f_n| \leq C_0 |\lambda_n \psi_n|.$$

By Lemma 3, one gets the following inequality

$$|f_n| \leq C_0 c \|\psi\|_{C^3([0,1])}. \tag{4.6}$$

From Equation (4.6), it is clear that the f_n is bounded. Hence, by using Weierstrass M-test we can say that $f(x)$ is a continuous function. Using Lemmas 3, 4 and Equation (4.4), one gets

$$|T_n'(t)| \leq \frac{C_0 c}{\lambda_n} \|\psi\|_{C^3([0,1])}. \tag{4.7}$$

By virtue of Equation (4.7), we can conclude that continuity of $u_t(x,t)$ is ensured due to Weierstrass M-test.

Next, we will show that the continuity of $u_{xxt}(x,t)$. For this, term by term differentiation of the series of $u(x,t)$ in (4.5), one gets

$$u_{xxt}(x,t) = \sum_{n=n_\alpha+1}^{\infty} T_n'(t) X_n''(x), \tag{4.8}$$

where

$$X_n''(x) = -\lambda_n X_n(x), \quad T_n'(t) = \frac{\psi_n}{A_n'(T)} A_n'(t).$$

Due to the fact $|X_n| \leq 1$, we obtain

$$|X_n''(x)| \leq \lambda_n. \tag{4.9}$$

From the relations (4.7) and Lemma 3, we can extrapolate that $u_{xxt}(x,t)$, given by (4.8), is bounded above by a convergent series. Under the assumptions a similar argument can be developed for uniform convergent of the series representation of ${}^{(g)}D^{\eta}_{0+,t}u_t(x,t)$. □

Uniqueness of the solution of the ISP-I: Uniqueness of $v(x,t)$ can be obtained by assuming $\omega(x,t)$ and $\nu(x,t)$ be the two regular solution sets of the ISP-I, and proving them equal i.e. $\omega(x,t) = \nu(x,t)$ by using the fact that set of bi-orthogonal system (3.3)–(3.4) of function form a complete set in $L^2([0,1])$.

5 Inverse Source Problem-II

In this section, we suppose the ISP-II for the system (1.1)–(1.2) with the source term, i.e., $F(x,t) = s(t)f(x,t)$. We will discuss the recovery of the time dependent source term $s(t)$ along with $v(x,t)$, under the given over-specified condition (1.4).

5.1 Construction of the Solution ISP-II

The solution of the system (1.5)–(1.7) can be written in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x), \quad f(x,t) = \sum_{n=1}^{\infty} f_n(t)X_n(x),$$

where $T_n(t)$ satisfy the following differential equation

$$\eta(t) * T^{'}(t) - \rho T_n(t) = -\lambda_n T^{'}(t) + s(t)f_n(t)$$

and $f_n(t) = \int_1^0 f(x,t)Y_n(x)dx,\ n = 1,2,3,....$ Taking Laplace transform and using initial condition (1.7), we have the expression for $T_n(t)$ as

$$T_n(t) = s(t)f_n(t) * A_n(t),$$

where $s(t)$ is still to be determined. Under the conditions that, $f(x,t) \in C^3_{n_\alpha}([0,1])\ \forall\, t \in [0,T]$, we have

$$\int_0^1 f(x,t)\sin(\sqrt{\lambda_n}(1-x))dx = 0,\ n = 0,1,2,...,n_\alpha,$$

implies that the Fourier coefficient $f_n(t) = 0,$ for $n = 0,1,2,...,n_\alpha$ and

$$f_n(t) = \frac{\sqrt{2\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}} \int_0^1 f(x)\sin(\sqrt{\lambda_n}(1-x))dx,\ n > n_\alpha,$$

are real functions. Therefore, the formal solution of the system (1.5)–(1.7) is the series

$$u(x,t) = \sum_{n=n_\alpha+1}^{\infty} T_n(t)X_n(x), \tag{5.1}$$

$$f(x,t) = \sum_{n=n_\alpha+1}^{\infty} f_n(t)X_n(x). \tag{5.2}$$

Now, we will discuss some special cases of the ISP-I by taking the several choices of the memory kernel $\eta(t)$.

Case I: Letting the memory kernel $\eta(t) = \delta(t)$ in Equation (1.5), we have

$$A_n(t) = \frac{1}{1+\lambda_n} e^{\frac{\rho}{1+\lambda_n} t}.$$

Case II: Taking the memory kernel $\eta(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, where $0 < \beta < 1$ in Equation (1.5), we get

$$A_n(t) = t^{\beta-1} E_{(\beta-1,\beta),\beta}(-\lambda_n t^{\beta-1}, \rho t^{\beta}).$$

Case III: Introducing the memory kernel $\eta(t) = \sum_{j=1}^{n} \frac{U_j t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $0 < \beta_n < ... < \beta_2 < \beta_1 < 1$, $U_j > 0$, $j = 1, 2, 3, ..., n$, in Equation (1.5), one can obtain

$$A_n(t) = \frac{1}{U_1} t^{\beta_1 - 1} E_{(\beta_1-\beta_n,...,\beta_1-\beta_2,\beta_1-1,\beta_1),\beta_1}(\nu),$$

where

$$(\nu) := \left(-\frac{U_n}{U_1} t^{\beta_1-\beta_n}, ..., -\frac{U_2}{U_1} t^{\beta_1-\beta_2}, ..., -\frac{\lambda_n}{U_1} t^{\beta_1-1}, \frac{\rho}{U_1} t^{\beta_1} \right).$$

Case IV: Let an exponential cut-off of the power-law memory kernel of the form $\eta(t) = e^{-dt} t^{-\beta}/\Gamma(1-\beta)$, $0 < \beta < 1$, where $d > 0$ is the truncation parameter and substitute in Equation (1.5),

$$A_n(t) = e^{-bt} t^{\beta-1} E_{(\beta-1,\beta),\beta}\left(-\lambda_n t^{\beta-1}, -(b\lambda_n + \rho)t^{\beta}\right).$$

5.2 Existence of the solution of the ISP-II

In this section, we are going to present the following theorem that state the conditions under which solution of the ISP-II has a regular solution.

Theorem 2. *Assume*

(1) $f(x,t) \in C(\bar{\Omega})$, $f(x,t) \in C^3_{n_\alpha}(\bar{\Omega})$, $\forall\ t \in [0,T]$, $f_{n_\alpha+k}(t) \geq 0$, $k = 1, 2, 3, ...$. *Furthermore* $\left(\int_0^1 f(x,t)dx \right)^{-1} \leq M_1$, $t \in [0,T]$ *for some positive constant* M_1.

(2) $E(t) \in C([0,T])$ *and satisfies the consistency condition* $E(0) = 0$.
$E(t) > 0, \ \forall\, t \in [0,T]$.

Then, there exist a unique local in time solution of the ISP-II.

Proof. To determine $s(t)$ of time dependent source term, we will use the over-specified condition (1.4), we have

$$\int_0^1 {}^{(g)}D^{\eta}_{0+,t} u(x,t)dx = {}^{(g)}D^{\eta}_{0+,t} E(t).$$

By virtue of (1.1), we have

$$\int_0^1 \left(\frac{\partial^2}{\partial x^2} \left(\frac{\partial u(x,t)}{\partial t} \right) + \rho u(x,t) + s(t) f(x,t) \right) dx = {}^{(g)}D^{\eta}_{0+,t} E(t),$$

which leads to the following expression of $q(t)$

$$s(t) = \left(\int_0^1 f(x,t)dx \right)^{-1} \left({}^{(g)}D^{\eta}_{0+,t} E(t) - \int_0^t K(t,\tau) s(\tau) d\tau \right). \tag{5.3}$$

Setting

$$K(t,\tau) = \rho f_n(\tau) B_n(t-\tau) - \lambda_n f_n(\tau) B'_n(t-\tau).$$

Due to Lemma 4, we can have some positive constant M_2 such that

$$\|K(t,\tau)\|_{C([0,T]\times[0,T])} \leq M_2.$$

Let us define the mapping $\mathcal{A} : C([0,T]) \to C([0,T])$ by $\mathcal{A}(s(t)) := s(t)$, where $s(t)$ is given by (5.3). The mapping $\mathcal{A}(s(t))$ is well defined due to uniformly convergent of $\|K(t,\tau)\|_{C([0,T]\times[0,T])}$. Next, we will show that under the assumption $T < \frac{1}{M_1 M_2}$, the mapping $\mathcal{A}(s(t))$ is contraction. Consider

$$|\mathcal{A}(s_1(t)) - \mathcal{A}(s_2(t))| = \left(\int_0^1 f(x,t)dx \right)^{-1} \left(\int_0^t K(t,\tau) |s_1(\tau) - s_2(\tau)| d\tau \right).$$

By assumptions of Theorem 2 , we obtain

$$\max_{0\leq t\leq T} |\mathcal{A}(s_1(t)) - \mathcal{A}(s_2(t))| \leq M_1 M_2 T \max_{0\leq t\leq T} |s_1(\tau) - s_2(\tau)|,$$
$$\|\mathcal{A}(s_1) - \mathcal{A}(s_2)\|_{C([0,T])} \leq M_1 M_2 T \|s_1 - s_2\|_{C([0,T])},$$

which shows that the mapping $\mathcal{A}(s(t))$ is a contraction. Hence, unique existence of $s(t)$ is guaranteed by Banach fixed point theorem. Since, $s \in C([0,T])$ and for some constant $M > 0$, we have

$$\|s\|_{C([0,T])} \leq M.$$

Existence of the solution of the ISP-II: To prove the existence of the solution of the ISP-II, we will establish the continuity of $u_t(x,t)$, $u_{xxt}(x,t)$,

${}^{(g)}D^{\eta}_{0_+,t}u(x,t)$. Let us first note that the estimate of the $f(x,t)$ given by Equation (5.2) in the norm space is defined as

$$\|f\|^2_{C^{2,0}(\bar{\Omega}))}=\int_0^T \|f\|^2_{C^2([0,1])}dt=\int_0^T \sum_{n=n_\alpha+1}^{\infty} |f_n(t)|^2 dt=\sum_{n=n_\alpha+1}^{\infty} \|f_n(t)\|^2_{C([0,T])}.$$

By using the Young inequality for the integral convolution and due to estimates (42), see [1], we have

$$\begin{aligned}\Big\| \int_0^t B_n(t-\tau)f_n(t)d\tau\Big\|^2_{C([0,T])} &\leq \Big(\int_0^T B_n(t)dt\Big)^2 \int_0^T |f_n(t)|^2 dt\\ &\leq \frac{1}{\lambda_n^2}\int_0^T |f_n(t)|^2 dt. \end{aligned} \tag{5.4}$$

By Lemmas 3, 4 and estimate (5.4), we have

$$|T_n'(t)| \leq \frac{M}{\lambda_n^2}\|f_n(t)\|^2_{C([0,T])}. \tag{5.5}$$

The uniform convergence of the series $u_t(x,t)$ given by Equation (5.1) is bounded above due to inequalities (5.5). Consequently, continuity of $u(x,t)$ is obtained by using Weierstrass M-test.

Next, we will show that the continuity of $u_{xxt}(x,t)$. From the relations of (5.5) and (4.9), we can deduce that $u_{xxt}(x,t)$ is bounded above by a convergent series. Hence, by Weierstrass M-test, $u_{xx}(x,t)$ represents a continuous function.

Similarly, we can prove that ${}^{(g)}D^{\eta}_{0_+,t}u(x,t)$ represent a continuous function.

Uniqueness of the solution of the ISP-II: Banach fixed point theorem has been used to prove the uniqueness of $u(t)$. Uniqueness of $v(x,t)$ can be obtained by assuming $z(x,t)$ and $r(x,t)$ be the two regular solution sets of the ISP-II, and proving them equal i.e. $z(x,t)=r(x,t)$ by using the fact that set of bi-orthogonal system (3.3)–(3.4) of function form a complete set in $L^2([0,1])$. □

6 Examples

In this section, we are going to provide two examples i.e., Example 1 and Example 2 related to the inverse source problem I and inverse source problem II, respectively.

Example 1. Let $\psi(x)=\sin^3(\pi x)$, then

$$\begin{aligned}\psi_n =&\frac{\sqrt{2}\sqrt{\lambda_n}}{4\left(\sqrt{\lambda_n}\cos\sqrt{\lambda_n}+\sin\sqrt{\lambda_n}\right)}\\ &\times\left(\frac{3\sin 1-3\sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{1-\lambda_n}-\frac{3\sin 3-\sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{9-\lambda_n}\right).\end{aligned}$$

Substituting the value of ψ_n in (4.5), one can obtain

$$
\begin{aligned}
u(x,t) = \sum_{n=n_\alpha+1}^{\infty} & \left(\frac{3\sin 1 - 3\sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{1-\lambda_n} - \frac{3\sin 3 - \sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{9-\lambda_n} \right) \\
& \times \frac{\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x)A_n(t)}{2A_n'(T)\left(\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}\right)}, \\
f(x) = \sum_{n=n_\alpha+1}^{\infty} & \left(\frac{3\sin 1 - 3\sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{1-\lambda_n} - \frac{3\sin 3 - \sqrt{\lambda_n}\sin\sqrt{\lambda_n}}{9-\lambda_n} \right) \\
& \times \frac{\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x)}{2A_n'(T)\left(\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}\right)}.
\end{aligned}
$$

Example 2. In ISP-I, we take the memory kernel (dirac delta function), that is, $\eta(t) = \delta$, $f(x,t) = (2 + \lambda_{n_\alpha+1}t)\sin(\sqrt{\lambda_{n_\alpha+1}}x)$, and over-specified condition is

$$\int_0^1 u(x,t)dx = \frac{2t^2}{\lambda_{n_\alpha+1}}.$$

Indeed, using (5.1) the solution of the system is given by

$$
\begin{aligned}
& u(x,t) = \sqrt{2}\{a(t)f_1(t) * e^{-\lambda_{n_\alpha+1}t}\}\sin(\sqrt{\lambda_{n_\alpha+1}}x), \\
& f_1(t) = (2 + \lambda_{n_\alpha+1}t)/\sqrt{2}.
\end{aligned}
$$

In Case I, the expression for $s(t)$ given by (5.3) takes the form

$$s(t) = \left(\int_0^1 f(x,t)dx\right)^{-1}\left(E'(t) - F(t) - \int_0^t K(t,\tau)s(\tau)d\tau\right),$$

where

$$
\begin{aligned}
\int_0^1 f(x,t)dx = & \frac{(2 + \lambda_{n_\alpha+1}t)(1 - \cos(\sqrt{\lambda_{n_\alpha+1}}x))}{\sqrt{\lambda_{n_\alpha+1}}}, \quad E'(t) = \frac{4t}{\lambda_{n_\alpha+1}}, \\
F(t) = & 0, \quad K(t,\tau) = \frac{(2 + \lambda_{n_\alpha+1}t)(1 - \cos(\sqrt{\lambda_{n_\alpha+1}}x))}{\sqrt{\lambda_{n_\alpha+1}}} e^{-\pi(t-\tau)}.
\end{aligned}
$$

In this case, we can find expression for $s(t)$ given by $s(t) = t$. Hence, we obtain

$$u(x,t) = \sqrt{2}t^2\sin(\sqrt{\lambda_{n_\alpha+1}}x).$$

Example 3. In ISP-II, we consider the power law memory kernel, that is,

$$\eta(t) = \frac{t^\beta}{\Gamma(1-\beta)}, \quad f(x,t) = \left(\frac{\Gamma(2)}{\Gamma(2-\beta)} + \lambda_{n_\alpha+1}t^{\beta_0}\right)\sin(\sqrt{\lambda_{n_\alpha+1}}x)$$

and over-specified condition is

$$\int_0^1 u(x,t)dx = \frac{2t}{\lambda_{n_\alpha+1}}.$$

In Case II, the solution of the ISP-II is given by

$$u(x,t) = \sqrt{2}\Big\{ a(t) f_1(t) * \mathcal{E}_{\beta,\beta}(t;\lambda_{n_\alpha+1}) \Big\} \sin(\sqrt{\lambda_{n_\alpha+1}}x),$$
$$f_1(t) = \left(\frac{\Gamma(2)}{\Gamma(2-\beta)} + \lambda_{n_\alpha+1} t^{\beta} \right).$$

In Case II, the expression for $s(t)$ takes the following form

$$s(t) = \left(\int_0^1 f(x,t)dx \right)^{-1} \left(D^{\beta}_{0_+,t} E(t) - \digamma(t) - \int_0^t K(t,\tau) s(\tau) d\tau \right),$$

where

$$\int_0^1 f(x,t)dx = \frac{1-\cos(\sqrt{\lambda_{n_\alpha+1}}x)}{\sqrt{\lambda_{n_\alpha+1}}} \left(\frac{\Gamma(2)}{\Gamma(2-\beta)} + \lambda_{n_\alpha+1} t^{\beta} \right),$$
$$D^{\beta}_{0_+,t} E(t) = \frac{4}{\lambda_{n_\alpha+1}} \frac{\Gamma(2)}{\Gamma(2-\beta)} t^{1-\beta}, \quad \digamma(t) = 0,$$
$$K(t,\tau) = \frac{1-\cos(\sqrt{\lambda_{n_\alpha+1}}x)}{\sqrt{\lambda_{n_\alpha+1}}} \left(\frac{\Gamma(2)}{\Gamma(2-\beta)} + \lambda_{n_\alpha+1} t^{\beta} \right) \mathcal{E}_{\beta,\beta}(t-\tau;\lambda_{n_\alpha+1}).$$

In this case, we can find expression for $s(t)$ given by $s(t) = t^{1-\beta}$. Hence, we obtain

$$u(x,t) = \sqrt{2} t \sin(\lambda_{n_\alpha+1} x).$$

7 Conclusions

Generalized integro-differential equation written as convolution of arbitrary memory kernel $\eta(t)$ is considered. Several well known nonlocal diffusion equations are special cases of Equation (1.1) and can be obtained by taking several choices of the kernel function $\eta(t)$. Two inverse problems namely, ISP-I and ISP-II defined for generalized diffusion Equation (1.1) with nonlocal boundary conditions involving a parameter $\beta > 0$ are considered. A bi-orthogonal system of functions obtained from spectral and its conjugate problems is used to construct the series representations for the solutions of inverse problems. With over-specified data given at some time T, the determination of space dependent source term along with diffusion concentration comprised the ISP-I. The existence and uniqueness of regular solution of the ISP-I is proved. An integral type over-determination condition is used for the recovery of a time dependent source term in ISP-II. The unique existence of the time dependent source term is obtained by using Banach fixed point theorem. The solution of the ISP-II exists under the certain assumptions (see Theorem 2). Several spacial cases of the inverse problems are discussed by taking particular choice of kernel $\eta(t)$. The proposal of regularizing algorithms for the inverse problems investigated in this article is an interesting topic to be considered.

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