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Composite Laguerre Pseudospectral Method for Fokker-Planck Equations

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Abstract. A composite generalized Laguerre pseudospectral method for the nonlinear Fokker-Planck equations on the whole line is developed. Some composite generalized Laguerre interpolation approximation results are established. As an application, a composite Laguerre pseudospectral scheme is provided for the problems of the relaxation of fermion and boson gases. Convergence and stability of the scheme are proved. Numerical results show the efficiency of this approach and coincide well with theoretical analysis.

Keywords: composite generalized Laguerre pseudospectral method, nonlinear Fokker-Planck equations, the whole line.

AMS Subject Classification: 65M99; 41A30; 35K55.

1 Introduction

Fokker-Planck equations describe the evolution of stochastic systems, such as the erratic motions of small particles immersed in fluids, and the velocity distributions of fluid particles in turbulent flows. Several methods have been developed for the linear Fokker-Planck equations, such as combined Hermite spectral-finite difference method, domain decomposition spectral method, a semi-analytical iterative technique and etc, see [1, 6, 9, 13, 22]. However, the nonlinear Fokker-Planck equations can better reflect nonlinear characteristics of the corresponding to physical problems. Applications could be found in various fields such as astrophysics, the physics of polymer fluids and particle beams, biophysics and population dynamics, see [7]. So it is interesting to solve various nonlinear problems, such as the nonlinear Fokker Planck equations, see [2,4,10,11,16,20].

Let $\mathbb{R} = \{ v \mid -\infty < v < \infty\}$ be the velocity of particles. Denote by $W(v,t)$ the probability density. Moreover, $W_0(v)$ is the initial state. For simplicity, let $\partial_z W = \frac{\partial W}{\partial z}$, etc. The nonlinear Fokker-Planck equations for the relaxation of fermion and boson gases are given as follows (cf. [7,14]):

$$\begin{aligned} &\partial_t W(v,t)=\partial_v(vW(v,t)(1+kW(v,t))) + \partial_v^2 W(v,t), \quad v \in \mathbb{R},\ 0 < t \leq T, \\ &W(v,t) \to 0, \quad |v| \to \infty, \quad 0 < t \leq T, \\ &W(v,0) = W_0(v), \quad v \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $k = 1$ for bosons and $k = -1$ for fermions. These models have been introduced as a simplification with respect to Boltzmann-based models as in [5,15]. The entropy method was applied for quantifying explicitly the exponential decay towards Fermi-Dirac and Bose-Einstein distributions in the one-dimensional case, see [3]. Further more, some numerical methods were developed for the problems (1.1). For example, the full-discrete generalized Hermite spectral and pseudo-spectral schemes were proposed in [4,21]. However, in the stability and convergence analysis of the fully discrete pseudospectral scheme, the aliasing error brings a certain difficulty. Also, a composite Laguerre-Legendre pseudospectral scheme is presented in [19]. Yet it requires more basis functions, which makes the calculation more complex. Recently, Wang [20] considered the composite Laguerre spectral method for the problems (1.1), in which the nonlinear term $\partial_v(vW(v,t)(1+kW(v,t)))$ exacerbates the difficulty of calculating the quadratures over the whole line. So we prefer composite generalized Laguerre interpolation method that only predicates on estimated values of unknown functions on the interpolation nodes, and handles nonlinear terms easily [18,24,25,27], which preserves the continuity and possesses the global spectral accuracy on the whole line [8,9,12,17,22,26].

This paper is focused on developing a semidiscrete composite generalized Laguerre pseudospectral method for problem (1.1). The method needs fewer basis functions and avoids the aliasing errors analysis caused by the second order difference term in [21]. To cope with the stationary solution

$$F_\infty(v) = 1/\big(\beta e^{v^2/2} - k\big),$$

whose value decays exponentially as $|v| \to \infty$ in [3], we take the generalized Laguerre functions as the base functions, which is a complete $L^2(\mathbb{R})$- orthogonal system in [9]. Also, we can greatly ameliorate the accuracy of numerical errors by selecting the scaling factor involved in the base functions. Moreover, the numerical analysis is simplified and the algorithm scheme has better stability by using the basis function of weight function $\chi(v) \equiv 1$.

This paper is organized as follows. In Section 2, we establish some results on the composite Laguerre interpolation approximation, which are pivotal to the error analysis of pseudospectral methods for various differential equations on the whole line. Then we construct a semidiscrete composite Laguerre pseudospectral scheme for (1.1) and present some numerical results to show the high accuracy of the proposed algorithm in Section 3. We prove its convergence and stability in Section 4. The final section is for some concluding remarks.

2 Preliminaries

In this section, we establish some basic results on the composite Laguerre-Gauss-Radau interpolations.

2.1 Generalized Laguerre-Gauss-Radau interpolations

Let $\mathbb{R}^+ = \{\ v\ |v \in (0,\infty)\}$ and $\chi(v)$ be a certain weight function. For any integer $r \geq 0$,

$$H^r_\chi(\mathbb{R}^+) = \{u \mid u \text{ is measurable on } \mathbb{R}^+ \text{ and } \|u\|_{r,\chi,\mathbb{R}^+} < \infty\},$$

equipped with the following inner product, semi-norm and norm:

$$(u,w)_{r,\chi,\mathbb{R}^+} = \sum_{0\leq k\leq r} \int_{\mathbb{R}^+} \partial_v^k u(v)\partial_v^k w(v)\chi(v)dv,$$

$$|u|_{r,\chi,\mathbb{R}^+} = \int_{\mathbb{R}^+} (\partial_v^r u(v))^2\chi(v)dv, \quad \|u\|_{r,\chi,\mathbb{R}^+} = (u,u)^{\frac{1}{2}}_{r,\chi,\mathbb{R}^+}.$$

In particular, $H^0_\chi(\mathbb{R}^+) = L^2_\chi(\mathbb{R}^+)$, with the inner product $(u,w)_{\chi,\mathbb{R}^+}$ and the norm $\|u\|_{\chi,\mathbb{R}^+}$. We omit the subscript χ in the notations when $\chi(v) \equiv 1$.

Let $\omega^1_{\alpha,\beta}(v) = v^\alpha e^{-\beta v}$, $\alpha > -1, \beta > 0$. The generalized Laguerre polynomial of degree l is defined by

$$\mathcal{L}^{(\alpha,\beta)}_l(v) = \frac{1}{l!}v^{-\alpha}e^{\beta v}\partial_v^l(v^{l+\alpha}e^{-\beta v}).$$

The set of $\mathcal{L}^{(\alpha,\beta)}_l(v)$ is a complete $L^2_{\omega^1_{\alpha,\beta}}(\mathbb{R}^+)-$ orthogonal system.

We denote by $\mathcal{P}_N(\mathbb{R}^+)$ the set of all polynomials of degree at most N. Let $\bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{0\}$ and $\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}$, $0 \leq j \leq N$, be the zeros of polynomial $v\partial_v\mathcal{L}^{(\alpha,\beta)}_{N+1}(v)$, which are arranged in ascending order. Denote by $\omega^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}$, $0 \leq j \leq N$, the corresponding Christoffel numbers such that

$$\int_{\mathbb{R}^+} \phi(v)\omega^1_{\alpha,\beta}(v)dv = \sum_{j=0}^{N} \phi(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})\omega^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}, \quad \forall\phi \in \mathcal{P}_{2N}(\bar{\mathbb{R}}^+). \tag{2.1}$$

We introduce the following discrete inner product and norm (cf. [22]):

$$(u,w)_{\omega^1_{\alpha,\beta},N,\mathbb{R}^+} = \sum_{j=0}^{N} u(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})w(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})\omega^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j},$$

$$\|u\|_{\omega^1_{\alpha,\beta},N,\mathbb{R}^+} = (u,u)^{\frac{1}{2}}_{\omega^1_{\alpha,\beta},N,\mathbb{R}^+}$$

By the exactness of (2.1),

$$\|\phi\|_{\omega^1_{\alpha,\beta},N,\mathbb{R}^+} = \|\phi\|_{\omega^1_{\alpha,\beta},\mathbb{R}^+}, \quad \forall\ \phi \in \mathcal{P}_N(\mathbb{R}^+).$$

Let

$$d(N,\alpha) = \sqrt{(N+1+\alpha)/(N+1)}.$$

By (2.4) of [22], we have

$$\|\phi\|_{\omega^1_{\alpha,\beta},N,\mathbb{R}^+} \le \sqrt{2}\max(d(N,\alpha),1)\,\|\phi\|_{\omega^1_{\alpha,\beta},\mathbb{R}^+}, \quad \forall \phi \in \mathcal{P}_{N+1}(\mathbb{R}^+).$$

Let $\bar{\mathbb{R}}^+$ be the same as before. For any $u \in C(\bar{\mathbb{R}}^+)$, the generalized Laguerre-Gauss-Radau interpolation $\mathcal{I}_{R,N,\alpha,\beta,\mathbb{R}^+}u \in \mathcal{P}_N(\bar{\mathbb{R}}^+)$ is determined by

$$\mathcal{I}_{R,N,\alpha,\beta,\mathbb{R}^+}u(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}) = u(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}), \quad 0 \le j \le N.$$

To design proper pseudospectral method for the Fokker-Planck equation and many other similar problems, we shall use the orthogonal system of generalized Laguerre functions, defined by

$$\tilde{\mathcal{L}}^{(\alpha,\beta)}_l(v) = e^{-\frac{1}{2}\beta v}\mathcal{L}^{(\alpha,\beta)}_l(v), \quad l = 0,1,2,\ldots.$$

The set of $\tilde{\mathcal{L}}^{(\alpha,\beta)}_l(v)$ is a complete $L^2_{v^\alpha}(\mathbb{R}^+)-$ orthogonal system.

We now consider the new generalized Laguerre-Gauss-Radau interpolation corresponding to the weight function v^α. Let $\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j} = \xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}, \tilde{\omega}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j} = e^{\beta\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}}\omega^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}$, $0 \le j \le N$. We introduce the following discrete inner product and norm (cf. [22]):

$$(u,w)_{v^\alpha,N,\mathbb{R}^+} = \sum_{j=0}^{N} u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})w(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})\tilde{\omega}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j},$$
$$\|u\|_{v^\alpha,N,\mathbb{R}^+} = (u,u)^{\frac{1}{2}}_{v^\alpha,N,\mathbb{R}^+}.$$

We set $\mathcal{Q}_{N,\beta}(\mathbb{R}^+) = \{\ e^{-\frac{1}{2}\beta v}\psi \mid \psi \in \mathcal{P}_N(\mathbb{R}^+)\}$. By (2.8) of [22], we give

$$(\phi,\psi)_{v^\alpha,N,\mathbb{R}^+} = (\phi,\psi)_{v^\alpha,\mathbb{R}^+},$$
$$\phi \in \mathcal{Q}_{m,\beta}(\mathbb{R}^+), \quad \psi \in \mathcal{Q}_{2N-m,\beta}(\mathbb{R}^+), \quad 0 \le m \le 2N. \tag{2.2}$$

In particular, (cf. (2.9) of [22])

$$\|\phi\|_{v^\alpha,N,\mathbb{R}^+} = \|\phi\|_{v^\alpha,\mathbb{R}^+}, \quad \forall \phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}^+).$$

Moreover, for any $\phi \in \mathcal{Q}_{N+1,\beta}(\mathbb{R}^+)$, we have (cf. (2.10) of [22])

$$\|\phi\|_{v^\alpha,N,\mathbb{R}^+} \le \sqrt{2}\max(d(N,\alpha),1)\,\|\phi\|_{v^\alpha,\mathbb{R}^+}.$$

For pseudospectral method of nonlinear problems with varying coefficient, we need the following result.

Lemma 1. *For any $\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}^+)$, there holds*

$$\|v\phi\|_{v^\alpha,\mathbb{R}^+} \le c\beta^{-1}(N+1)\,\|\phi\|_{v^\alpha,\mathbb{R}^+}. \tag{2.3}$$

Proof. Following the same line as in the proof of Lemma 2.1 of [19], we can obtain the desired result. □

For any $u \in C(\bar{\mathbb{R}}^+)$, the generalized Laguerre-Gauss-Radau interpolation $\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}u \in \mathcal{Q}_{N,\beta}(\mathbb{R}^+)$ is determined by

$$\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}) = u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}), \quad 0 \le j \le N.$$

Furthermore, by the definitions of $\mathcal{I}_{R,N,\alpha,\beta,\mathbb{R}^+}$ and $\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}$, we have that

$$\begin{aligned}
&\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+j}) = u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j})\\
&= e^{-\frac{1}{2}\beta\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}}(e^{\frac{1}{2}\beta\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}}u(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}))\\
&= e^{-\frac{1}{2}\beta\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}}(\mathcal{I}_{R,N,\alpha,\beta,\mathbb{R}^+}(e^{\frac{1}{2}\beta\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}}u(\xi^{(\alpha,\beta)}_{R,N,\mathbb{R}^+,j}))), \;\; 0 \le j \le N.
\end{aligned}$$

This means $\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}u = e^{-\frac{1}{2}\beta v}\mathcal{I}_{R,N,\alpha,\beta,\mathbb{R}^+}(e^{\frac{1}{2}\beta v}u)$. Therefore, we have that if $u \in C(\bar{\mathbb{R}}^+)$, $\partial_v^r(e^{\frac{1}{2}\beta v}u) \in L^2_{\omega^1_{r+\alpha,\beta}}(\mathbb{R}^+) \cap L^2_{\omega^1_{r+\alpha-1,\beta}}(\mathbb{R}^+)$, integer $r \ge 1$ and $r > \alpha + 1$, then (cf. [22])

$$\begin{aligned}
\big\|\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^+}u - u\big\|_{v^\alpha,\mathbb{R}^+} \le{}& c(\beta N)^{\frac{1-r}{2}}\big(\beta^{-1}\big\|\partial_v^r(e^{\frac{1}{2}\beta v}u)\big\|_{\omega^1_{r+\alpha-1,\beta},\mathbb{R}^+}\\
&+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\big\|\partial_v^r(e^{\frac{1}{2}\beta v}u)\big\|_{\omega^1_{r+\alpha,\beta},\mathbb{R}^+}\big).
\end{aligned} \tag{2.4}$$

In particular, if $|\alpha| < 1$, then the above result holds for any integer $r \ge 1$.

Remark 1. If $u \in H^r_{v^{\alpha+r}}(\mathbb{R}^+)$, then the norm $\big\|\partial_v^r(e^{\frac{1}{2}\beta v}u)\big\|_{\omega^1_{r+\alpha,\beta},\mathbb{R}^+}$ is finite.

We now consider the interpolation on the subdomain $\mathbb{R}^- = (-\infty, 0)$. The space $H^r_\chi(\mathbb{R}^-)$ is defined as usual, with the inner product $(u,w)_{r,\chi,\mathbb{R}^-}$, the semi-norm $|u|_{r,\chi,\mathbb{R}^-}$ and the norm $\|u\|_{r,\chi,\mathbb{R}^-}$. Especially, $H^0_\chi(\mathbb{R}^-) = L^2_\chi(\mathbb{R}^-)$, with the inner product $(u,w)_{\chi,\mathbb{R}^-}$ and the norm $\|u\|_{\chi,\mathbb{R}^-}$.

Let $\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j}$, $0 \le j \le N$, be the zeros of polynomial $-v\tilde{\mathcal{L}}^{(\alpha+1,\beta)}_N(-v)$, which are arranged in descending order. Denote by $\tilde{\omega}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j}$, $0 \le j \le N$, the corresponding Christoffel numbers. For $v \in \mathbb{R}^-$, we introduce the discrete inner product and norm as follows (cf. [22]):

$$\begin{aligned}
(u,w)_{(-v)^\alpha,N,\mathbb{R}^-} &= \sum_{j=0}^{N} u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j})w(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j})\tilde{\omega}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j},\\
\|u\|_{(-v)^\alpha,N,\mathbb{R}^-} &= (u,u)^{\frac{1}{2}}_{(-v)^\alpha,N,\mathbb{R}^-}.
\end{aligned}$$

We set $\mathcal{Q}_{N,\beta}(\mathbb{R}^-) = \{ e^{\frac{1}{2}\beta v}\psi \mid \psi \in \mathcal{P}_N(\mathbb{R}^-)\}$. Like (2.2)–(2.3), we have that

$$\begin{aligned}
&(\phi,\psi)_{(-v)^\alpha,N,\mathbb{R}^-} = (\phi,\psi)_{(-v)^\alpha,\mathbb{R}^-},\\
&\forall\phi \in \mathcal{Q}_{m,\beta}(\mathbb{R}^-),\ \ \psi \in \mathcal{Q}_{2N-m,\beta}(\mathbb{R}^-), \quad 0 \le m \le 2N, \qquad (2.5)\\
&\|\phi\|_{(-v)^\alpha,N,\mathbb{R}^-} = \|\phi\|_{(-v)^\alpha,\mathbb{R}^-}, \quad \forall\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}^-),\\
&\|\phi\|_{(-v)^\alpha,N,\mathbb{R}^-} \le \sqrt{2}\max(d(N,\alpha),1)\,\|\phi\|_{(-v)^\alpha,\mathbb{R}^-}, \ \ \forall\phi \in \mathcal{Q}_{N+1,\beta}(\mathbb{R}^-),\\
&\|v\phi\|_{(-v)^\alpha,\mathbb{R}^-} \le c\beta^{-1}(N+1)\,\|\phi\|_{(-v)^\alpha,\mathbb{R}^-}, \quad \forall\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}^-). \qquad (2.6)
\end{aligned}$$

Let $\bar{\mathbb{R}}^- = \mathbb{R}^- \cup \{0\}$. For any $u \in C(\bar{\mathbb{R}}^-)$, the generalized Laguerre-Gauss - Radau interpolation $\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^-}u \in \mathcal{Q}_{N,\beta}(\bar{\mathbb{R}}^-)$ is determined by

$$\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^-}u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j}) = u(\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^-,j}), \quad 0 \le j \le N.$$

Let $\omega^2_{\alpha,\beta}(v){=}\omega^1_{\alpha,\beta}(-v){=}(-v)^\alpha e^{\beta v}$. If $u{\in}C(\bar{\mathbb{R}}^-)$, $\partial_v^r(e^{-\frac{1}{2}\beta v}u){\in}L^2_{\omega^2_{r+\alpha,\beta}}(\mathbb{R}^-) \cap L^2_{\omega^2_{r+\alpha-1,\beta}}(\mathbb{R}^-)$, integer $r{\ge}1$ and $r > \alpha{+}1$, then (cf. [22]),

$$\begin{aligned}
\big\|\tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^-}u{-}u\big\|_{(-v)^\alpha,\mathbb{R}^-} \le{}& c(\beta N)^{\frac{1-r}{2}}(\beta^{-1}\big\|\partial_v^r(e^{-\frac{1}{2}\beta v}u)\big\|_{\omega^2_{r+\alpha-1,\beta},\mathbb{R}^-}\\
&+ (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}\big\|\partial_v^r(e^{-\frac{1}{2}\beta v}u)\big\|_{\omega^2_{r+\alpha,\beta},\mathbb{R}^-}). \qquad (2.7)
\end{aligned}$$

In particular, if $|\alpha| < 1$, then the above result holds for any integer $r \ge 1$.

Remark 2. If $u \in L^2_{(-v)^{r+\alpha}}(\mathbb{R}^-)$, then $\left\|\partial_v^r(e^{-\frac{1}{2}\beta v}u)\right\|_{\omega^2_{\alpha+r,\beta},\mathbb{R}^-}$ is finite.

2.2 Composite generalized Laguerre interpolation on the whole

We are now in position of studying the composite generalized Laguerre-Gauss-Radau interpolation on the whole line $\mathbb{R} = \bar{\mathbb{R}}^+ \cup \bar{\mathbb{R}}^-$. The space $L^2_\chi(\mathbb{R})$ is defined as usual, with the inner product $(u,w)_{\chi,\mathbb{R}}$ and the norm $\|u\|_{\chi,\mathbb{R}}$. We omit the subscript χ in the notations when $\chi(v) \equiv 1$. Further, let

$$\begin{aligned}
&H^1_{1,1+v^2}(\mathbb{R}) = \{ u \mid u \in L^2_{v^2+1}(\mathbb{R}),\ \partial_v u \in L^2(\mathbb{R})\},\\
&\mathcal{Q}_{N,\beta}(\mathbb{R}) = H^1_{1,1+v^2}(\mathbb{R}) \cap \{ \phi \mid \phi\,|_{\mathbb{R}^+} \in \mathcal{Q}_{N,\beta}(\mathbb{R}^+), \quad \phi\,|_{\mathbb{R}^-} \in \mathcal{Q}_{N,\beta}(\mathbb{R}^-)\}.
\end{aligned}$$

We introduce the discrete inner product and norm as

$$(u,w)_{|v|^\alpha,N,\mathbb{R}}{=}(u,w)_{v^\alpha,N,\mathbb{R}^+}{+}(u,w)_{(-v)^\alpha,N,\mathbb{R}^-}, \ \ \|u\|_{|v|^\alpha,N,\mathbb{R}}{=}(u,u)^{\frac{1}{2}}_{|v|^\alpha,N,\mathbb{R}}.$$

By virtue of (2.2)–(2.3) and (2.5)–(2.6), we have that

$$\begin{aligned}
&(\phi,\psi)_{|v|^\alpha,N,\mathbb{R}} = (\phi,\psi)_{|v|^\alpha,\mathbb{R}}, \quad \forall\phi\psi \in \mathcal{Q}_{2N,\beta}(\mathbb{R}), \qquad (2.8)\\
&\|\phi\|_{|v|^\alpha,N,\mathbb{R}} = \|\phi\|_{|v|^\alpha,\mathbb{R}}, \quad \forall\phi\ \in \mathcal{Q}_{N,\beta}(\mathbb{R}), \qquad (2.9)\\
&\|\phi\|_{|v|^\alpha,N,\mathbb{R}} \le \sqrt{2}\max(d(N,\alpha),1)\,\|\phi\|_{|v|^\alpha,\mathbb{R}}, \quad \forall\phi \in \mathcal{Q}_{N+1,\beta}(\mathbb{R}), \qquad (2.10)\\
&\|v\phi\|_{|v|^\alpha,\mathbb{R}} \le c\beta^{-1}(N+1)\,\|\phi\|_{|v|^\alpha,\mathbb{R}}, \quad \forall\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}).
\end{aligned}$$

The composite interpolation $I_{N,\alpha,\beta,\mathbb{R}}u(v) \in \mathcal{Q}_{N,\beta}(\mathbb{R})$ is defined by

$$I_{N,\alpha,\beta,\mathbb{R}}u(v)|_{\mathbb{R}^{\pm}} = \tilde{\mathcal{I}}_{R,N,\alpha,\beta,\mathbb{R}^{\pm}}u(v), \quad v \in \{\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^{\pm},k},\ 0 \le k \le N\}.$$

Let $\omega_{\alpha,\beta}|_{\mathbb{R}^{\pm}} = \omega^{j}_{\alpha,\beta}, j = 1, 2$. By using (2.4) and (2.7), we obtain that

$$\begin{aligned}\left\|I_{N,\alpha,\beta,\mathbb{R}}u - u\right\|^2_{|v|^{\alpha},\mathbb{R}} &\le c(\beta N)^{1-r}\big(\beta^{-2}\left\|\partial^r_v(e^{\frac{1}{2}\beta|v|}u)\right\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}}\\ &+ (1+\beta^{-1})\ln N\left\|\partial^r_v(e^{\frac{1}{2}\beta|v|}u)\right\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}}\big).\end{aligned} \tag{2.11}$$

In particular, if $|\alpha| < 1$, then the above result holds for any integer $r \ge 1$.

Remark 3. If $u \in L^2_{|v|^{r+\alpha}}(\mathbb{R})$, then $\left\|\partial^r_v(e^{\frac{1}{2}\beta|v|}u)\right\|_{\omega_{r+\alpha,\beta},\mathbb{R}}$ is finite.

In numerical analysis of the composite pseudospectral method for the Fokker-Planck equation on the whole line, we need a non-standard projection $P^1_{N,\beta,\mathbb{R}} : H^1_{1,v^2+1}(\mathbb{R}) \to Q_{N,\beta}(\mathbb{R})$, which is defined by

$$(\partial_v(P^1_{N,\beta,\mathbb{R}}u - u), \partial_v\phi)_{\mathbb{R}} + (P^1_{N,\beta,\mathbb{R}}u - u, \phi)_{v^2+1,\mathbb{R}} = 0, \quad \forall\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}).$$

By (2.11) of [20], we have that if $u\in H^1_{1,v^2+1}(\mathbb{R}),\ \partial^{r+1}_v(e^{\frac{1}{2}\beta|v|}u)\in L^2_{\omega_{r+1,\beta}}(\mathbb{R})$ and integer $r \ge 1$, then

$$\begin{aligned}&\left\|\partial_v(P^1_{N,\beta,\mathbb{R}}u - u)\right\|_{\mathbb{R}} + \left\|P^1_{N,\beta,\mathbb{R}}u - u\right\|_{v^2+1,\ \mathbb{R}}\\ &\quad\le c(\beta+\beta^{-2})(\beta N)^{\frac{1-r}{2}}\left\|\partial^{r+1}_v(e^{\frac{1}{2}\beta|v|}u)\right\|_{\omega_{r+1,\beta},\mathbb{R}}.\end{aligned} \tag{2.12}$$

We need the following embedding inequality (cf. [21]).

Lemma 2. *If $u \in H^1(\mathbb{R})$, then,*

$$\|u\|_{L^{\infty}(\mathbb{R})} \le \sqrt{2}\,\|u\|^{\frac{1}{2}}_{\mathbb{R}}\,|u|^{\frac{1}{2}}_{1,\mathbb{R}}. \tag{2.13}$$

By Lemma 2 and (2.12), we can get the estimate on $L^{\infty}(\mathbb{R})$-norm of $P^1_{N,\beta,\mathbb{R}}u$.

$$\left\|P^1_{N,\beta,\mathbb{R}}u\right\|_{L^{\infty}(\mathbb{R})} \le \|u\|_{L^{\infty}(\mathbb{R})} + c(\beta+\beta^{-2})(\beta N)^{\frac{1-r}{2}}\left\|\partial^{r+1}_v(e^{\frac{1}{2}\beta|v|}u)\right\|_{\omega_{r+1,\beta},\mathbb{R}}. \tag{2.14}$$

In the end of this section, we need some inverse inequality which will be used in the sequel (cf. [20]).

Proposition 1. *For any $\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R})$ and $1 \le p \le q \le \infty$, there holds*

$$\left\|\phi e^{(\frac{\beta}{2}-\frac{\beta}{q})|v|}\right\|_{L^q(\mathbb{R})} \le c(\beta N)^{(\frac{1}{p}-\frac{1}{q})}\left\|\phi e^{(\frac{\beta}{2}-\frac{\beta}{p})|v|}\right\|_{L^p(\mathbb{R})}.$$

Proposition 2. *For any $\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R})$, there hold*

$$\|\partial_v\phi\| \le c\beta(N+1)\,\|\phi\|, \quad \|v\phi\| \le c\beta^{-1}(N+1)\,\|\phi\|. \tag{2.15}$$

Proposition 3. *By Proposition 1, for any $\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R})$ and $q \ge 1$, there holds*

$$\|\phi\|_{L^{2q}(\mathbb{R})} \le \left\|\phi e^{(\frac{\beta}{2}-\frac{\beta}{2q})|v|}\right\|_{L^{2q}(\mathbb{R})} \le c(\beta N)^{(\frac{1}{2}-\frac{1}{2q})}\,\|\phi\|_{\mathbb{R}}. \tag{2.16}$$

3 Composite pseudospectral method

In this section, we propose the composite pseudospectral method for the nonlinear Fokker-Planck equations on the whole line. We also describe the implementation and present some numerical results.

3.1 Pseudospectral scheme

Now, we deduce the pseudospectral scheme of (1.1), whose weak formulation is to seek W which belongs to the space $L^\infty(0,T;L^2(\mathbb{R}))\cap L^2(0,T;H^1_{1,v^2+1}(\mathbb{R}))$ such that

$$\begin{cases} (\partial_t W(t),u)_{\mathbb{R}}+(vW(t),\partial_v u)_{\mathbb{R}}+k(vW^2(t),\partial_v u)_{\mathbb{R}} \\ \quad +(\partial_v W(t),\partial_v u)_{\mathbb{R}}=0, \qquad \forall u\in H^1_{1,v^2+1}(\mathbb{R}), \quad 0<t\le T, \\ W(0)=W_0. \end{cases} \tag{3.1}$$

We now design the composite generalized Laguerre pseudospectral scheme for (3.1). It is to find $w_N(t)\in\mathcal{Q}_{N,\beta}(\mathbb{R})$ for all $0\le t\le T$, such that

$$\begin{cases} (\partial_t w_N(t),\phi)_{N,\mathbb{R}}+(vw_N(t),\partial_v\phi)_{N,\mathbb{R}}+k(vw_N^2(t),\partial_v\phi)_{N,\mathbb{R}} \\ \quad +(\partial_v w_N(t),\partial_v\phi)_{N,\mathbb{R}}=0, \qquad \forall\phi\in\mathcal{Q}_{N,\beta}(\mathbb{R}), \quad 0<t\le T, \\ w_N(0)=I_{N,0,\beta,\mathbb{R}}W_0 \text{ or } P^1_{N,\beta,\mathbb{R}}W_0. \end{cases}$$

Thanks to (2.8), the above problem is equivalent to

$$\begin{cases} (\partial_t w_N(t),\phi)_{\mathbb{R}}+(vw_N(t),\partial_v\phi)_{\mathbb{R}}+k(vw_N^2(t),\partial_v\phi)_{N,\mathbb{R}} \\ \quad +(\partial_v w_N(t),\partial_v\phi)_{\mathbb{R}}=0, \qquad \forall\phi\in\mathcal{Q}_{N,\beta}(\mathbb{R}), \quad 0<t\le T, \\ w_N(0)=I_{N,0,\beta,\mathbb{R}}W_0 \text{ or } P^1_{N,\beta,\mathbb{R}}W_0. \end{cases} \tag{3.2}$$

Next, we describe the implementations for pseudospectral scheme (3.2) with a nonhomogeneous term $f(v,t)$, we use the Crank-Nicolson discretization in time t, with the mesh size τ.

For simplicity of statements, we use the notation

$$a_{N,\mathbb{R}}(z,\phi)=(vz,\partial_v\phi)_{\mathbb{R}}+k(vz^2,\phi)_{N,\mathbb{R}}+(\partial_v z,\partial_v\phi)_{\mathbb{R}}.$$

The fully discrete scheme of (3.2) is as follows:

$$\begin{cases} \frac{1}{\tau}(w_N(t+\tau)-w_N(t),\phi)_{\mathbb{R}}+\frac{1}{2}a_{N,\mathbb{R}}(w_N(t+\tau)+w_N(t),\phi) \\ \quad =\frac{1}{2}(f(t+\tau)+f(t),\phi)_{N,\mathbb{R}}, \qquad t=0,\tau,\dots,T-\tau, \\ w_N(0)=P^1_{N,\beta,\mathbb{R}}W_0. \end{cases} \tag{3.3}$$

Let

$$A_{N,\mathbb{R}}(z,u)=\frac{1}{2}\tau a_{N,\mathbb{R}}(z,u)+(z,u)_{\mathbb{R}}, \quad \bar{A}_{N,\mathbb{R}}(z,u)=-\frac{1}{2}\tau a_{N,\mathbb{R}}(z,u)+(z,u)_{\mathbb{R}}.$$

Then, at each time step, we need to solve the following nonlinear equation:

$$A_{N,\mathbb{R}}(w_N(t),\phi)=\bar{A}_{N,\mathbb{R}}(w_N(t-\tau),\phi)+\frac{\tau}{2}(f(t)+f(t-\tau),\phi)_{N,\mathbb{R}}, \ \forall\phi\in\mathcal{Q}_{N,\beta}(\mathbb{R}).$$

For notational convenience, let $\tilde{\mathcal{L}}_l^{(\beta)}(v) = \tilde{\mathcal{L}}_l^{(0,\beta)}(v)$. Moreover, set

$$\psi_{1,l}^{(\beta)}(v)=\tilde{\mathcal{L}}_l^{(\beta)}(v)-\tilde{\mathcal{L}}_{l+1}^{(\beta)}(v),\ \ \psi_{2,l}^{(\beta)}(v)=\tilde{\mathcal{L}}_l^{(\beta)}(-v)-\tilde{\mathcal{L}}_{l+1}^{(\beta)}(-v),\ \ 0\le l\le N-1,$$

and

$$G_l^{(1,\beta)}(v) = \begin{cases} \psi_{1,l}^{(\beta)}(v), & v\in\mathbb{R}^+, \\ 0, & v\in\mathbb{R}^-, \end{cases} \qquad G_l^{(2,\beta)}(v) = \begin{cases} 0, & v\in\mathbb{R}^+, \\ \psi_{2,l}^{(\beta)}(v), & v\in \mathbb{R}^-. \end{cases}$$

Obviously, $G_l^{(1,\beta)}(v), G_l^{(2,\beta)}(v) \in \mathcal{Q}_{N,\beta}(\mathbb{R})$. Furthermore, let

$$G^{(\beta)}(v) = e^{-\frac{1}{2}\beta|v|} \in \mathcal{Q}_{N,\beta}(\mathbb{R}).$$

The functions $G_l^{(j,\beta)}(v),\ 0\le l\le N-1, j=1,2$ and $G^{(\beta)}(v)$ form a basis of $\mathcal{Q}_{N,\beta}(\mathbb{R})$.

In actual computation, we expand the numerical solution as

$$w_N(v,t) = \sum_{l=0}^{N-1} v_l^{(1)}(t)G_l^{(1,\beta)}(v) + \sum_{l=0}^{N-1} v_l^{(2)}(t)G_l^{(2,\beta)}(v) + v^{(0)}(t)G^{(\beta)}(v).$$

3.2 Numerical results

In this subsection, we present some numerical results confirming the theoretical analysis. We use scheme (3.3) to solve (1.1) with $k=1$. The numerical errors are measured by the discrete norm

$$E_{N,\tau}(t) = \|W(t) - w_N(t)\|_{N,\mathbb{R}}\,.$$

Now, we take the test function which decays exponentially at infinity,

$$W(v,t) = \tfrac{1}{4\sqrt{t+1}}\tanh\left(\tfrac{\sqrt{6}}{12}(v-\tfrac{5\sqrt{6}}{6}t)\right)e^{-\frac{1}{2}v^2}.$$

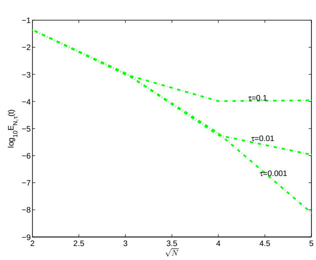

Figure 1. L^2- errors against $\sqrt{N}$ with different τ.

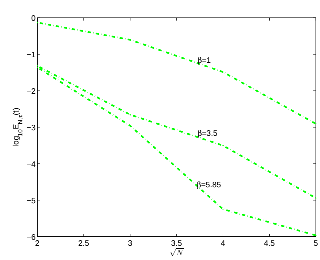

Figure 2. L^2- errors against $\sqrt{N}$ with different β.

In Figure 1, we plot the errors $\log_{10} E_{N,\tau}(t)$ with $t=10$ and $\beta=5.85$ vs. $\sqrt{N}$. Clearly, the errors decay fast when N increases and τ decreases. The above facts coincide very well with theoretical analysis in Theorem 1 on page 16. In particular, they show the spectral accuracy in the space of scheme (3.3).

In Figure 2, we plot $\log_{10} E_N(t)$ at $t = 10$, $\tau = 0.01$ and different values of parameter β vs. $\sqrt{N}$. It seems that the errors with suitably bigger β are smaller than those with smaller β. However, how to choose the best parameter β is still an open problem. Roughly speaking, if the exact solution decays faster as v increases, then it is better to take suitably bigger β.

In Figure 3, we plot the numerical errors $\log_{10} E_N(t)$ with $\beta = 7, N = 36, \tau = 0.001$. It indicates the stability of scheme (3.3).

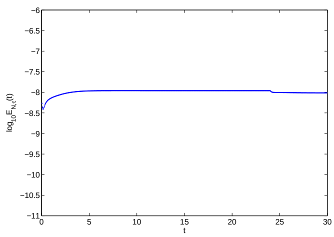

Figure 3. L^2- errors against t with $\tau = 0.001$.

Next, considering the following test function which decays algebraically at infinity(cf. [21])

$$W(v,t) = \frac{v^2+\cos t}{(v^2+1)^5}.$$

In Figure 4, we present the errors $\log_{10} E_{N,\tau}(t)$ with $t = 10$ and $\tau = 0.0001$ vs. the $\sqrt{N}$ with the different scaling factor β. They show that the solution of (1.1) with decays algebraically at infinity can also be solved numerically by the scheme (3.3).

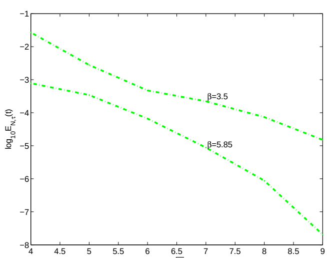

Figure 4. L^2- errors against $\sqrt{N}$ with different β.

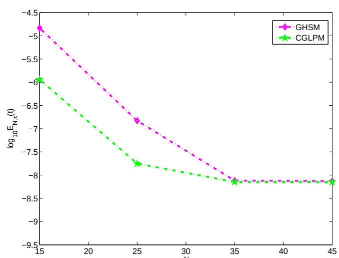

Figure 5. L^2- errors against $\sqrt{N}$ with $\tau = 0.001$.

In Figure 5, we plot L^2- errors of the composite generalized Laguerre pseudospectral method (CGLPM) with $\beta = 5.75, \tau = 0.001, N = 15, 25, 35, 45$ and $t = 1$ vs. L^2- errors of the generalized Hermite spectral method(GHSM) of [4], which show numerically the CGLPM method's superiority compared with the GHSM method in [4] for $N \leq 35$.

4 Convergence and stability analysis of pseudospectral scheme

In this section, we consider the convergence and stability of scheme (3.2).

4.1 Convergence analysis

We next deal with the convergence of scheme (3.2). Let $W_N = P^1_{N,\beta,\mathbb{R}}W$. We derive from (3.1) that

$$\begin{aligned}&(\partial_t W_N(t),\phi)_{\mathbb{R}} + (vW_N(t),\partial_v\phi)_{\mathbb{R}} + k(vW_N^2(t),\partial_v\phi)_{N,\mathbb{R}} + (\partial_v W_N(t),\partial_v\phi)_{\mathbb{R}}\\ &\qquad + \sum_{j=1}^{4} G_j(t,\phi) = 0, \quad \forall\ \phi\in\mathcal{Q}_{N,\beta}(\mathbb{R}), \quad 0<t\le T,\\ &W_N(0) = P^1_{N,\beta,\mathbb{R}}W_0, \end{aligned}\tag{4.1}$$

where

$$\begin{aligned}&G_1(t,\phi) = (\partial_t W(t) - \partial_t W_N(t),\phi)_{\mathbb{R}},\quad G_2(t,\phi) = (v(W(t)-W_N(t)),\partial_v\phi)_{\mathbb{R}},\\ &G_3(t,\phi) = (\partial_v(W(t)-W_N(t)),\partial_v\phi)_{\mathbb{R}},\\ &G_4(t,\phi) = k(vW^2(t),\partial_v\phi)_{\mathbb{R}} - k(vW_N^2(t),\partial_v\phi)_{N,\mathbb{R}}.\end{aligned}$$

Taking $\widetilde{W}_N = w_N - W_N$ and subtracting (4.1) from (3.2), we obtain that

$$\begin{aligned}&(\partial_t\widetilde{W}_N(t),\phi)_{\mathbb{R}} + (v\widetilde{W}_N(t),\partial_v\phi)_{\mathbb{R}} + (\partial_v\widetilde{W}_N(t),\partial_v\phi)_{\mathbb{R}}\\ &\qquad = \sum_{j=1}^{5} G_j(t,\phi), \quad \forall\ \phi\in\mathcal{Q}_{N,\beta}(\mathbb{R}), \quad 0<t\le T,\\ &\widetilde{W}_N(0) = I_{N,0,\beta,\mathbb{R}}W_0 - P^1_{N,\beta,\mathbb{R}}W_0 \text{ or } 0,\end{aligned}\tag{4.2}$$

where $G_5(t,\phi) = k(v(W_N^2(t) - w_N^2(t)),\partial_v\phi)_{N,\mathbb{R}}$. Take $\phi = 2\widetilde{W}_N(t)$ in (4.2), we deduce that for $0<t\le T$,

$$\partial_t\big\|\widetilde{W}_N(t)\big\|^2_{\mathbb{R}} + 2\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}} = 2\sum_{j=1}^{5} G_j(t,\widetilde{W}_N(t)) + \big\|\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}.\tag{4.3}$$

Therefore, it suffices to estimate the terms $|G_j(t,\widetilde{W}_N)|$. Firstly, we use the Cauchy inequality and (2.12) to verify that for integers $r\ge 1$,

$$\begin{aligned}&|G_1(t,\widetilde{W}_N(t))| = 2|(\partial_t W(t) - \partial_t W_N(t),\widetilde{W}_N(t))_{\mathbb{R}}|\\ &\quad\le \big\|\partial_t(W(t)-W_N(t))\big\|^2_{\mathbb{R}} + \big\|\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}\\ &\quad\le c(\beta+\beta^{-2})^2(\beta N)^{1-r}\big\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}\partial_t W(t))\big\|^2_{\omega_{r+1,\beta},\mathbb{R}} + \big\|\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}.\end{aligned}\tag{4.4}$$

Similarly,

$$\begin{aligned}&|G_2(t,\widetilde{W}_N(t)) + G_3(t,\widetilde{W}_N(t))|\\ &\quad\le c(\beta+\beta^{-2})^2(\beta N)^{1-r}\big\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W(t))\big\|^2_{\omega_{r+1,\beta},\mathbb{R}} + \frac{1}{4}\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}.\end{aligned}$$

Obviously,

$$|G_4(t,\widetilde{W}_N(t))| \le 2|(v(W^2(t)-I_{N,\alpha,\beta,\mathbb{R}}W^2(t)),\partial_v\widetilde{W}_N(t))_{\mathbb{R}}| \\ +|(v(I_{N,\alpha,\beta,\mathbb{R}}W^2(t)-W_N^2(t)),\partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}|.$$

According to (2.11) with $\alpha=1$, it is easy to derive that

$$\begin{aligned}&2|(v(W^2(t)-I_{N,\alpha,\beta,\mathbb{R}}W^2(t)),\,\partial_v\widetilde{W}_N(t))_{\mathbb{R}}|\\ &\quad\le \frac{1}{8}\big\|\partial_v\widetilde{W}_N(t)\big\|_{\mathbb{R}}^2 + c(\beta N)^{1-r}(\beta^{-2}\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\big\|^2_{\omega_{r,\beta},\mathbb{R}}\\ &\qquad+(1+\beta^{-1})\ln N\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\big\|^2_{\omega_{r+1,\beta},\mathbb{R}}).\end{aligned}$$

By $I_{N,\alpha,\beta,\mathbb{R}}u^2(v)=u^2(v)=(I_{N,\alpha,\beta,\mathbb{R}}u(v))^2,\ v\in\{\tilde{\xi}^{(\alpha,\beta)}_{R,N,\mathbb{R}^\pm,k},\ 0\le k\le N\}$, (2.13), (2.14) with $r=1$, (2.11) with $\alpha=2$ and (2.12), we have that

$$\begin{aligned}&2\big|(v(I_{N,\alpha,\beta,\mathbb{R}}W^2(t)-W_N^2(t)),\partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}\big| = 2\big|(v(I_{N,\alpha,\beta,\mathbb{R}}W(t)\\ &\quad+W_N(t))(I_{N,\alpha,\beta,\mathbb{R}}W(t)-W_N(t)),\partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}\big| \le c(\|W(t)\|_\infty^2\\ &\quad+\|W_N(t)\|^2_\infty)\,\|v(I_{N,\alpha,\beta,\mathbb{R}}W(t)-W_N(t))\|^2_{N,\mathbb{R}}+\frac{1}{8}\left\|\partial_v\widetilde{W}_N(t)\right\|^2_{\mathbb{R}}\\ &\le c\max(d^2(N,2),1)(\|W(t)\|^2_\infty+\|W_N(t)\|^2_\infty)\,\|v(I_{N,\alpha,\beta,\mathbb{R}}W(t)-W_N(t))\|^2_{\mathbb{R}}\\ &\quad+\frac{1}{8}\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}\le c\max(d^2(N,2),1)\big(1+\beta^2+\big(\frac{\beta}{2}\big)^4\big)\big(\,\|W(t)\|^2_{1,\mathbb{R}}\\ &\quad+\||v|W(t)\|^2_{2,\mathbb{R}})(\|v(I_{N,\alpha,\beta,\mathbb{R}}W(t)-W(t))\|^2_{\mathbb{R}}+\|v(W(t)-W_N(t))\|^2_{\mathbb{R}})\\ &\quad+\frac{1}{8}\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}\le c\max(d^2(N,2),1)\big(1+\beta^2+(\frac{\beta}{2})^4)(\|W(t)\|^2_{1,\mathbb{R}}\\ &\quad+\||v|W(t)\|^2_{2,\mathbb{R}}\,)(\beta N)^{1-r}\big((\beta+\beta^{-2})^2\left\|\partial_v^{r+1}(e^{\frac{1}{2}\beta v}W(t))\right\|^2_{\omega_{r+1,\beta},\mathbb{R}}\\ &\quad+\beta^{-2}\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\big\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}}\\ &\quad+(1+\beta^{-1})\ln N\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\big\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}}\big)+\frac{1}{8}\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}.\end{aligned}$$

A combination of the above two estimates gives that

$$\begin{aligned}&|G_4(t,\widetilde{W}_N(t))|\le c\max(d^2(N,2),1)(1+\beta^2+(\frac{\beta}{2})^4)\big(\,\|W(t)\|^2_{1,\mathbb{R}}\\ &\quad+\||v|W(t)\|^2_{2,\mathbb{R}}\,)(\beta N)^{1-r}\big(\beta^{-2}\big(\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\big\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}}\\ &+\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\big\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}})+(1+\beta^{-1})\ln N\big(\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\big\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}}\\ &+\big\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\big\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}})+(\beta+\beta^{-2})^2\big\|\partial_v^{r+1}(e^{\frac{1}{2}\beta v}W(t))\big\|^2_{\omega_{r+1,\beta},\mathbb{R}}\big)\\ &\quad+\frac{1}{4}\big\|\partial_v\widetilde{W}_N(t)\big\|^2_{\mathbb{R}}.\end{aligned}$$

By (2.8) and (2.9) with $\alpha=0$, we deduce that

$$|G_5(t,\widetilde{W}_N(t))| = 2|k(v(W_N^2(t)-w_N^2(t)),\partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}|$$

$$= 2|(v\widetilde{W}_N^2(t) + 2v\widetilde{W}_N(t)W(t), \partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}$$
$$+ 2v\widetilde{W}_N(t)(W_N(t) - I_{N,\alpha,\beta,\mathbb{R}}W(t)), \partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}|.$$

Using (2.15), (2.16) with $q = \infty$ and (2.13), we derive that

$$|(v\widetilde{W}_N^2(t), \partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}| \le \left\|\widetilde{W}_N(t)\right\|_{L^\infty(\mathbb{R})} \left\|v\widetilde{W}_N(t)\right\|_{\mathbb{R}} \left\|\partial_v\widetilde{W}_N(t))\right\|_{\mathbb{R}}$$
$$\le c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 \left\|\partial_v\widetilde{W}_N(t))\right\|_{\mathbb{R}}$$
$$\le c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}} (\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \left\|\partial_v\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2),$$
$$|(2v\widetilde{W}_N(t)W(t), \partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}| \le c\left\|vW(t)\right\|_{L^\infty(\mathbb{R})}^2 \left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \frac{1}{4}\left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2$$
$$\le c\left\|vW(t)\right\|_{1,\mathbb{R}}^2 \left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \frac{1}{4}\left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2,$$

and

$$\begin{aligned}
&(2v\widetilde{W}_N(t)(W_N(t) - I_{N,\alpha,\beta,\mathbb{R}}W(t)), \partial_v\widetilde{W}_N(t))_{N,\mathbb{R}}| \\
&\le 2\left\|\widetilde{W}_N(t)\right\|_{L^\infty(\mathbb{R})} \left\|v(W_N(t) - I_{N,\alpha,\beta,\mathbb{R}}W(t))\right\|_{N,\mathbb{R}} \left\|\partial_v\widetilde{W}_N(t)\right\|_{N,\mathbb{R}} \\
&\le c(\beta N)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 \left\|v(W_N(t) - I_{N,\alpha,\beta,\mathbb{R}}W(t))\right\|_{\mathbb{R}}^2 + \tfrac{1}{4}\left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 \\
&\le c(\beta N)^{1-r}(\beta N)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 \Big((\beta+\beta^{-2})^2\left\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W(t))\right\|_{\omega_{r+1,\beta},\mathbb{R}}^2 \\
&+\beta^{-2}\left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W(t))\right\|_{\omega_{r+1,\beta},\mathbb{R}}^2 + (1+\beta^{-1})\ln N\left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W(t))\right\|_{\omega_{r+2,\beta},\mathbb{R}}^2\Big) \\
&\quad+\frac{1}{4}\left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2.
\end{aligned}$$

For fully big N, and $r > 1$, we have

$$|G_5(t, \widetilde{W}_N(t))| \le c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}(\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \left\|\partial_v\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2)$$
$$+ c(\left\|vW(t)\right\|_{1,\mathbb{R}}^2 + d(W(t),\beta))\left\|\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2 + \frac{1}{2}\left\|\partial_v\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2, \tag{4.5}$$

where $d(W(t), \beta)$ is a positive constant, depend on $\left\|\partial_v^r(e^{\frac{1}{2}\beta v}W(t))\right\|_{\omega_{r+2,\beta},\mathbb{R}}$, β and $\left\|\partial_v^{r+j}(e^{\frac{1}{2}\beta|v|}W(t))\right\|_{\omega_{r+1,\beta},\mathbb{R}}, j = 0, 1$. By inserting (4.4)–(4.5) into (4.3), we find that

$$\begin{aligned}
\partial_t\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 &\le c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}} \\
&\times\left(\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2\right) + c(\left\|vW(t)\right\|_{1,\mathbb{R}}^2 + d(W(t),\beta) + 1) \\
&\times\left\|\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2 + c(\beta+\beta^{-2})^2(\beta N)^{1-r}\mathcal{A}_{r,\beta}(W(t)),
\end{aligned}$$

or equivalently,

$$\partial_t\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + (\left\|\partial_v\widetilde{W}_N(t)\right\|_{\mathbb{R}}^2 + \left\|\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2) \tag{4.6}$$
$$\le c(\left\|vW(t)\right\|_{1,\mathbb{R}}^2 + d(W(t),\beta) + 2)\left\|\widetilde{W}_N(t))\right\|_{\mathbb{R}}^2 + c(\beta N)^{1-r}\mathcal{A}_{r,\beta}(W(t))$$

$$+ c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\left\|\widetilde{W}_N(t)\right\|_{\mathbb{R}}(\left\|\widetilde{W}_N(t)\right\|^2_{\mathbb{R}} + \left\|\partial_v\widetilde{W}_N(t))\right\|^2_{\mathbb{R}}),$$

where

$$\begin{aligned}
\mathcal{A}_{r,\beta}(W(t)) &= (\beta+\beta^{-2})^2\big(\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}\partial_t W(t))\|^2_{\omega_{r+1,\beta},\mathbb{R}} \\
&+ \|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W(t))\|^2_{\omega_{r+1,\beta},\mathbb{R}}\big) + \max(d^2(N,2),1) \\
&\times \big(1+\beta^2+(\beta/2)^4\big)\big(\,\|W(t)\|^2_{1,\mathbb{R}} + \||v|W_N(t)\|^2_{2,\mathbb{R}}\,\big) \\
&\times \big(\beta^{-2}(\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}} + \|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\|^2_{\omega_{r+\alpha-1,\beta},\mathbb{R}}) \\
&+ (1+\beta^{-1})\ln N\big(\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W(t))\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}} + \|\partial_v^r(e^{\frac{1}{2}\beta|v|}W^2(t))\|^2_{\omega_{r+\alpha,\beta},\mathbb{R}}\big) \\
&+ (\beta+\beta^{-2})^2\|\partial_v^{r+1}(e^{\frac{1}{2}\beta v}W(t))\|^2_{\omega_{r+1,\beta},\mathbb{R}}\big).
\end{aligned}$$

Integrating (4.6) with respect to t, we deduce that

$$\begin{aligned}
&\|\widetilde{W}_N(t)\|^2_{\mathbb{R}} + \int_0^t \big(1 - c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1)\|\widetilde{W}_N(\tau))\|_{\mathbb{R}}\big)\big(\|\partial_v\widetilde{W}_N(\tau)\|^2_{\mathbb{R}} \\
&+ \|\widetilde{W}_N(\tau))\|^2_{\mathbb{R}}\big)d\tau \le c(\beta N)^{1-r}\Big(\int_0^t \mathcal{A}_{r,\beta}(W(\tau))d\tau + \lambda\big((\beta+\beta^{-2})^2 \\
&\times \|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r+1,\beta},\mathbb{R}} + \beta^{-2}\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r-1,\beta},\mathbb{R}} + (1+\beta^{-1})\ln N \\
&\times \|\partial_v^r(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r,\beta},\mathbb{R}}\big)\Big) + c\int_0^t (\|vW(\tau)\|_\infty + d(W(t),\beta)+2)\|\widetilde{W}_N(\tau))\|^2_{\mathbb{R}}d\tau,
\end{aligned} \tag{4.7}$$

where $\lambda = 1$ for $w_N(0) = I_{N,0,\beta,\mathbb{R}}W_0$, and $\lambda = 0$ for $w_N(0) = P^1_{N,\beta,\mathbb{R}}W_0$. We shall use the following lemma (cf. Lemma 3.1 of [11]).

Lemma 3. *Assume that a) the constants $b_1 > 0, b_2 \ge 0, b_3 \ge 0$ and $d \ge 0$,*
b) $Z(t)$ and $A(t)$ are non-negative functions of t,
c) $d \le b_1^2/b_2^2 e^{-b_3 t_1}$ for $t_1 > 0$,
d) for all $t \le t_1$, $Z(t) + \int_0^t (b_1 - b_2 Z^{\frac{1}{2}}(\tau))A(\tau)d\tau \le d + b_3\int_0^t Z(\tau)d\tau$.
Then, for all $t \le t_1$,

$$Z(t) \le de^{b_3 t}.$$

Let $\||vW(t)|\|_\infty = \sup_{0\le t\le T}\|vW(t)\|_{1,\mathbb{R}}$. We now take in Lemma 4, $b_1 = \frac{1}{2}$, $b_2 = c\beta^{-1}(\beta N)^{\frac{1}{2}}(N+1), b_3 = c(\||vW(t)|\|^2_\infty + d(W(t),\beta)+2)$, and

$$\begin{aligned}
d &= c(\beta N)^{1-r}\Big(\int_0^t \mathcal{A}_{r,\beta}(W(\tau))d\tau + \lambda\big((\beta+\beta^{-2})^2\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r+1,\beta},\mathbb{R}} \\
&+ \beta^{-1}\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r-1,\beta},\mathbb{R}} + (1+\beta^{-1})\ln N\|\partial_v^r(e^{\frac{1}{2}\beta|v|}W_0)\|^2_{\omega_{r,\beta},\mathbb{R}}\big)\Big),
\end{aligned}$$

$$Z(t) = \|\widetilde{W}_N(t)\|^2_{\mathbb{R}} + \frac{1}{2}\int_0^t \|\widetilde{W}_N(\tau)\|^2_{1,\mathbb{R}}d\tau,\ A(t) = \|\partial_v\widetilde{W}_N(\tau)\|^2_{\mathbb{R}} + \|\widetilde{W}_N(\tau)\|^2_{\mathbb{R}}.$$

Moreover, if $r > 1$ and $\|\mathcal{A}_{r,\beta}(W)\|_{L^1(0,T)} < \infty$, then $d \to 0$ as $N \to \infty$. Therefore, applying Lemma 4 to (4.7), we obtain that for any $T > 0$,

$$\begin{aligned}
&\left\|\widetilde{W}_N(t)\right\|^2_{\mathbb{R}} + \frac{1}{2}\int_0^t \left\|\widetilde{W}_N(\tau))\right\|^2_{1,\mathbb{R}} d\tau \le ce^{b_3 T}(\beta N)^{1-r} \\
&\quad \times (\lambda((\beta+\beta^{-2})^2 \left\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r+1,\beta},\mathbb{R}} + \beta^{-1}\left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r-1,\beta},\mathbb{R}} \\
&\quad + (1+\beta^{-1})\ln N \left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r,\beta},\mathbb{R}}) + \int_0^t \mathcal{A}_{r,\beta}(W(\tau))d\tau). \qquad (4.8)
\end{aligned}$$

Finally, a combination of (2.12) and (4.8) leads to the following result.

Theorem 1. *For $0 \le t \le T$, we have*

$$\begin{aligned}
&\left\|W(t) - w_N(t)\right\|^2_{\mathbb{R}} + \frac{1}{2}\int_0^t (\|\partial_v(W(\tau) - w_N(\tau))\|^2_{\mathbb{R}} + \|W(\tau) - w_N(\tau)\|^2_{\mathbb{R}})d\tau \\
&\le c(\beta N)^{1-r}\Big(e^{b_3 T}\big(\lambda\big((\beta+\beta^{-2})^2\left\|\partial_v^{r+1}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r+1,\beta},\mathbb{R}} \\
&+ \beta^{-1}\left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r-1,\beta},\mathbb{R}} + (1+\beta^{-1})\ln N\left\|\partial_v^{r}(e^{\frac{1}{2}\beta|v|}W_0)\right\|^2_{\omega_{r,\beta},\mathbb{R}}) \\
&+ \int_0^t \mathcal{A}_{r,\beta}(W(\tau))d\tau\big) + \left\|\partial_v^{r+1}(e^{\frac{1}{2}\beta v}W(t))\right\|^2_{\omega_{r+1,\beta},\mathbb{R}}\Big),
\end{aligned}$$

provided that the norms appearing in the previous statements are finite.

4.2 Stability analysis

We now consider the stability of scheme (3.2), which might be of the generalized stability as described in [10]. Suppose that W_0 has the errors $\tilde{W}_0$. They induce the error of w_N denoted by $\tilde{w}_N$. Then, we obtain from (3.2) that for all $\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R})$ satisfy

$$\begin{cases} (\partial_t \tilde{w}_N(t), \phi)_{\mathbb{R}} + (v\tilde{w}_N(t), \partial_v\phi)_{\mathbb{R}} + (\partial_v\tilde{w}_N(t), \partial_v\phi)_{\mathbb{R}} \\ +k(v(2w_N(t)\tilde{w}_N(t) + \tilde{w}_N^2(t)), \partial_v\phi)_{N,\mathbb{R}} = 0, \quad \forall\phi \in \mathcal{Q}_{N,\beta}(\mathbb{R}), \quad 0 < t \le T, \\ \tilde{w}_N(0) = \tilde{w}_0. \end{cases} \qquad (4.9)$$

Taking $\phi = 2\tilde{w}_N(t)$ in (4.9), we derive that for $0 < t \le T$,

$$\begin{aligned}
&\partial_t \|\widetilde{w}_N(t)\|^2_{\mathbb{R}} + 2\|\partial_v\widetilde{w}_N(t)\|^2_{\mathbb{R}} \\
&= -2k(v(2w_N(t)\tilde{w}_N(t) + \tilde{w}_N^2(t)), \partial_v\tilde{w}_N(t))_{N,\mathbb{R}} + \|\widetilde{w}_N(t)\|^2_{\mathbb{R}}.
\end{aligned} \qquad (4.10)$$

Thanks to Cauchy inequality, (2.10), (2.16) with $q = 2$ and Proposition 2, it yields that

$$\begin{aligned}
&|-2k(v(2w_N(t)\tilde{w}_N(t) + \tilde{w}_N^2(t)), \partial_v\tilde{w}_N(t))_{N,\mathbb{R}}| \\
&= |2(2vw_N(t)\tilde{w}_N(t), \partial_v\tilde{w}_N(t))_{N,\mathbb{R}} + (v\tilde{w}_N^2(t), \partial_v\tilde{w}_N(t))_{N,\mathbb{R}}| \\
&\le c(\beta N)^{\frac{1}{2}}(N+1)\|\widetilde{w}_N(t)\|_{\mathbb{R}}(\|\widetilde{w}_N(t))\|^2_{\mathbb{R}} + \|\partial_v\widetilde{w}_N(t)\|^2_{\mathbb{R}}) \\
&\quad + \|vw_N(t)\|^2_{L^\infty(\mathbb{R})}\|\widetilde{w}_N(t)\|^2_{\mathbb{R}} + \|\partial_v\widetilde{w}_N(t)\|^2_{\mathbb{R}} \\
&\le c(\beta N)^{\frac{1}{2}}(N+1)\|\widetilde{w}_N(t)\|_{\mathbb{R}}(\|\widetilde{w}_N(t))\|^2_{\mathbb{R}} + \|\partial_v\widetilde{w}_N(t)\|^2_{\mathbb{R}}) \\
&\quad + \|vw_N(t)\|^2_{1,\mathbb{R}}\|\widetilde{w}_N(t)\|^2_{\mathbb{R}} + \|\partial_v\widetilde{w}_N(t)\|^2_{\mathbb{R}}.
\end{aligned}$$

Substituting above inequality into (4.10), we deduce that

$$
\begin{aligned}
&\partial_t \|\widetilde{w}_N(t)\|_{\mathbb{R}}^2 + \|\partial_v \widetilde{w}_N(t)\|_{\mathbb{R}}^2 + \|\widetilde{w}_N(t)\|_{\mathbb{R}}^2 \\
&\leq c(\beta N)^{\frac{1}{2}} (N+1) \|\widetilde{w}_N(t)\|_{\mathbb{R}} (\|\widetilde{w}_N(t))\|_{\mathbb{R}}^2 + \|\partial_v \widetilde{w}_N(t)\|_{\mathbb{R}}^2) \\
&\quad + (\|v w_N(t)\|_{1,\mathbb{R}}^2 + 2) \|\widetilde{w}_N(t)\|_{\mathbb{R}}^2 .
\end{aligned} \tag{4.11}
$$

Integrating the inequality (4.11) from 0 to t with respect to t, we obtain that

$$
\begin{aligned}
&\|\tilde{w}_N(t)\|_{\mathbb{R}}^2 + \int_0^t \big(1 - c(\beta N)^{\frac{1}{2}} (N{+}1) \|\widetilde{w}_N(t)\|_{\mathbb{R}} \big)(\|\widetilde{w}_N(t))\|_{\mathbb{R}}^2 + \|\partial_v \widetilde{w}_N(t)\|_{\mathbb{R}}^2) d\xi \\
&\quad \leq \|\tilde{w}_0\|_{\mathbb{R}}^2 + \int_0^t (\|v w_N(t)\|_{1,\mathbb{R}}^2 + 2) \|\widetilde{w}_N(\xi)\|_{\mathbb{R}}^2 \, d\xi.
\end{aligned}
$$

Let $Z(t)$ be the same as before (see the page 16 of the paper). We take in Lemma 4,

$$
d = \|\tilde{w}_0\|_{\mathbb{R}}^2, \quad b_1 = 1/2, \; b_2 = c(\beta N)^{\frac{1}{2}} \text{ and } b_3 = \|v w_N(t)\|_{1,\mathbb{R}}^2 + 2.
$$

Finally, applying Lemma 4, we get the following result of stability.

Theorem 2. *Suppose that*

$$
\|\tilde{w}_0\|_{\mathbb{R}}^2 \leq \frac{1}{c(\beta N)(N+1)^2} \exp(-(\|v w_N(t)\|_{1,\mathbb{R}}^2 + 2)T),
$$

then for all $0 \leq t \leq T$,

$$
\|\tilde{w}_N(t)\|_{\mathbb{R}}^2 + \frac{1}{2} \int_0^t \|\tilde{w}_N(\xi)\|_{\mathbb{R}}^2 \, d\xi \leq \|\tilde{w}_0\|_{\mathbb{R}}^2 \exp(C_1(w_N(t))T),
$$

where $C_1(w_N(t))$ *depends on* $\|v w_N(t)\|_{1,\mathbb{R}}^2$.

5 Conclusions

In this paper, we developed the composite generalized Laguerre pseudospectral method for the nonlinear Fokker-Planck equation on the whole line, which is distinguished from the methods as mentioned in the references in Section 1. The numerical results demonstrated spectral accuracy in space, and well confirmed the theoretical analysis.

The main advantages of the proposed approach are as follows:

- By using different generalized Laguerre interpolation approximations on different subdomains, we could deal with non-standard types of PDEs on the whole line properly. This trick also simplifies actual computations, especially for large modes N.

- With the aid of composite generalized Laguerre interpolation approximations coupled with domain decomposition, we could exactly match the numerical solutions on the common boundary $v = 0$ of adjacent subdomains $\mathbb{R}^+$ and $\mathbb{R}^-$, and the singularities of coefficients appearing in the underlying differential equations. Consequently, we could deal with the nonlinear Fokker-Planck equation on the whole domain properly.

- The adjustable parameter β involved in the generalized Laguerre interpolation approximations provides flexibility to match the asymptotic behaviors of the exact solutions as $|x| \to \infty$.

We can apply the main idea and techniques developed in this paper to many other nonlinear problems of multiple dimensions. In particular, the results on interpolation approximations are very appropriate for various pseudospectral methods with domain decomposition for to solve partial differential equations defined on unbounded domains or exterior problems. Also, we will extend the idea and technique of this article to the partial differential equations with variable coefficients [23], and report relevant results in the near future.

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