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Compensation Problem in Linear Fractional Order Disturbed Systems

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Abstract. In this paper, we study fractional-order linear, finite-dimensional disturbed systems. The fundamental objective of this work is to study the remediability or compensation problem in linear fractional-order time-invariant perturbed systems. The remediability was introduced with the aim of finding an appropriate control that steers the output of the perturbed system towards normal observation at the final moment. We begin first by giving some characterizations of compensation, and then we prove that a rank condition is sufficient to assure the remediability of our system. The relationship between controllability and compensation is also given, and we provide some examples to illustrate our results.

Keywords: fractional order, disturbed systems, controllability, remediability, observation.

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1 Introduction

The perturbations are fundamentally errors and faults in the particular computations and can cause a dynamic system to sustain considerable harm (infections, pollution, etc.) in various domains of physics, chemistry, and biology. Disturbed systems have continued to play an important role in recent years.

Undefined perturbations are dissected by observation, and various studies have been dedicated to their identification and reconstruction according to observation (see [1, 2, 4] and [6]).
However, it is not only needed for the detection of disturbances in the system, but it is also necessary to intervene with adequate control to reduce the effects of the disturbances and thus regularize their impact on the system. The notion of remediability is introduced in the objective to find a suitable control that ensures the compensation of the perturbations by reducing them, and this is done by steering the observation of the perturbed system back to its natural state in the absence of disturbance.
The notion of remediability is studied and treated initially for a class of parabolic systems with a finite time horizon, then for the asymptotic case, discrete systems, hyperbolic systems, and regional cases [1,10]. In [9], the gradient remediability of perturbed parabolic systems and the report on gradient controllability are studied.
The regional compensation is studied for a certain distributed nonlinear system; the fixed point theorem was used to solve this problem. L. Afifi et al. [2] have dealt with remediability for a type of localized linear system. The study of remediability for distributed systems with delays has been the subject of several works [10]. Thus, recently [7], the problem of compensation with minimal energy for a class of perturbed linear systems at varied times has been studied.
A number of researchers have discussed the controllability and observability of fractional-order differential systems. I. Ahmad et al. [3] give for a fractional order delay dynamic system of the implicit type some results of controllability and observability. In [8], M. Mohan Raja et al. are investigating a note on results for existence and controllability for fractional integrodifferentials. The approximate controllability of fractional nonlinear systems is studied for semilinear fractional differential systems.
For finite-dimensional linear fractional-order systems, the remediability has not yet been processed. The objective of this work is to give some characteristics of compensation for linear fractional-order systems. We give some conditions for the remediability of finite-dimensional linear fractional-order systems, and we discuss the assumptions with an appropriate control operator to remove the impact of the disturbance. A comparison between the controllability and the remediability is given, and we show that if the system is controllable, then the remediability is verified, but the inverse is not true. To illustrate our work, some examples are presented.
The organization of our paper is defined as follows: in Section 2, we introduce a model of perturbed fractional order systems and give the problem statement. Then, we determine and describe the controllability and remediability. In Section 3, we give some properties for the characterization of the remediability of the fractional-order system and some examples to confirm the procured results. In Section 4, we discuss the connection between the notions of controllability and remediability. Finally, a conclusion is given in Section 5.

2 Problem statement

Let's consider the linear fractional order control systems given by

$$\begin{cases} {}_0^c\mathfrak{D}_\theta^\lambda x(\theta) = \mathcal{A}x(\theta) + \mathfrak{B}u(\theta) + g(\theta),\ 0 < \theta < \tau\ ;\ 0 < \lambda < 1, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$, $\mathfrak{B} \in \mathcal{M}_{n,p}(\mathbb{R})$, $u \in L^2(0,\tau;\mathbb{R}^p)$, $g \in L^2(0,\tau;\mathbb{R}^n)$ and ${}_0^c\mathfrak{D}_\theta^\lambda$ denotes the Caputo fractional order derivative, where

$${}_0^c\mathfrak{D}_\theta^\lambda x(\theta) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \displaystyle\int_0^\theta (\theta - r)^{-\lambda} \dot{x}(r)dr, & 0 < \lambda < 1, \\ \dot{x}(\theta), & \lambda = 1, \end{cases}$$

where Γ is Gamma function. The corresponding output is given by

$$y(\theta) = \mathfrak{C}x(\theta), \forall \theta \in]0, \tau[\tag{2.2}$$

with $\mathfrak{C} \in \mathcal{M}_{q,n}(\mathbb{R})$. We have

$$x(\theta) = \Psi_0(\theta)x_0 + H_\theta^\lambda u + G_\theta^\lambda g,$$

where

$$\Psi_0(\theta) = \sum_{m=0}^{\infty} \frac{\mathcal{A}^m \theta^{m\lambda}}{\Gamma(m\lambda + 1)}.$$

Then,

$$y(\theta) = \mathfrak{C}\Psi_0(\theta)x_0 + \mathfrak{C}H_\theta^\lambda u + \mathfrak{C}G_\theta^\lambda g,$$

where H_θ^λ and G_θ^λ are the following operators given by

$$\begin{aligned} H_\theta^\lambda : L^2(0,\theta;\mathbb{R}^p) &\longrightarrow \mathbb{R}^n, \\ u &\longrightarrow \int_0^\theta \Psi(\theta - r)\mathfrak{B}u(r)dr \end{aligned}$$

and

$$\begin{aligned} G_\theta^\lambda : L^2(0,\theta;\mathbb{R}^n) &\longrightarrow \mathbb{R}^n, \\ g &\longrightarrow \int_0^\theta \Psi(\theta - r)g(r)dr \end{aligned}$$

with

$$\Psi(\theta) = \sum_{m=0}^{\infty} \frac{\mathcal{A}^m \theta^{(m+1)\lambda - 1}}{\Gamma[(m+1)\lambda]}.$$

Let us define that

$\mathcal{R}(.)$ denotes the range of a map.

$\mathcal{N}(.)$ denotes the null space of a map.

We thereafter define controllability and its characterization for linear fractional order system.

DEFINITION 1. The system

$$\begin{cases} {}_0^c\mathfrak{D}_\theta^\lambda x(\theta) = \mathcal{A}x(\theta) + \mathfrak{B}u(\theta), \quad 0 < \theta < \tau \; ; \; 0 < \lambda < 1, \\ x(0) = x_0 \end{cases} \tag{2.3}$$

is controllable on $[0, \tau]$, if for every $(x_0, x_d) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $u \in L^2(0, \tau; \mathbb{R}^p)$ such that the solution $x \in C^0(0, \tau; \mathbb{R}^n)$ of the system (2.3) satisfies

$$x(\tau) = x_d.$$

If and only if

$$\mathcal{R}(H_\tau^\lambda) = \mathbb{R}^n,$$

also, the matrix

$$\Delta^\lambda(\tau) = \int_0^\tau \Psi(\tau - r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau - r)^*(\tau - r)^{2(1-\lambda)}dr$$

is invertible or the famous Kalman rank condition for controllability is given by

$$rank \begin{pmatrix} \mathfrak{B} & \mathcal{A}\mathfrak{B} & \ldots & \mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = n.$$

In the case where the disturbance and control are $g = 0$ and $u = 0$, the corresponding output is given by

$$y_{0,0}(\theta) = \mathfrak{C}\Psi_0(\theta)x_0$$

and if the system is perturbed by disturbance g, the observation becomes

$$\begin{array}{rcl} y_{0,g}(\theta) & = & \mathfrak{C}\Psi_0(\theta)x_0 + \int_0^\theta \mathfrak{C}\Psi(\theta - r)g(r)dr \\ & \neq & \mathfrak{C}\Psi_0(\theta)x_0. \end{array}$$

Then, we insert a control operator $\mathfrak{B}u$ for suppressing at final time τ the impact of this disturbance, i.e., $y_{u,g}(\tau) = y_{0,0}(\tau)$.

DEFINITION 2. The system (2.1) with the output function (2.2), or (2.1)+ (2.2) is remediable on $[0, \tau]$, if for any $g \in L^2(0, \tau; \mathbb{R}^n)$, there is a control $u \in L^2(0, \tau; \mathbb{R}^p)$ such that

$$\mathfrak{C}H_\tau^\lambda u + \mathfrak{C}G_\tau^\lambda g = 0.$$

3 Characterization results

We define the following properties findings.

Proposition 1. *The following characteristics are similar*

i *(2.1)+ (2.2) is remediable on* $[0, \tau]$;

ii $\mathcal{R}(\mathfrak{C}G_\tau^\lambda) \subset \mathcal{R}(\mathfrak{C}H_\tau^\lambda)$;

iii $\mathcal{R}(\mathfrak{C}H_\tau^\lambda) = \mathcal{R}(\mathfrak{C})$;

iv $\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) = \mathcal{N}(G_\tau^{\lambda^*}\mathfrak{C}^*)$;

v $\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) = (\mathcal{R}(\mathfrak{C}))^\perp$;

vi $\mathcal{N}(\mathfrak{B}^* G_\tau^{\lambda^*}\mathfrak{C}^*) = \mathcal{N}(G_\tau^{\lambda^*}\mathfrak{C}^*)$;

vii *There exists* $\gamma > 0$ *such that for every* $\omega \in \mathbb{R}^q$*, we have*

$$\|\Psi(\tau - .)^* \mathfrak{C}^* \omega\|_{L^2(0,\tau;\mathbb{R}^n)} \leq \gamma \ \|\mathfrak{B}^* \Psi(\tau - .)^* \mathfrak{C}^* \omega\|_{L^2(0,\tau;\mathbb{R}^p)} . \qquad (3.1)$$

Proof. Determined from Definition 2, by considering the orthogonal, the fact that

$$\begin{aligned}\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) =& \mathcal{N}(\mathfrak{B}^*\Psi(\tau - .)^*\mathfrak{C}^*),\\ \mathcal{N}(G_\tau^{\lambda^*}\mathfrak{C}^*) =& \mathcal{N}(\Psi(\tau - .)^*\mathfrak{C}^*),\end{aligned}$$

and also the result of R. F. Curtain [5]. □

Lemma 1. *Let* $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z}$ *be Banach reflexive spaces,* $\mathcal{P} \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ *and* $\mathcal{Q} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$*. We have*

$$\mathcal{R}(\mathcal{P}) \subset \mathcal{R}(\mathcal{Q})$$

if and only if

$$\exists \gamma > 0, \quad \forall z^* \in \mathcal{Z}' \quad / \quad \|\mathcal{P}^* z^*\|_{\mathcal{X}'} \leq \gamma \|\mathcal{Q}^* z^*\|_{\mathcal{Y}'} .$$

We consider now the remediability Gramian of the system (2.1)+(2.2).

Definition 3. Let $q \geq 1$, the remediability Gramian of system (2.1)+(2.2) is the symmetric $q \times q$-matrix defined by

$$\Theta^\lambda(\tau) = \int_0^\tau \mathfrak{C}\Psi(\tau - r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{2(1-\lambda)}dr.$$

Remark 1. We have, for all $\beta \in \mathbb{R}^q$,

$$\beta^*\Theta^\lambda(\tau)\beta = \int_0^\tau \left\|\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{1-\lambda}\beta\right\|^2 dr.$$

Hence, the remediability Gramian $\Theta^\lambda(\tau)$ is a nonnegative symmetric matrix.

We give here after a second characterization of the notion remediability.

Theorem 1. *Let* $\bar{\Theta}^\lambda(\tau) = \Theta^\lambda(\tau)_{|_{\mathcal{R}(\mathfrak{C})}}$*, (2.1)+(2.2) is remediable on* $[0, \tau]$ *if and only if, the matrix* $\bar{\Theta}^\lambda(\tau)$ *is invertible in* $\mathcal{R}(\mathfrak{C})$*.*

Proof. We first assume that $\bar{\Theta}^\lambda(\tau)$ is invertible in $\mathcal{R}(\mathfrak{C})$ and prove that (2.1)+(2.2) is remediable on $[0, \tau]$. Let $u \in L^2(0, \tau; \mathbb{R}^p)$ be defined by

$$u(r) = \mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{2(1-\lambda)}\bar{\Theta}^\lambda(\tau)^{-1}(-\mathfrak{C}G_\tau^\lambda g),$$

for $r \in [0, \tau]$. Then,

$$y(\tau) = \mathfrak{C}\Psi_0(\tau)x_0 + \int_0^\tau \mathfrak{C}\Psi(\tau - r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{2(1-\lambda)}dr\bar{\Theta}^\lambda(\tau)^{-1} \times (-\mathfrak{C}G_\tau^\lambda g) + \mathfrak{C}G_\tau^\lambda g = \mathfrak{C}\Psi_0(\tau)x_0.$$

Hence, (2.1)+ (2.2) is remediable on $[0, \tau]$.
Let us now assume that $\bar{\Theta}^\lambda(\tau)$ is not invertible in $\mathcal{R}(\mathfrak{C})$. Then there exists $\beta \in \mathcal{R}(\mathfrak{C})\backslash\{0\}$ such that $\bar{\Theta}^\lambda(\tau)\beta = 0$. In particular, $\beta^*\bar{\Theta}^\lambda(\tau)\beta = 0$, that is,

$$\int_0^\tau \beta^*\mathfrak{C}\Psi(\tau - r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{2(1-\lambda)}\beta dr = 0. \tag{3.2}$$

But the left hand side of (3.2) is equal to

$$\int_0^\tau \left\|\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*(\tau - r)^{1-\lambda}\beta\right\|^2 dr.$$

Hence, (3.2) implies that

$$\beta^*(\tau - r)^{1-\lambda}\mathfrak{C}\Psi(\tau - r)\mathfrak{B} = 0, \qquad r \in [0, \tau],$$

from which we get

$$\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*\beta = 0,$$

Consequently, (3.1) does not hold. This concludes the proof of Theorem 1. □

We give next an adequate condition confirming the remediability of (2.1)+(2.2) on $[0, \tau]$.

Proposition 2. *If*

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & \dots & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q,$$

then (2.1)+(2.2) is remediable on $[0, \tau]$.

Proof. Let us now give a first proof of Proposition 2. Using Cayley-Hamilton theorem, we have

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & \dots & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q$$

$$\Longleftrightarrow \forall \beta \in \mathbb{R}^q, \begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix}_{(np,q)} \beta = 0 \quad \Longrightarrow \beta = 0 \Longleftrightarrow \mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) = \{0\}.$$

Hence, if $\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) = \{0\}$, then $\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) \subset \mathcal{N}(G_\tau^{\lambda^*}\mathfrak{C}^*)$, that implies (2.1)+ (2.2) is remediable on $[0, \tau]$.
Our second proof. We assume that

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & \dots & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q,$$

and (2.1)+ (2.2) is not remediable on $[0,\tau]$. Then, $\bar{\Theta}^\lambda(\tau)$ is not invertible in $\mathcal{R}(\mathfrak{C})$. That implies there exists $\beta \in \mathcal{R}(\mathfrak{C})\backslash\{0\}$ such that $\bar{\Theta}^\lambda(\tau)\beta = 0$. In particular, $\beta^*\bar{\Theta}^\lambda(\tau)\beta = 0$, that is,

$$\int_0^\tau \beta^*\mathfrak{C}\Psi(\tau-r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau-r)^*\mathfrak{C}^*(\tau-r)^{2(1-\lambda)}\beta dr = 0. \tag{3.3}$$

But the left hand side of (3.3) is equal to

$$\int_0^\tau \left\|\mathfrak{B}^*\Psi(\tau-r)^*\mathfrak{C}^*(\tau-r)^{1-\lambda}\beta\right\|^2 dr.$$

Hence, (3.3) implies that

$$\mathfrak{B}^*\Psi(\tau-r)^*\mathfrak{C}^*(\tau-r)^{1-\lambda}\beta = 0, \qquad r \in [0,\tau],$$

from which we get

$$\sum_{m=0}^{\infty} \frac{\mathfrak{B}^*(\mathcal{A}^*)^m(\tau-r)^{(m+1)\lambda-1}}{\Gamma[(m+1)\lambda]}\mathfrak{C}^*(\tau-r)^{1-\lambda}\beta = 0.$$

One gets

$$\sum_{m=0}^{\infty} \frac{\mathfrak{B}^*(\mathcal{A}^*)^m\mathfrak{C}^*(\tau-r)^{m\lambda}}{\Gamma[(m+1)\lambda]}\beta = 0,$$

which implies

$$\frac{\mathfrak{B}^*\mathfrak{C}^*}{\Gamma[\lambda]}\beta + \frac{\mathfrak{B}^*\mathcal{A}^*\mathfrak{C}^*(\tau-r)^\lambda}{\Gamma[2\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^2\mathfrak{C}^*(\tau-r)^{2\lambda}}{\Gamma[3\lambda]}\beta + ... = 0, \tag{3.4}$$

or putting $r = \tau$ in (3.4), one gets

$$\mathfrak{B}^*\mathfrak{C}^*\beta = 0.$$

The expression (3.4) becomes

$$\frac{\mathfrak{B}^*\mathcal{A}^*\mathfrak{C}^*(\tau-r)^\lambda}{\Gamma[2\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^2\mathfrak{C}^*(\tau-r)^{2\lambda}}{\Gamma[3\lambda]}\beta + ... = 0,$$

that implies

$$(\tau-r)^\lambda\left[\frac{\mathfrak{B}^*\mathcal{A}^*\mathfrak{C}^*}{\Gamma[2\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^2\mathfrak{C}^*(\tau-r)^\lambda}{\Gamma[3\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^3\mathfrak{C}^*(\tau-r)^{2\lambda}}{\Gamma[4\lambda]}\beta + ...\right] = 0.$$

Then,

$$\frac{\mathfrak{B}^*\mathcal{A}^*\mathfrak{C}^*}{\Gamma[2\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^2\mathfrak{C}^*(\tau-r)^\lambda}{\Gamma[3\lambda]}\beta + \frac{\mathfrak{B}^*(\mathcal{A}^*)^3\mathfrak{C}^*(\tau-r)^{2\lambda}}{\Gamma[4\lambda]}\beta + ... = 0,$$

for $r = \tau$, one gets

$$\mathfrak{B}^*\mathcal{A}^*\mathfrak{C}^*\beta = 0.$$

Repeated this procedure $(n-1)$ times, one has

$$\mathfrak{B}^*(\mathcal{A}^*)^{n-1}\mathfrak{C}^*\beta = 0,$$

then,

$$\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix}\beta = 0. \tag{3.5}$$

But $\beta \neq 0$, so (3.5) contradicts the assumption

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q,$$

this concludes our second proof of Proposition 2. □

Let us give the following remarks

Remark 2. i) One can has

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q$$

even if the system (2.3) is not controllable on $[0, \tau]$.
ii) (2.1)+(2.2) can be remediable on $[0, \tau]$ without having

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = q.$$

Following is an example that shows this.

Example 1. i) Let's consider $n = 2$, $p = q = 1$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathfrak{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathfrak{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The following is the controllability matrix by

$$\begin{pmatrix} \mathfrak{B} & \mathcal{A}\mathfrak{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and its rank is $1 < 2$. As a result, the corresponding system is uncontrollable on $[0, \tau]$. On the other hand,

$$\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

its rank is $1 = q$, then (2.1)+(2.2) is remediable on $[0, \tau]$.
ii) Now, for $n = 2$, $p = 1$, $q = 2$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \mathfrak{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$
$$\mathfrak{C} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \quad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

one gets

$$\Psi(\tau - r) = \begin{pmatrix} \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} & \dfrac{(\tau - r)^{2\lambda-1}}{\Gamma(2\lambda)} \\ 0 & \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} \end{pmatrix}.$$

We have

$$\begin{aligned} \Psi(\tau - r)^* \mathfrak{C}^* \omega &= \begin{pmatrix} \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} & 0 \\ \dfrac{(\tau - r)^{2\lambda-1}}{\Gamma(2\lambda)} & \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} (\omega_1 + \omega_2) \end{pmatrix} \end{aligned}$$

and $\mathfrak{B}^* \Psi(\tau - r)^* \mathfrak{C}^* \omega = \dfrac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} (\omega_1 + \omega_2)$, then

$$\|\Psi(\tau - r)^* \mathfrak{C}^* \omega\|^2_{L^2(0,\tau;\mathbb{R}^2)} = \int_0^\tau \left(\frac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} (\omega_1 + \omega_2) \right)^2 dr$$

and

$$\|\mathfrak{B}^* \Psi(\tau - r)^* \mathfrak{C}^* \omega\|^2_{L^2(0,\tau;\mathbb{R})} = \int_0^\tau \left(\frac{(\tau - r)^{\lambda-1}}{\Gamma(\lambda)} (\omega_1 + \omega_2) \right)^2 dr.$$

Hence

$$\|\Psi(\tau - r)^* \mathfrak{C}^* \omega\|_{L^2(0,\tau;\mathbb{R}^2)} \leq \|\mathfrak{B}^* \Psi(\tau - r)^* \mathfrak{C}^* \omega\|_{L^2(0,\tau;\mathbb{R})}$$

with $\gamma = 1$, and then, (2.1)+(2.2) is remediable on $[0, \tau]$, even if

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} \end{pmatrix} = rank \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 1 \neq 2.$$

In the next, we present a rank condition for the remediability.

Theorem 2. *(2.1)+(2.2) is remediable on* $[0, \tau]$ *if and only if*

$$rank \begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & \ldots & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = rank \begin{pmatrix} \mathfrak{C} \end{pmatrix}.$$

Proof. Let us now give a first proof of Theorem 2. Through the properties of Proposition 1, we have (2.1)+(2.2) is remediable on $[0, \tau]$ if and only if

$$\mathcal{N}(H_\tau^{\lambda^*} \mathfrak{C}^*) = \mathcal{N}(G_\tau^{\lambda^*} \mathfrak{C}^*).$$

Then, from Cayley-Hamilton theorem, we deduce that

$$\beta \in \mathcal{N}(H_\tau^{\lambda^*} \mathfrak{C}^*) \Longleftrightarrow \begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix}_{(np,q)} \beta = 0.$$

Since

$$\mathcal{N}\left[(G_\tau^\lambda)^*\mathfrak{C}^*\right] = \mathcal{N}\left(\mathfrak{C}^*\right),$$

then (2.1)+ (2.2) is remediable on $[0,\tau]$ if and only if

$$\mathcal{N}\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix} = \mathcal{N}\left(\mathfrak{C}^*\right)$$

or equivalently

$$\mathcal{R}\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = \mathcal{R}(\mathfrak{C}).$$

Our second proof. We assume that

$$rank\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} \neq rank\begin{pmatrix} \mathfrak{C} \end{pmatrix}$$

and prove that (2.1)+(2.2) is not remediable on $[0,\tau]$. We have

$$rank\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} \neq rank\begin{pmatrix} \mathfrak{C} \end{pmatrix}.$$

Then, there exists $\beta \in \mathbb{R}^q$ such that

$$\beta \in \quad \mathcal{N}\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix} \setminus \mathcal{N}\left(C^*\right),$$

i.e.,

$$\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix}\beta = 0,$$

and

$$\mathfrak{C}^*\beta \neq 0.$$

Using Cayley-Hamilton theorem, one gets

$$\mathfrak{B}^*\Psi(\tau - r)^*\mathfrak{C}^*\beta = 0,$$

which implies $\beta \in \mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*)$ but $\beta \notin \mathcal{N}(\mathfrak{C}^*)$, one gets $\mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*) \neq \mathcal{N}\left(\mathfrak{C}^*\right)$, then (2.1)+(2.2) is not remediable on $[0,\tau]$.
Conversely, we assume that

$$rank\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} & ... & \mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B} \end{pmatrix} = rank\begin{pmatrix} \mathfrak{C} \end{pmatrix}.$$

By duality,

$$\mathcal{N}\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix} = \mathcal{N}(\mathfrak{C}^*).$$

Let $\beta \in \mathcal{N}(H_\tau^{\lambda^*}\mathfrak{C}^*)$, then

$$\begin{pmatrix} (\mathfrak{C}\mathfrak{B})^* \\ (\mathfrak{C}\mathcal{A}\mathfrak{B})^* \\ \vdots \\ (\mathfrak{C}\mathcal{A}^{n-1}\mathfrak{B})^* \end{pmatrix} \beta = 0.$$

Hence

$$\mathcal{N}\left[(G_\tau^\lambda)^*\mathfrak{C}^*\right] = \mathcal{N}\left(\mathfrak{C}^*\right),$$

therefore, (2.1)+(2.2) is remediable on $[0, \tau]$. This concludes our second proof of Theorem 2. □

4 Remediability and controllability

We give in the following an important result for the relation between controllability and remediability

Proposition 3. *i) If the system (2.3) is controllable on* $[0, \tau]$*, then (2.1)+(2.2) is remediable on* $[0, \tau]$*.*

ii) The opposite is not true.

Proof. We suppose that the linear control system (2.3) is controllable on $[0, \tau]$ $\iff \mathcal{R}(H_\tau^\lambda) = \mathbb{R}^n$, then,

$$\mathcal{R}\left(\mathfrak{C}H_\tau^\lambda\right) = \mathcal{R}\left(\mathfrak{C}\right),$$

consequently (2.1)+(2.2) is remediable on $[0, \tau]$.

Counter example: Let's consider the matrix $\mathcal{A}$ defined by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}).$$

We consider the case where $p = q = 1$ and

$$\mathfrak{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathfrak{C} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} \mathfrak{C}\mathfrak{B} & \mathfrak{C}\mathcal{A}\mathfrak{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

its rank is $1 = rank \begin{pmatrix} \mathfrak{C} \end{pmatrix}$, consequently (2.1)+(2.2) is remediable on $[0, \tau]$. On the other hand, the controllability matrix is given by

$$\begin{pmatrix} \mathfrak{B} & \mathcal{A}\mathfrak{B} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and its rank is $1 < 2$. Then (2.3) is not controllable on $[0, \tau]$. □

Remark 3. In the case (2.3) is controllable on $[0,\tau]$, let $\bar{u} \in L^2(0,\tau;\mathbb{R}^p)$ be defined by:

$$\bar{u}(r) = \mathfrak{B}^* \Psi(\tau - r)^* (\tau - r)^{2(1-\lambda)} \Delta^\lambda(\tau)^{-1}(-G^\lambda_\tau f);$$

for $r \in [0,\tau]$ and $\bar{x} \in C^0(0,\tau;\mathbb{R}^n)$ be the solution of the fractional control system

$$\begin{cases} {}^c_0\mathfrak{D}^\lambda_\theta \bar{x}(\theta) = \mathcal{A}\bar{x}(\theta) + \mathfrak{B}\bar{u}(\theta) + f(\theta), \quad 0 < \theta < \tau \; ; \; 0 < \lambda < 1, \\ \bar{x}(0) = x_0. \end{cases} \tag{4.1}$$

Let consider the output equation of system (4.1)

$$\bar{y}(\theta) = \mathfrak{C}\bar{x}(\theta) \; ; \;\; 0 < \theta < \tau,$$

then,

$$\begin{aligned} \bar{x}(\tau) =& \Psi_0(\tau)x_0 + \int_0^\tau \Psi(\tau - r)\mathfrak{B}\mathfrak{B}^*\Psi(\tau - r)^* \\ & (\tau - r)^{2(1-\lambda)} \Delta^\lambda(\tau)^{-1}(-G^\lambda_\tau f)dr + G^\lambda_\tau f = \Psi_0(\tau)x_0. \end{aligned}$$

One has

$$\bar{x}(\tau) - \Psi_0(\tau)x_0 = 0.$$

And we have

$$\bar{x}(\tau) = \Psi_0(\tau)x_0 + H^\lambda_\tau \bar{u} + G^\lambda_\tau f.$$

Consequently,

$$\mathfrak{C}H^\lambda_\tau \bar{u} + \mathfrak{C}G^\lambda_\tau f = 0,$$

then (2.1)+(2.2) is remediable on $[0,\tau]$.

4.1 Numerical simulations

Let us define $\mathcal{A}$, where $n = 2$ by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One has with $\lambda = \frac{1}{2}$

$$\Psi(\tau - \theta) = \begin{pmatrix} \dfrac{(\tau-\theta)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})} & 1 \\ 0 & \dfrac{(\tau-\theta)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})} \end{pmatrix}.$$

We consider the case where $p = 1, q = 2$ and

$$\mathfrak{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \mathfrak{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with the following disturbance

$$f(\theta) = \begin{pmatrix} \Gamma(\frac{1}{2}) \\ 0 \end{pmatrix}.$$

Using Remark 3, one gets, where $\tau = 10$,

$$u(\theta) = -36\tau^{\frac{-3}{2}}(\tau - \theta) + 24\tau^{-1}(\tau - \theta)^{\frac{1}{2}}.$$

We suppose the initial state is null $x_0 = 0$, then $y_{(0,0)} = 0$. And

$$y_{(u,f)}(\theta) = \begin{pmatrix} -18\tau^{\frac{-3}{2}}\theta^2 + 16\tau^{-1}\theta^{\frac{3}{2}} + 2\theta^{\frac{1}{2}} \\ \dfrac{-24\tau^{\frac{-3}{2}}\theta^{\frac{3}{2}}}{\sqrt{\pi}} + \dfrac{24\tau^{-1}\theta}{\sqrt{\pi}} \end{pmatrix},$$

$$y_{(0,f)}(\theta) = \begin{pmatrix} 2\theta^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

We get the following numerical attainment which perform the previous developments.

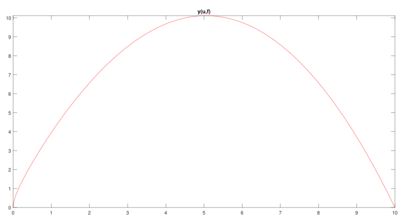

Figure 1. Representation of $y_{(u,f)}$.

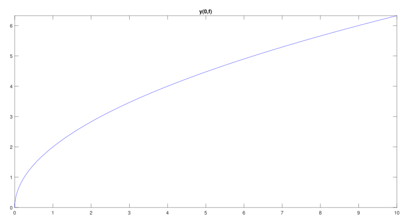

Figure 2. Representation of $y_{(0,f)}$.

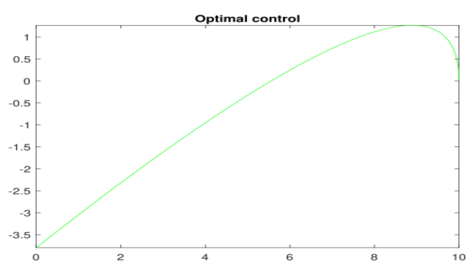

Figure 3. Representation of the optimal control u.

Then, Figures 1 and 2, give an illustration of observations $y_{(u,f)}$ and $y_{(0,f)}$. Figure 1 proves the impact of the control, which steer the output at the normal one without perturbation at time $\tau = 10$, i.e., $y_{(u,f)}(\tau) = y_{(0,0)}(\tau) = 0$ and Figure 3 gives the evolution of optimal control u.

5 Conclusions

In this paper, we define linear fractional-order dynamical systems. The notion Remediability is a significant concept in perturbation theory. In particular, this involves studying the attenuation of the impact of any perturbation via observation. We give some conditions for the remediability of fractional orders, systems, we study the possibility of eliminating the perturbation impact with an appropriate control operator. The relationship between controllability and compensation is also given. And some examples to illustrate our work are given.

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