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A new method to solve multi-objective linear fractional programming problem in fuzzy stochastic environment

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Abstract. Fuzzy stochastic optimization has emerged as an effective approach for dealing with probabilistic and imprecise uncertainties, which makes it useful for problems when data is simultaneously impacted by vagueness and randomness. When these uncertainties involve in decision making problem where, it is required to determine the relative merits between different alternatives, we have often used the fuzzy stochastic fractional programming problem. This paper developed a new approach to derive the acceptable range of objective values for a Multi-objective fuzzy stochastic linear fractional programming problem (MOFSLFPP). In this problem, the fuzzy random variables coefficient is involved as the parameters of the objective function as well as system constraints. The proposed method constructs an expectation model based on the mean of the fuzzy random variable. For the satisfaction level of decision-makers, the level set properties of the fuzzy set are applied in the objective function. The chance-constrained programming method is utilized to transform the MOFSLFPP into its equivalent crisp form. For validation of the proposed methodology, an existing numerical has been solved, and the comparison of the proposed methodology has been discussed with the existing one. Also to demonstrate the practical application of this methodology, an inventory management problem has been discussed.

Keywords: stochastic programming; linear fractional programming; chance constrained programming; expectation model; fuzzy random variables.

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1 Introduction

In many real-life decision-making situations, a specialized area of mathematical optimization known as fractional programming is associated with the optimization of objective functions that are defined as ratios of functions. Martos and Whinston [18], a Hungarian mathematicians developed linear fractional programming problem in 1960s to optimize the ratio-based objectives such as profit/investment, cost/benefit, output/employ, assets/capital, etc. The decision-making problems often depend on human judgment and intuition which makes it challenging to describe parameter values in mathematical models. Although the early fractional programming models were mostly deterministic and regularly encountered difficulties when applied to large-scale, non-linear problems, making it difficult for them to handle uncertain situations. Furthermore, because of the imprecise situations, the uncertainties need to be appropriately managed to ensure that the established mathematical model retains the uncertainty in determining the solution. Stochastic programming (SP) and the fuzzy mathematical programming (FMP) are the two major approaches for encountering uncertainties in decision-making problems. SP is an extended form of mathematical programming that discusses decision problems whose coefficients (input data) may not be known with certainty but may be represented as chances or probabilities. Randomness associated with resource parameters in the constraints is the form of uncertainty that gets major attention in real-world optimization problems. When making decisions in the real world problems, sometimes data can be measured by fuzzy, stochastic, and both fuzzy and stochastic (fuzzy random variable) uncertainty. Fuzzy random variable occurs in such problems where both types of uncertainties, imprecision and unpredictability present.

As the Multi-objective linear fractional programming problem (MOLFPP) may not attain the optimum solutions for each objective simultaneously, So, there is always a scope for developing a methodology to find an alternate way for obtaining a compromise solution that may be better or computational complexity is lesser. In recent decades, there has been a significant increase in research work in developing efficient algorithms and solution techniques for fractional programming problems. Decision-makers may handle the ingrained uncertainties and imprecision associated with real-world decision-making by using MOFSLFPP.

To better explain the importance of using this environment of uncertainty, we can discuss a situation of supply chain optimization in which a company must decide which raw material quantities to order from several suppliers in the best possible amounts. The company wants to keep costs as low as possible without compromising quality of service. On the other hand, the problem involves uncertainty regarding supplier reliability and demand estimations. The supplier's delivery time frames are imprecise because of variable manufacturing rates (fuzzy uncertainty), and the demand is unpredictable because of market fluctuations (random uncertainty). To handle this problem, one can employ a multi-objective fractional programming method with fuzzy random variables involved. The goal is to maximize the cost-to-service ratio, in which the cost (numerator) represents the expected costs for transportation, raw materials, and probable penalties for delivery delays, and the service level (denominator) indicates the dependability of meeting customer requirements. The fuzzy random variables consider the supplier's uncertain supply times and the market's fluctuating demand. We use the expectation model based on fuzzy data to determine each supplier's estimated delivery timings. We use the chance-constrained method to account for the randomness in demand by ensuring that constraints on stock levels and delivery times are fulfilled with a given probability. Additionally, to find the acceptance range of the objective of the company, utilize the level set approach and allow the company to derive different levels of confidence in an uncertain environment.

The objective of this research is to develop a novel approach that handles robust and practical, real-world applications based on MOLFPP that involve both types of uncertain situations like fuzzy (imprecision) and random (stochastic) uncertainty. We have considered a situation of MOFSLFPP in which both the objective functions and subject-to-chance constraints are taken to be fuzzy random variables. The objective of the problem is approximated by the expectation model of stochastic programming. This study employs the level set properties of the fuzzy set, which transform the fuzzy parameters into their crisp form. Using level set properties from fuzzy set theory allows for flexible modeling of decision makers' preferences, providing a range of solutions based on their risk tolerance. The chance-constrained approach transforms probabilistic constraints into their deterministic form, making the model more practical and solvable with conventional optimization techniques.

The research demonstrates the practical relevance and advantages of this innovative method by applying the developed models to specific application areas, such as finance, resource allocation, inventory management problems, transportation problems, or environmental planning. The proposed approach allows decision-makers to adjust their satisfaction level and risk tolerance by providing more flexible ranges of objective value.

This paper includes the following sections: Section 1 provide the brief introduction about the research work. Motivation of the research work is included in Section 2. The literature review of this study is include in Section 3. Some fundamental definitions that help to construct the method are provided in Section 4. The description of the problem and a methodology for solving it are discussed in Section 5, and a stepwise computing procedure is given in Section 6. Section 7 of the manuscript includes a numerical example and a practical real application of inventory management problem, and Section 8 highlights the result obtained. Section 9 conclude the proposed work and in Section 10, some future scope has been given.

2 Motivation

Decision-making in modern situations of real life can often be affected by uncertainties that traditional models are unable to accurately represent. It might be difficult to find robust solutions when parameters in optimization problems are random and imprecise due to imprecise or unknown variables. Optimization methods that can handle such uncertainties are obviously needed as sectors continue to get more connected and dependent on real-time data. Traditional optimization models, especially those that handle probabilistic uncertainty, have limitation in managing the imprecision introduced by human estimations, imprecise predictions, or expert opinions. This gap in traditional methods has grown even wider in the modern era, when decision-making must account for not only random fluctuations (such as market dynamics or supply chain disruptions) but also fuzzy elements (such as estimated demand or predicted weather conditions). The use of fuzzy random variables in optimization models has become an essential approach because of this increasing complexity. Incorporating both randomness and fuzziness, this approach provides a more comprehensive strategy to modelling the uncertainties that decision-makers face in many fields, including supply chain management, inventory management problems, energy, healthcare, finance etc.

This study is particularly required in fields like supply chain management, where worldwide connectivity brings with it unpredictable events (such as extreme weather, economic instability, and unexpected disruptions in transportation networks) that impact both inaccurate forecasting and the unpredictability of transportation. Similar to this, in the healthcare industry, decisions about the distribution of medical supplies or the deployment of people are greatly impacted by both uncertain, random events (like pandemics or sudden outbreaks) and imprecise data (like fluctuating patient needs or uncertain disease progression). Additionally, market randomness (economic swings, stock price fluctuation) and subjective variables (investor satisfaction, economic expectations) both contribute to the constant volatility of financial markets.

This growing requirement to effectively simulate real-world uncertainties, along with the increasing challenges in obtaining reliable solutions, motivated us to focus on multi-objective fuzzy stochastic fractional programming problems. The primary goal of this study is to create an optimization structures which allow decision-makers to navigate complex, uncertain environments while optimizing multiple conflicting objectives expressed as fractional ratios. Essentially, this study fills important gaps in modern decision-making, since organizations want more reliable models to manage two types of uncertainties: imprecise and random. This approach improves the accuracy and robustness of decision-making models in uncertain situations by including fuzzy random variables into optimization. Chance constraints have been introduced to ensure that the solutions obtained may be used in real-world situations, covering against potential risks and obtaining the best possible resource allocation. This work is thus highly relevant in addressing the critical needs of modern industries, providing decision-makers with more accurate, reliable, and adaptable tools to solve complex, uncertain optimization problems across a variety of sectors.

3 Literature review

Researchers developed various methods to solve the linear fractional programming problem (LFPP). Charnes and Cooper [8] proposed a variable transformation method that transforms the fractional objective into its linear form with some additional constraints. The principle of maximizing or minimizing ratios in many real-world situations is effectively represented by this framework. Many real-world decision-making problems, such as hospital planning, corporate and financial planning, healthcare and production planning, multi-product inventory management problems and others where multiple rates must be simultaneously optimized, can be solved using multi-objective fractional programming [4]. Using various approaches and methodologies, including the first-order Taylor series, numerous scholars have attempted to convert the MOLFPP into a LFPP in the literatures. Chakraborty and Gupta [6] introduced a different approach for solving MOLFPP that always produce an efficient solution and lowers the complexity of the problem, in this approach, an equivalent multi-objective linear programming problem (MOLPP) has been formulated using the appropriate transformations and the required problem has been solved using a set theoretic approach.

The FMP is effective in handling the situation for the objective function and constraints are imprecise along with decision problems with a fuzzy goal and constraints [2]. Mehra et al. [19] proposed a method to solve the LFPP with fuzzy coefficients, in this approach, they transform the fuzzy programming problem into two non-fuzzy linear fractional programming problems based on the grade of satisfaction of objective function and constraint set. To formulate MOLFPP into its corresponding linear form and create a method of solution, Mishra [22] used the weighting sum method to create a set of non-dominated bi-level LFPP. Singh et al. [29] developed an approach to solve intuitionistic fuzzy LFPP with cost coefficient and resource parameters are taken as triangular intuitionistic fuzzy numbers, by using the concept of component wise optimization, the given problem is transformed into a deterministic MOLFPP, afterthat the fuzzy mathematical programming approach is used to solve the transformed MOLFPP. Recently, Singh et al. [30] discussed an integrated production-transportation problem based on scalarizing fuzzy MOLFPP, apply the Charnes-Cooper transformation method, the problem is reduced into deterministic MOLPP and then the required problem is scalarized by using Gamma-connective and minimum bounded sum operator techniques and get the required solution. Solomon et al. [31] proposed an intuitionistic fuzzy optimization method for solving MOLFPP, using the concept of parametric function, the LFPP is converted into a suitable non-fractional programming problem. To find the best possible solution at which a certain level of satisfying optimality is attain by all the objective functions, termination conditions are imposed on all the objective function by the decision maker. Nayak and Ojha [24] developed a method for solving MOLFPP with fuzzy coefficients present in both objectives and constraints. They determine an acceptable range of objective values for different values of α and β chosen by the decision-makers. Dealing with stochastic fuzzy programming problems is more efficient than dealing with deterministic ones. (Charnes and Cooper, 1959) [7] developed a chance-constraint technique that allowed probabilistic constraints to be satisfied with some probability. Sharma et al. [28] developed an approach for solving a multi-objective bi-level chance-constrained hierarchical optimization problem, the triangular intuitionistic fuzzy number involved in objective function whereas the normally distributed random variables is present in the coefficient of constraints, the concept of component wise optimization, alpha-cut method, chance-constrained programming and interval programming is used to transform and get the crisp multi-objective bi-level linear programming problem, then the required problem is solved by applying the TOPSIS method. Biswas et al. [3] developed an approach to solve fuzzy multi-objective chance constrained programming model for land allocation problems with the help of fuzzy goal programming problem. For the proper utilization of different farming resources and total cultivating land, optimal production of seasonal crops and their related expenditures are considered. Some resource coefficients are associated with the normally distributed fuzzy random variables.

Various model require an appropriate structure to simultaneously address the two types of uncertainty (fuzziness and randomness), which restrict their practical applications. In situations that require decisions like finance and supply chain management, where these uncertainties frequently occur simultaneously, it is becoming increasingly important to integrate fuzzy and random uncertainty, as illustrated by more recent work by Li and Zhang (2021) [34]. Nasseri et al. [23] introduced an approach for solving fuzzy stochastic LFPP where fuzzy random variables involve in the chance constraints and the parameters in the objective function is characterized by triangular fuzzy numbers. Using Zadeh extension principle and chance constrained programming method, the standard problem is transformed into crisp form and solve the required problem by the Fuzzy Mathematical programming approach. Recently Kumar et al. [15] developed an approach for solving multi-objective mixed fuzzy-stochastic optimization problems, in this approach, fuzzy random uncertainty is involve in the chance constraints of the production programming problem, the realisation of the random variable is taken as fuzzy number with gaussian membership function, the chance-constrained programming method and apply the fuzzy programming technique, solve and get the required solutions. Acharya et al. [1] also developed a method for handling MOLFPP involving two parameters of Cauchy distribution.

MOFSLFPP addresses problems with making decisions under uncertainty and imprecision that combines the concepts of fuzzy theory, stochastic optimization, and multi-objective linear fractional programming problems (MOLFPP). Osman et al. [25] considered a "fuzzy goal programming approach" for solving MOFPP involving fuzzy stochastic uncertainty. The approach is based on α- cut method and chance-constrained technique to transform SP into its deterministic form. Mehra & Chandra [19] also developed a method based on the fuzzy coefficient to find the $(\alpha - \beta)$ acceptable optimal values for LFPP in fuzzy environment. Kumar and Dutta [16] propose an approach for solving multi-objective linear fractional inventory model of multi-product in fuzzy environment. Fuzzy goal programming approach is used to solve this model. Khalifa et al. [14] considered an application of fuzzy random-based inventory management problem based on multi-objective fractional programming. To handle the fuzzy random parameters involve in objective functions and constraints of the application, they used α-cut approach and classify the required problem into the subproblems with their different criteria.

In many prior studies, the solution space was restricted giving decision-makers a single or limited number of feasible choices without taking into account their level of satisfaction or preferences. Frequently, these models were unable to provide a way to modify the conclusion according to the decision maker's risk tolerance or degree of satisfaction. This research significantly improves practical applicability by handling both fuzzy and random uncertainty and combining the predicted value of fuzzy parameters. It gives decision-makers a more thorough and dependable instrument for making decisions that they may employ in unpredictable situations where outcome optimization requires both fuzziness and unpredictability, such as financial markets or manufacturing systems. The proposed approach provides an acceptable range of objectives that is wider and adaptable to different decision-maker satisfaction levels, giving it a distinct advantage over traditional methods that give limited flexibility to handle uncertainty.

4 Preliminaries

4.1 Linear fractional programming problem (LFPP)

The general form of linear fractional programming problem which initially presented by Dinkelbach [10] in (1967) as follows

$$\text{Maximize/Minimize } f(x) = \frac{ax+\alpha}{cx+\beta}, \quad \text{s.t. } Ax \leq b, \quad x \geq 0,$$

where a, c and $x \in R^n$, A is a coefficient matrix and b is a resource vector, α, β are scalars. It is assumed that $cx+\beta > 0$, $\forall\ x$.

4.2 Division operation of the two fuzzy numbers

Let us assume that $F(R^+)$ and $F(R^{++})$ represent the set of non-negative fuzzy numbers and the set of nonzero positive fuzzy numbers respectively. If $\tilde{r}, \tilde{s} \in F(R^+)$ be two non-negative fuzzy numbers, then by applying the "Interval arithmetic operations for fuzzy numbers" [19], $\tilde{r_\alpha} \cdot \tilde{s_\alpha} = [\tilde{r^l_\alpha} \tilde{s^l_\alpha}, \tilde{r^u_\alpha} \tilde{s^u_\alpha}]$ and if $\tilde{r} \in F(R^+)$ and $\tilde{s} \in F(R^{++})$, then $\frac{\tilde{r_\alpha}}{\tilde{s_\alpha}} = (\frac{\tilde{r^l_\alpha}}{\tilde{s^u_\alpha}}, \frac{\tilde{r^u_\alpha}}{\tilde{s^l_\alpha}}), \forall \alpha \in [0,1]$, where, $\tilde{r^l_\alpha}, \tilde{s^l_\alpha}, \tilde{r^u_\alpha}, \tilde{s^u_\alpha}$ be the lower and upper α-cut of $\tilde{r}$ and $\tilde{s}$.
Let $\tilde{r} = (\underline{r}, r_0, \bar{r})$ and $\tilde{s} = (\underline{s}, s_0, \bar{s})$ be two triangular fuzzy numbers and α-cut of these fuzzy numbers be,

$$\tilde{r_\alpha} = \{\underline{r} + \alpha(r_0 - \underline{r}), \bar{r} - \alpha(\bar{r} - r_0)\} \text{ and } \tilde{s_\alpha} = \{\underline{s} + \alpha(s_0 - \underline{s}), \bar{s} - \alpha(\bar{s} - s_0)\}$$

Then, α level set of the division of these triangular fuzzy numbers becomes

$$\frac{\tilde{r_\alpha}}{\tilde{s_\alpha}} = \left\{ \frac{\underline{r} + \alpha(r_0 - \underline{r})}{\bar{s} - \alpha(\bar{s} - s_0)}, \frac{\bar{r} - \alpha(\bar{r} - r_0)}{\underline{s} + \alpha(s_0 - \underline{s})} \right\}.$$

4.3 Fuzzy random variable (FRV)

Kwakernaak [17] has introduced the term fuzzy random variable (FRV) as a random variable whose parameters are not real but a fuzzy number. Let (Ω, F, P) be a probability space and $\mathcal{F}(R)$ denote the set of all fuzzy numbers in R. Then the fuzzy random variable is a mapping from probability space to the set of all fuzzy numbers. This concept has been developed by numerous researchers, including Puri et al. [26] and others, based on various measurability criteria. A continuous probability distribution's density function may contain unknown parameters. Buckely [5] created a fuzzy probability density function for a continuous random variable by expressing these uncertainties as fuzzy numbers.

4.4 Stochastic programming problem

Stochastic programming (SP) is an efficient optimization approach developed to handle situations in which unpredictable conditions are explicitly represented by probabilistic information. Dantzig [9] introduced stochastic programming in 1955 with their seminal publication "Linear Programming under Uncertainty". The primary advancement of SP is how it uses probability theory and optimization approaches to deal with uncertainty. Decision-makers can assess and analyze the risk associated with various decisions using probabilistic distributions. SP is a widely used method for solving optimization problems under uncertainty. These problems occur when a decision-maker considers a problem that is not completely known, such as unpredictable demand, or weather conditions. This uncertain information is represented by a random distribution based on past data or expert opinion. Doshi and Trivedi [11] provide an approach for solving a stochastic programming problem with some parameters that follow uniform distribution, present in the objective function while other coefficients follow continuous probability distributions with known mean and variance involved in constraint. Numerous real-world applications involving decision-making under uncertainty make use of stochastic programming. It optimizes portfolio allocation in finance by taking risk and unpredictable market conditions into consideration.

4.5 Chance constrained programming method

Chance-constrained programming, initially proposed by Charnes and Cooper [7] in 1959, is an important approach in stochastic optimization that deals with uncertainty by ensuring constraints can be satisfied with a predetermined probability. Constraints with random parameters can be expressed probabilistically in this framework, providing decision-makers to take into consideration uncertainty while preserving the desired level of reliability. The general form of a chance constraint assures that the probability of the constraint being violated is less than a specified threshold, hence balancing risk and feasibility in the solution. Chance-constrained programming (CCP) is extensively used in supply chain management to ensure that production capabilities or inventory levels correspond to uncertain demand with a high probability while allowing random violations or stockouts within a given risk tolerance [33]. To manage financial risk in portfolio optimization, Chance constraint programming is used. Recent studies, investigate how CCP can regulate the probability that the return on a portfolio will decrease below a specific threshold. CCP makes it possible to construct a robust portfolio that is compatible with an investor's risk tolerance by incorporating stochastic components such asset price fluctuations [32]. The chance-constrained can be written in mathematical form as

$$\text{Minimize } Z = \sum_{j=1}^{n} C_j x_j,$$

such that

$$P(\sum_{j=1}^{n} a_{ij} x_j \leq b_i) \geq p_i, \quad x_j \geq 0, \quad i = 1, 2, \ldots, m, \quad 0 \leq p_i \leq 1,$$

where P denotes the probability up to which constraint will be satisfied with the level of satisfaction p_i.

4.6 Expected values of fuzzy numbers

We can compute the expected interval for each fuzzy number by the method given by Heilpern [12] and Mishra et al. [21] as

$$EI(\tilde{f}_i) = \left[\int_0^1 f_{i\alpha}^l \, d\alpha, \int_0^1 f_{i\alpha}^u \, d\alpha \right].$$

Here, $f_{i\alpha}^l$ and $f_{i\alpha}^u$ are upper and lower bounds of the α-cut for fuzzy number and expected value of the fuzzy number $\widetilde{f_i}$ is defined as,

$$EV(\tilde{f}_i) = \frac{[\int_0^1 f_{i\alpha}^l d\alpha + \int_0^1 f_{i\alpha}^u d\alpha]}{2}.$$

5 Problem formulations and methodology

A fuzzy stochastic multi-objective linear fractional programming (FSMOLFP) model having r number of objectives involving fuzzy random variable coefficients in objective as well as chance constraints, as characterized by Nasseri et al. in (2018) [23] as follows:

$$\text{Max } Z_k = \left\{ \frac{\sum_{j=1}^{n} \widetilde{\bar{f}_{kj}} x_j + \tilde{\bar{r}}_k}{\sum_{j=1}^{n} \widetilde{\bar{g}_{kj}} x_j + \tilde{\bar{s}}_k} \right\} = \frac{\widetilde{\widetilde{f_k(\bar{x})}}}{\widetilde{\widetilde{g_k(\bar{x})}}}, \tag{5.1}$$

s.t.

$$P(\sum_{j=1}^{n} \widetilde{\bar{a}_{ij}} x_j \preccurlyeq \tilde{\bar{b}}_i) \succcurlyeq \gamma_i, \quad x_j \geq 0, \quad i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, r, \quad 0 \leq \gamma_i \leq 1.$$

In the above problem, $\widetilde{\bar{f}_{kj}}, \widetilde{\bar{g}_{kj}}, \tilde{\bar{r}}_k, \tilde{\bar{s}}_k, \widetilde{\bar{a}_{ij}}, \tilde{\bar{b}}_i$ are normally distributed N(0,1) FRVs, whose parameters (mean and variance) are triangular fuzzy number and

γ_i be the satisfying probability level of the chance constraints.

The proposed methodology for solving the problem (5.1) has been discussed in the following section.

6 The fuzzy stochastic multi-objective linear fractional programming algorithm

***Step 1*: Fuzzy expectation model of objective function.**
Here, we have used the expectation model of the stochastic programming problem. Let $\widetilde{E\left(f_{kj}\right)}$, $\widetilde{E\left(g_{kj}\right)}$, $\widetilde{E(r_k)}$ and $\widetilde{E(s_k)}$ be the mean values of FRVs $\widetilde{\bar{f}_{kj}}, \widetilde{\bar{g}_{kj}}, \widetilde{\bar{r}_k}$ and $\widetilde{\bar{s}_k}$ which is to be taken as a triangular fuzzy number.

$$\text{Max } E\left(Z_k\right) = \left\{ \frac{\sum_{j=1}^{n} \widetilde{E\left(f_{kj}\right)} x_j + \widetilde{E(r_k)}}{\sum_{j=1}^{n} \widetilde{E\left(g_{kj}\right)} x_j + \widetilde{E(s_k)}} \right\} = \frac{\widetilde{E(f_k(x))}}{\widetilde{E(g_k(x))}}. \tag{6.1}$$

"A triangular fuzzy number $\widetilde{A}$ can be represented by a triplet of three real numbers as, $\widetilde{A} = (a^l, a, a^u)$. The membership function of the triangular fuzzy number is of the form",

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a^l}{a-a^l}, & x \in [a^l, a], \\ \frac{a^u-x}{a^u-a}, & x \in [a, a^u], \\ 0, & x < a^l \text{ and } x > a^u, \end{cases}$$

where a^l and a^u represent the left and right tolerance value of the fuzzy number $\widetilde{A}$ respectively. Considering the above representation of a triangular fuzzy number associated with the mean and variance of the fuzzy random variable.

$$\widetilde{E\left(f_{kj}\right)} = \left\{f_{kj}^l, f_{kj}, f_{kj}^u\right\}, \widetilde{E\left(g_{kj}\right)} = \left\{g_{kj}^l, g_{kj}, g_{kj}^u\right\}, \widetilde{E(r_k)} = \left\{r_k^l, r_k, r_k^u\right\},$$
$$\widetilde{E(s_k)} = \left\{s_k^l, s_k, s_k^u\right\}.$$

***Step 2*: The equivalent deterministic form of objective function.**
Now, we obtain the required crisp objective functions by using the β level set properties of the fuzzy set for a particular value of β. Since we assume that numerator $\widetilde{f(x)} \in F(R^+)$ and denominator $\widetilde{g(x)} \in F(R^{++})$ and using the division operation of the fuzzy number as discussed in Section (4.2), in maximization type problem, the objective numerator and the denominator parameters would be replaced by upper bound and the lower bound of the expectable range of β-cut respectively as follows: [25]

$$\text{Max}\left(z_k\right)_\beta = \frac{\left(f_{kj}(x)\right)_\beta^u}{\left(g_{kj}(x)\right)_\beta^l} = \frac{\left(f_{k1}\right)_\beta^u x_1 + \left(f_{k2}\right)_\beta^u x_2 + \ldots + \left(f_{kn}\right)_\beta^u x_n + \left(r_k\right)_\beta^u}{\left(g_{k1}\right)_\beta^l x_1 + (\left(g_{k2}\right)_\beta^l x_2 + \ldots + \left(g_{kn}\right)_\beta^l x_n + \left(s_k\right)_\beta^l}. \tag{6.2}$$

***Step 3*: A Method for converting MOLFPP objective to MOLPP objective.** [20, 27]

As we have assumed in Step 2, the denominator $\widetilde{g(x)} \in F(R^{++})$, i.e., $\widetilde{g(x)} > 0, \forall x$. So, the objective of linear fractional programming of the form, $Max\ (z_k) = \frac{f_k(x)}{g_k(x)}$ can be equivalently considered as the linear objective

$$\text{Max}\,(z_k) = \text{Max}[f_k(x) - g_k(x)]. \tag{6.3}$$

The required fractional objective function (6.2) is equivalent to,

$$\begin{aligned}\text{Max}\,(z_k)_\beta =& \left\{(f_{k1})_\beta^u x_1 + (f_{k2})_\beta^u x_2 + (f_{k3})_\beta^u x_3 + \ldots + (f_{kn})_\beta^u x_n + (\ r_k)_\beta^u\right\}\\ &- \left\{(g_{k1})_\beta^l x_1 + ((g_{k2})_\beta^l\ x_2 + (g_{k3})_\beta^l x_3 +\ \ldots + (g_{kn})_\beta^l x_n + (\ s_k)_\beta^l\right\}.\end{aligned} \tag{6.4}$$

***Step 4*: Conversion of the chance-constraint**

We assume that $\widetilde{\widetilde{a_{ij}}}$ and $\widetilde{\widetilde{b_i}}$ are normally distributed fuzzy random variables in problem (5.1). $\widetilde{m_{b_i}}$ and $\widetilde{\sigma^2_{b_i}}$ be the mean and variance of $\widetilde{\widetilde{b_i}}$, $\widetilde{m_{a_{ij}}}$ be the mean of $\widetilde{\widetilde{a_{ij}}}$ and $V_{\widetilde{a}_i}$ be the variance-covariance matrix. Moreover, assume that $\widetilde{\widetilde{a_{ij}}}$ and $\widetilde{\widetilde{b_i}}$ are independent to each other. Thus, the set of fuzzy stochastic constraints of (5.1) can be determined and transformed to their fuzzy equivalents as follows (Hulsurkar et al. [13]):

$$\sum_{j=1}^{n} \widetilde{m_{a_{ij}}} x_j - \varphi^{-1}\left(1-\gamma_i\right)\sqrt{\widetilde{\sigma^2_{b_i}} + x^T V_{a_i} x} \le \widetilde{m_{b_i}}\ , \quad i = 1, 2, \ldots, m.$$

Since we have taken the mean and variance of the fuzzy random variables as triangular fuzzy numbers, the constraint can be written as

$$\begin{aligned}&\sum_{j=1}^{n} (m^l_{a_{ij}}, m_{a_{ij}}, m^u_{a_{ij}}) x_j - \varphi^{-1}\left(1-\gamma_i\right)\sqrt{(\sigma^{2l}_{b_i}, \sigma^2_{b_i}, \sigma^{2u}_{b_i}) + x^T V_{a_i} x}\\ &\qquad \le (m^l_{b_i}, m_{b_i}, m^u_{b_i}).\end{aligned} \tag{6.5}$$

***Step 5*:** For a fixed value of β, Problem (6.4) can be solved with the nonlinear constraint (6.5) for considering different α-cut. Further, expected values of each objective function as the method discussed in (4.6) can be obtained to construct the pay-off matrix for these expected values. From these values, the expected lower and expected upper bound for each objective function can be obtained.

***Step 6*:** Construct "Linear membership function" for expected values of each objective function defined as follows for $\exp z^{(1)}$ by using the bounds obtained in Step 5.

$$\mu_{\exp z^{(1)}}(x) = \begin{cases} 0, & \exp z^{(1)} \le \min\ \exp z^{(1)}, \\ \frac{\exp z^{(1)} - \min\ \exp z^{(1)}}{\max \exp z^{(1)} - \min \exp z^{(1)}}, & \min \exp z^{(1)} \le \exp z^{(1)} \le \max \exp z^{(1)}, \\ 1, & \exp z^{(1)} \ge \max\ \exp z^{(1)}. \end{cases}$$

***Step 7*:** Calculate the "Pareto optimal solution" of the required multi-objective non-linear programming problem (MONLPP), using the fuzzy programming method.
***Step 8*:** Repeat Step 2 to Step 7 for distinct values of β.

Remark 1. Let Ω be the set of feasible solution of (5.1). A feasible solution $x^* \in \Omega$ is said to be an efficient solution of (5.1) if there is no $\bar{x} \in \Omega$ such that $Z_k(\bar{x}) \geq Z_k(x^*), k = 1, 2, \ldots, K$ and $Z_k(\bar{x}) > Z_k(x^*)$ for at least one k.

Theorem 1. *The efficient solution of* (6.3) *subject to the constraint set of* (6.5) *is also an efficient solution of the problem* (5.1)*.*

Proof. Suppose $x^* \in \Omega$ is an efficient solution of (6.3), so there is no $\bar{x} \in \Omega$ such that

$$z_k(\bar{x}) \geq z_k(x^*), k = 1, 2, \ldots, K \text{ and } z_k(\bar{x}) > z_k(x^*) \text{ for at least one k.}$$
$$\implies f_k(\bar{x}) - g_k(\bar{x}) \geq (f_k(x^*) - g_k(x^*)), k = 1, 2, \ldots, K \text{ and}$$
$$(f_k(\bar{x}) - g_k(\bar{x}) > (f_k(x^*) - g_k(x^*)) \text{ for at least one k.}$$

Since, $g_k(\bar{x}) > 0, k = 1, 2, \ldots, K$ as given in Step 3. From the above inequalities we get,
$\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1 \geq \frac{f_k(x^*)}{g_k(\bar{x})} - \frac{g_k(x^*)}{g_k(\bar{x})}, k = 1, 2, \ldots, K$ and $\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1 > \frac{f_k(x^*)}{g_k(\bar{x})} - \frac{g_k(x^*)}{g_k(\bar{x})}$, for at least one k.
We can rewrite the above inequality as

$$\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1 \geq \frac{f_k(x^*)}{g_k(x^*)} \cdot \frac{g_k(x^*)}{g_k(\bar{x})} - \frac{g_k(x^*)}{g_k(\bar{x})}, k = 1, 2, \ldots, K,$$
$$\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1 \geq (\frac{f_k(x^*)}{g_k(x^*)} - 1)\frac{g_k(x^*)}{g_k(\bar{x})}, k = 1, 2, \ldots, K.$$

Similarly, $\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1 > (\frac{f_k(x^*)}{g_k(x^*)} - 1)\frac{g_k(x^*)}{g_k(\bar{x})}$, for at least one k.

We know that when we maximize the fraction $\frac{f(x)}{g(x)}$, this implies, we have to maximize $f(x)$ and minimize $g(x)$. So, in the above inequalities we have minimize $g_k(x)$ and (x^*) is the minimum point.

Therefore, $g_k(x^*) \leq g_k(x), \forall x \in \Omega$. So, $\nexists \bar{x} \in \Omega$ such that $g_k(x^*) \geq g_k(\bar{x}), k = 1, 2, \ldots, K$ and $g_k(x^*) > g_k(\bar{x})$ for at least one k.

Therefore, $\frac{g_k(x^*)}{g_k(\bar{x}} \geq 1, k = 1, 2, \ldots, K$ and $\frac{g_k(x^*)}{g_k(\bar{x}} > 1$ for at least one k.

From above inequality,

$$\left(\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1\right) \geq \left(\frac{f_k(x^*)}{g_k(x^*)} - 1\right)\frac{g_k(x^*)}{g_k(\bar{x})} \geq \left(\frac{f_k(x^*)}{g_k(x^*)} - 1\right)$$

implies $(\frac{f_k(\bar{x})}{g_k(\bar{x})} - 1) \geq (\frac{f_k(x^*)}{g_k(x^*)} - 1)$.

Therefore, there does not exist $\bar{x} \in \Omega$ such that
$(\frac{f_k(\bar{x})}{g_k(\bar{x})}) \geq (\frac{f_k(x^*)}{g_k(x^*)})$ for $k = 1, 2, \ldots, K$ and $(\frac{f_k(\bar{x})}{g_k(\bar{x})}) > (\frac{f_k(x^*)}{g_k(x^*)})$ for at least one k.
Hence, x^* is an efficient solution of (5.1). □

A diagram for a better understanding of the algorithm is shown in Figure 1.

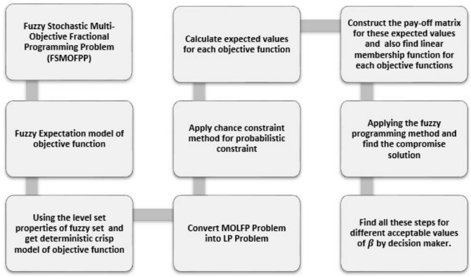

Figure 1. Diagram for algorithm used to solve (FSMOLFPP).

7 Examples

In this section, we have illustrated a numerical example and a practical application of the three-product inventory management problem based on the proposed methodology.

7.1 Numerical example

For numerical illustration to validate the proposed method, we consider a MOLFPP, which is undertaken by Nayak et al. [24] in the fuzzy stochastic environment as

$$
\begin{aligned}
\text{Max } f_1(x) &= \frac{\widetilde{\bar{c}}_{11}x_1 + \widetilde{\bar{c}}_{12}x_2 + \widetilde{\bar{c}}_{13}x_3 + \widetilde{\bar{r}}_{11}}{\widetilde{\bar{d}}_{11}x_1 + \widetilde{\bar{d}}_{12}x_2 + \widetilde{\bar{d}}_{13}x_3 + \widetilde{\bar{s}}_{11}} = \frac{\widetilde{\bar{7}}x_1 + \widetilde{\bar{9}}x_2 + \widetilde{\bar{5}}x_3 + \widetilde{\bar{5}}}{\widetilde{\bar{5}}x_1 + \widetilde{\bar{5}}x_2 + \widetilde{\bar{6}}x_3 + \widetilde{\bar{5}}},\\
\text{Max } f_2(x) &= \frac{\widetilde{\bar{c}}_{21}x_1 + \widetilde{\bar{c}}_{22}x_2 + \widetilde{\bar{c}}_{23}x_3 + \widetilde{\bar{r}}_{21}}{\widetilde{\bar{d}}_{21}x_1 + \widetilde{\bar{d}}_{22}x_2 + \widetilde{\bar{d}}_{23}x_3 + \widetilde{\bar{s}}_{21}} = \frac{\widetilde{\bar{9}}x_1 + \widetilde{\bar{7}}x_2 + \widetilde{\overline{10}}x_3 + \widetilde{\bar{4}}}{\widetilde{\overline{10}}x_1 + \widetilde{\bar{5}}x_2 + \widetilde{\overline{12}}x_3 + \widetilde{\bar{5}}},\\
\text{Max } f_3(x) &= \frac{\widetilde{\bar{c}}_{31}x_1 + \widetilde{\bar{c}}_{32}x_2 + \widetilde{\bar{c}}_{33}x_3 + \widetilde{\bar{r}}_{31}}{\widetilde{\bar{d}}_{31}x_1 + \widetilde{\bar{d}}_{32}x_2 + \widetilde{\bar{d}}_{33}x_3 + \widetilde{\bar{s}}_{31}} = \frac{\widetilde{\bar{5}}x_1 + \widetilde{\bar{7}}x_2 + \widetilde{\overline{11}}x_3 + \widetilde{\bar{5}}}{\widetilde{\bar{3}}x_1 + \widetilde{\bar{8}}x_2 + \widetilde{\bar{9}}x_3 + \widetilde{\bar{3}}},
\end{aligned}
\tag{7.1}
$$

s.t.

$$P[\widetilde{\bar{3}}x_1 + \widetilde{\bar{5}}x_2 + \widetilde{\bar{5}}x_3 \leq \widetilde{\bar{8}}] \geq 0.90, \quad x_1, x_2, x_3 \geq 0.$$

Step 1. In this above problem (7.1), the parameter involved in objective as well as subject to chance constraints is normally distributed fuzzy random variables, so the expectation model is used for objective function as

$$
\begin{aligned}
\text{Max } E(f_1(x)) &= \frac{\widetilde{E(7)}x_1 + \widetilde{E(9)}x_2 + \widetilde{E(5)}x_3 + \widetilde{E(5)}}{\widetilde{E(5)}x_1 + \widetilde{E(5)}x_2 + \widetilde{E(6)}x_3 + \widetilde{E(5)}},\\
\text{Max } E(f_2(x)) &= \frac{\widetilde{E(9)}x_1 + \widetilde{E(7)}x_2 + \widetilde{E(10)}x_3 + \widetilde{E(4)}}{\widetilde{E(10)}x_1 + \widetilde{E(5)}x_2 + \widetilde{E(12)}x_3 + \widetilde{E(5)}},
\end{aligned}
\tag{7.2}
$$

$$\text{Max } E(f_3(x)) = \frac{\widetilde{E(5)}x_1 + \widetilde{E(7)}x_2 + \widetilde{E(11)}x_3 + \widetilde{E(5)}}{\widetilde{E(3)}x_1 + \widetilde{E(8)}x_2 + \widetilde{E(9)}x_3 + \widetilde{E(3)}},$$

s.t.

$$P[\tilde{\tilde{3}}x_1 + \tilde{\tilde{5}}x_2 + \tilde{\tilde{5}}x_3 \leq \tilde{\tilde{8}}] \geq 0.90,$$
$$x_1, x_2, x_3 \geq 0.$$

Expected values of the fuzzy stochastic parameters of the objective functions has been considered to be a triangular fuzzy number as,
$\widetilde{E(7)} = (4,7,11), \widetilde{E(9)} = (6,9,15), \widetilde{E(5)} = (3,5,9), \widetilde{E(5)} = (2,5,7),$
$\widetilde{E(5)} = (2,5,6), \widetilde{E(5)} = (4,5,7), \widetilde{E(6)} = (3,6,8), \widetilde{E(5)} = (2,5,8),$
$\widetilde{E(9)} = (6,9,11), \widetilde{E(7)} = (5,7,9), \widetilde{E(10)} = (8,10,12), \widetilde{E(4)} = (3,4,5),$
$\widetilde{E(10)} = (8,10,12), \widetilde{E(5)} = (3,5,7), \widetilde{E(12)} = (8,12,14), \widetilde{E(5)} = (4,5,8),$
$\widetilde{E(5)} = (3,5,12), \widetilde{E(7)} = (5,7,8), \widetilde{E(11)} = (8,11,15), \widetilde{E(5)} = (2,5,6),$
$\widetilde{E(3)} = (1,3,4), \widetilde{E(8)} = (2,8,14), \widetilde{E(9)} = (3,9,12), \widetilde{E(3)} = (1,3,5).$
The mean and variance of the fuzzy random variables present in constrained are also taken as triangular fuzzy numbers, which are given as

$$\widetilde{m_1(3)} = (1,3,5), \widetilde{m_2(5)} = (2,5,7), \widetilde{m_3(5)} = (3,5,6), \widetilde{m_4(8)} = (6,8,12),$$
$$\widetilde{\sigma_1^2(3)} = (1,2,3), \widetilde{\sigma_2^2(5)} = (1,3,5), \widetilde{\sigma_3^2(5)} = (2,3,4), \widetilde{\sigma_4^2(8)} = (3,5,7).$$

Step 2. Now, using the level set property of fuzzy set which is given in Step 2 of Section 6, and for an appropriate value of β, let us assume that the decision maker accepts an β-level of 0.5. The expectation model of the objective function would be replaced by its deterministic form and problem (7.2) can be rewritten as

$$\begin{aligned} \text{Max } f_1(x) &= \frac{9x_1 + 12x_2 + 7x_3 + 6}{3.5x_1 + 4.5x_2 + 4.5x_3 + 3.5}, \\ \text{Max } f_2(x) &= \frac{10x_1 + 8x_2 + 11x_3 + 4.5}{9x_1 + 4x_2 + 10x_3 + 4.5}, \\ \text{Max } f_3(x) &= \frac{8.5x_1 + 7.5x_2 + 13x_3 + 5.5}{2x_1 + 5x_2 + 6x_3 + 2}, \end{aligned} \tag{7.3}$$

s.t.

$$P[\tilde{3}x_1 + \tilde{5}x_2 + \tilde{5}x_3 \leq \tilde{8}] \geq 0.90,$$
$$x_1, x_2, x_3 \geq 0.$$

Steps 3, 4. By using Equation (6.4) for the objective, and for the constraint set as discussed in (6.5). The problem (7.3) is transform into the MONLPP as

$$\begin{aligned} \text{Max } f_1(x) &= 5.5x_1 + 7.5x_2 + 2.5x_3 + 2.5, \\ \text{Max } f_2(x) &= x_1 + 4x_2 + x_3, \\ \text{Max } f_3(x) &= 6.5x_1 + 2.5x_2 + 7x_3 + 3.5, \end{aligned} \tag{7.4}$$

s.t.

$$(1,3,5)\,x_1+(2,5,7)\,x_2+(3,5,6)\,x_3+1.28$$
$$\times\sqrt{(1,2,3)\,x_1^2+(1,3,5)\,x_2^2+(2,3,4)\,x_3^2+(3,5,7)}\le(6,8,12),\;\; x_1,x_2,x_3\ge 0.$$

Now, by considering one objective at a time of the problem (7.4), for different

Table 1. Solution obtained for Max $f_1(x)$ of (7.4) for different values of α.

α	x_1	x_2	x_3	f_1	f_2	f_3
0	1.4279	0.7676	0	2.3479	1.2202	2.6908
0.2	1.2904	0.5748	0	2.3117	1.1949	2.7873
0.4	1.1922	0.4606	0	2.2838	1.1777	2.8543
0.6	1.1204	0.3849	0	2.2617	1.1649	2.9049
0.8	1.0667	0.3309	0	2.2437	1.1549	2.9455
1	1.0256	0.2902	0	2.2289	1.1468	2.9795
0.8	0.9635	0.357	0	2.2357	1.1639	2.8646
0.6	0.9120	0.4159	0	2.2419	1.1792	2.7731
0.4	0.8677	0.4676	0	2.2474	1.1930	2.6974
0.2	0.8285	0.5142	0	2.2524	1.2058	2.6330
0	0.7930	0.5570	0	2.2570	1.2178	2.5769
			EV	2.2651	1.2093	2.8069

values of α, we get the solution for Max $f_1(x)$ using MATLAB (free online version), and also get the corresponding values of $f_2(x)$ and $f_3(x)$, results obtained has been summarized in Table 1.

Step 5. Now, calculate the expected values for each objective as the procedure discussed in Subsection 4.6.

$$\int_0^1 f^l_{1_\alpha}\,d\alpha=\int_0^{0.2}\frac{0.4695-0.0362\alpha}{0.2}d\alpha+\int_{0.2}^{0.4}\frac{0.4678-0.0279\alpha}{0.2}d\alpha$$
$$+\int_{0.4}^{0.6}\frac{0.4655-0.0221\alpha}{0.2}d\alpha+\int_{0.6}^{0.8}\frac{0.4631-0.018\alpha}{0.2}d\alpha+\int_{0.8}^{1}\frac{0.4605-0.0148\alpha}{0.2}d\alpha$$

gives $\int_0^1 f^l_{1_\alpha}\,d\alpha=2.2772$, similarly, $\int_0^1 f^u_{1_\alpha}\,d\alpha=2.4434$, $EI\,(f_1)=[2.2772,\;2.4434]$. Thus, $EV(f_1)=2.2651, EV(f_2)=1.2093, EV(f_3)=2.8069$.

Step 6. Similarly, calculate the expected values for Max f_2 and Max f_3 for different values of α and construct the pay-off matrix of these expected values as

$$\begin{bmatrix}2.2651 & 1.2093 & 2.8069\\ 2.2645 & 1.4868 & 1.8442\\ 2.2011 & 1.0760 & 3.5354\end{bmatrix}.$$

With the above matrix, the Max and Min expected values for each of the objective functions are as

$$EV(f_1)^{\max}=2.2651,\qquad EV(f_1)^{\min}=2.2011,$$
$$EV(f_2)^{\max}=1.4868,\qquad EV(f_2)^{\min}=1.0760,$$
$$EV(f_3)^{\max}=3.5354,\qquad EV(f_3)^{\min}=1.8442.$$

Step 7. Now, by constructing the linear membership function for the expected values of each objective and by using the fuzzy programming technique, problem (7.4) can be converted to non-linear programming problem (NLPP) as

$$\text{Maximize} \ \ \lambda \quad \text{s.t.}$$

$$\begin{aligned}
&5.5x_1 + \ 7.5x_2 + 2.5x_3 + 2.5 - 0.064\lambda \geq 2.2011, \\
&x_1 + \ 4x_2 + x_3 - 0.4108\lambda \geq 1.0760, \\
&6.5x_1 + \ 2.5x_2 + 7x_3 + 3.5 - 1.6912\lambda \geq 1.8442, \\
&(5-2\alpha)x_1 + \ (7-2\alpha)x_2 + \ (6-\alpha)x_3 \\
&\quad + 1.28\sqrt{\ (3-\alpha)\, x_1^2 + \ (5-2\alpha)\, x_2^2 + (4-\alpha)\, x_3^2 + (7-2\alpha)} \leq (6+2\alpha).
\end{aligned} \tag{7.5}$$

By using MATLAB (online free version), for $\alpha = 0.5$, the NLPP (7.5) has been solved and the solution is obtained as follows

$$\begin{aligned}
&x_1 = 0.1758, \quad x_2 = 0.3661, \quad x_3 = 0.0897, \lambda = 1. \\
&f_1\,(x) = 2.0439, \quad f_2\,(x) = 1.2049, \quad f_3\,(x) = 2.3105.
\end{aligned}$$

Similarly, one can solve the problem (7.5) for distinct values of α.
Step 8. We have solved problem (7.2) for different values of β, and the obtained results are placed in Table 2.

7.2 Application of three-product inventory management problem

The inventory management problem helps organizations manage their inventory effectively while maintaining a balance between costs and profits. In this example, we focus on a three product inventory management problem in which the manager wants to maximise the two objectives within a given organisational perspective. The first objective is to maximize the overall profit from inventory sales. For every unit of inventory, the management wants to optimise the ratio of the profit cost to the back order quantity. In order to ensure effective use of storage resources, the second objective is to minimize the holding cost to total inventory ratio. In order to avoid overstocking, the constraint of the problem include a maximum budget limit for inventory purchases, a maximum storage capacity for products, and an upper limit on the number of orders.
Market price fluctuations, imprecise demand predictions, and varying storage conditions, parameters like selling price, purchasing cost, holding cost and ordering cost exhibit significant uncertainty and imprecision. As consequently, these factors have been represented as fuzzy random variables to allow for the combined impacts of randomness and fuzziness, which provides a more realistic depiction of the decision-making situation. we have consider a three-product inventory management problem as Kumar and Dutta [16].

Table 2. Solution of problem (7.2) for different acceptable values of β.

	α	x_1	x_2	x_3	f_1	f_2	f_3
	0	0.4277	0	0.0570	4.0370	1.3187	7.4982
	0.2	0.5877	0	0.0301	4.2060	1.3224	8.0476
$\beta = 0$	0.5	0.8548	0	0	4.4217	1.3288	8.7651
	0.8	1.1501	0	0	4.5698	1.3371	9.2094
	1	1.3945	0	0	4.6647	1.3420	9.4942
				EV	4.5398	1.3295	8.7629
	0	0.2064	0.1195	0.1141	2.7881	1.2716	4.1324
	0.2	0.3891	0.0897	0.1047	2.9013	1.2548	4.5333
$\beta = 0.2$	0.5	0.7167	0.0382	0.0805	3.0725	1.2351	5.1814
	0.8	1.1501	0	0	3.2790	1.2258	5.9768
	1	1.3945	0	0	3.3448	1.2304	6.1395
				EV	3.0775	1.2430	5.1923
	0	0	0.3524	0	2.0112	1.23855	2.1645
	0.2	0.0478	0.3695	0.0311	2.0259	1.2317	2.1991
$\beta = 0.5$	0.5	0.1758	0.3661	0.0897	2.0439	1.2049	2.3105
	0.8	0.3077	0.3938	0.1430	2.0730	1.1971	2.3754
	1	0.4397	0.4034	0.1827	2.0943	1.1879	2.4426
				EV	2.0497	1.21174	2.2945
	0	0.2341	0.1283	0.0392	1.4002	1.0259	1.8429
	0.2	0.2857	0.1529	0.0607	1.4199	1.0422	1.8215
$\beta = 0.8$	0.5	0.4103	0.1801	0.0923	1.4498	1.0671	1.8198
	0.8	0.5158	0.2523	0.1260	1.4866	1.0927	1.7779
	1	0.6336	0.2788	0.1612	1.5027	1.1048	1.7788
				EV	1.45169	1.0666	1.8079
	0	0.0818	0.0538	0.1864	1.0283	0.8549	1.4637
	0.2	0.0989	0.0600	0.2436	1.0267	0.8593	1.4397
$\beta = 1$	0.5	0.1172	0.0735	0.3492	1.0222	0.8633	1.4037
	0.8	0.1456	0.0860	0.4607	1.0195	0.8677	1.3780
	1	0.1606	0.972	0.5532	1.0163	0.8693	1.3605
				EV	1.0223	0.8627	1.4088

Notations and assumptions

The following nomenclature is used to deal the proposed approach:
k = Number of items, ϕ = Fixed cost per item,
$\tilde{\bar{S}}_k$ = Selling price for the k^{th} item, $\tilde{\bar{P}}_k$ = Purchasing price for k^{th} item,
Q_k = Order quantity for item k, (decision variables),
$\tilde{\bar{R}}_k$= Holding cost for k^{th} item, Y_0= Maximum number of orders placed,
W = Maximum available space for all the items,
$\widetilde{\bar{D}}_k$= Demand per unit time for k^{th}item, B= Maximum available budget

for all items, $\tilde{\bar{w_k}}$ = Space required for per unit of k^{th} item,
$O\tilde{\bar{C}}_k$ = Ordering cost for k^{th} item.

Assumptions

1. The time horizon is considered as infinite.
2. The lead time is taken as zero.
3. The holding cost is constant for each product.
4. Purchase cost is constant for each item, i.e., discount is not available.
5. The demand is inversely proportional to the selling price of each item.

$\widetilde{\bar{D}}_k = \delta(\widetilde{\bar{S}}_k)^{-\theta}$, where, $\delta > 0$ is scaling constant and $\theta > 1$ is price-elasticity parameter.

With the above mentioned assumptions, the multi-objective fuzzy stochastic linear fractional inventory model is formulated as follows:

$$\text{Max } \widetilde{\bar{Z}}_1 = \frac{\sum_{k=1}^{n} (\widetilde{\bar{S}}_k - \widetilde{\bar{P}}_k) Q_k}{\sum_{k=1}^{n} (\delta(\widetilde{\bar{S}}_k)^{-\theta} - Q_k)} = \frac{\text{Total profit}}{\text{Back order quantity}},$$

$$\text{Min } \widetilde{\bar{Z}}_2 = \frac{\sum_{k=1}^{n} \frac{\widetilde{\bar{R}}_k Q_k}{2}}{\sum_{k=1}^{n} Q_k} = \frac{\text{Holding cost}}{\text{Total ordering cost}},$$

s.t.

$$P\{\sum_{k=1}^{n} \widetilde{\bar{P}}_k Q_k \leq B\} \geq \beta \qquad \text{(Chance constraint on total budget)},$$

$$P\{\sum_{k=1}^{n} \widetilde{\bar{w}}_k Q_k \leq W\} \geq \beta \qquad \text{(Chance constraint on storage space)},$$

$$\sum_{k=1}^{n} \left(\widetilde{\bar{S}}_k\right)^{\theta} Q_k \geq \frac{\delta}{Y_0} \qquad \text{(Upper limit on number of order)},$$

$$Q_k \geq \frac{\phi\delta}{\widetilde{\mathrm{O\bar{C}}}_k (\widetilde{\bar{S}}_k)^{\theta}} \qquad \text{(Constraint on ordering cost of each item)},$$

$$Q_k > 0, \forall k = 1, 2, \ldots, n, Y_0 > 0, \delta > 0, \theta > 1, 0 \leq \beta \leq 1.$$

Here, the maximum budget $B = 100,000$, $\delta = 70,000$, $Y_0 = 10$, $\phi = 9$, the maximum available space $W = 200$, $\theta = 1.1$ and the probability level β= 0.5. Since, the parameters involved in the objective function and the constraint of the problem are fuzzy random variables. The fuzzy random coefficient and their expected value is given in Tables 3 and 4 respectively. We have used the expectation model and level set properties as given in Sections 6.1–6.2 and transformed the required problem into the MOLFPP. Further, we have transformed MOLFPP into the deterministic form of MONLPP by using the Sections 6.3–6.5. To solve the MONLPP, we have used the fuzzy programming technique and the required NLPP is formulated as

$$\text{Maximize } \lambda \quad \text{s.t.}$$

$$76Q_1 + 46Q_2 + 66Q_3 - 2773.03\lambda \geq 1747.64,$$
$$2.5Q_1 + 2.5Q_2 + 4.5Q_3 + 84.32\lambda \leq 169.90,$$
$$(360 - 10\alpha)Q_1 + (370 - 10\alpha)Q_2 + (400 - 10\alpha)Q_3$$
$$+ \phi^{-1}(0.5)\sqrt{(110-5\alpha)\,Q_1^2 + (113-5\alpha)\,Q_2^2 + (122-5\alpha)\,Q_3^2} \leq 100,000,$$
$$(4 - \alpha)Q_1 + (6 - \alpha)Q_2 + (5 - \alpha)Q_3$$
$$+ \phi^{-1}(0.5)\sqrt{(2 - \alpha)\,Q_1^2 + (4 - \alpha)\,Q_2^2 + (3 - \alpha)\,Q_3^2} \leq 200,$$
$$768.37Q_1 + 728.22Q_2 + 828.96Q_3 \geq 7000,$$
$$Q_1 \geq 10.24, \quad Q_2 \geq 9.61, \quad Q_3 \geq 7.99.$$

By using MATLAB (online free version), for $\alpha = 0.5$, the NLPP has been solved and the solution is obtained as follows: $Q_1 = 27.76$, $Q_2 = 9.61$, $Q_3 = 7.99$, Max $Z_1 = 13.40$, Min $Z_2 = 2.85$, Total Profit= 3034.28, Total back order quantity = 226.3, Holding cost=129.38 and Total ordered quantity = 45.36.

Table 3. Fuzzy random coefficients for different products.

	$\tilde{\tilde{S}}_k$	$\tilde{\tilde{R}}_k$	$\tilde{\tilde{P}}_k$	$\tilde{\tilde{OC}}_k$	$\tilde{\tilde{w}}_k$
$k = 1$	$\tilde{\tilde{420}}$	$\tilde{\tilde{6}}$	$\tilde{\tilde{350}}$	$\tilde{\tilde{80}}$	$\tilde{\tilde{3}}$
$k = 2$	$\tilde{\tilde{400}}$	$\tilde{\tilde{6}}$	$\tilde{\tilde{360}}$	$\tilde{\tilde{90}}$	$\tilde{\tilde{5}}$
$k = 3$	$\tilde{\tilde{450}}$	$\tilde{\tilde{10}}$	$\tilde{\tilde{390}}$	$\tilde{\tilde{95}}$	$\tilde{\tilde{4}}$

Table 4. Expected value and the variance of fuzzy random variables.

$E(\tilde{\tilde{S}}_1)$	$(400, 420, 440)$	$E(\tilde{\tilde{S}}_2)$	(380,400,420)	$E(\tilde{\tilde{S}}_3)$	(430,450,470)
$E(\tilde{\tilde{P}}_1)$	$(340, 350, 360)$	$E(\tilde{\tilde{P}}_2)$	$(350, 360, 370)$	$E(\tilde{\tilde{P}}_3)$	$(380, 390, 400)$
$E(\tilde{\tilde{R}}_1)$	(4,6,8)	$E(\tilde{\tilde{R}}_2)$	(4,6,8)	$E(\tilde{\tilde{R}}_3)$	(8,10,12)
$E(\tilde{\tilde{w}}_1)$	$(2, 3, 4)$	$E(\tilde{\tilde{w}}_2)$	$(4, 5, 6)$	$E(\tilde{\tilde{w}}_3)$	$(3, 4, 5)$
$E(\tilde{\tilde{OC}}_1)$	(60,80,100)	$E(\tilde{\tilde{OC}}_2)$	$(70, 90, 110)$	$E(\tilde{\tilde{OC}}_3)$	(75,95,115)
$\sigma^2\left(\tilde{\tilde{P}}_1\right)$	(100,105,110)	$\sigma^2\left(\tilde{\tilde{P}}_2\right)$	(103,108,113)	$\sigma^2\left(\tilde{\tilde{P}}_3\right)$	(112,117,122)
$\sigma^2\left(\tilde{\tilde{w}}_1\right)$	(0,1,2)	$\sigma^2\left(\tilde{\tilde{w}}_2\right)$	(2,3,4)	$\sigma^2\left(\tilde{\tilde{w}}_3\right)$	(1,2,3)

8 Result and discussion

8.1 Explanation of the result

By using the different values for α and β, decision makers can choose from a wide range of objective values with effectively manage both types of uncertain situations. Maximizing the acceptable range of β- level from the perspective of decision-makers within the framework of the level set properties of fuzzy set, which is used for conversion of fuzziness into its deterministic form. With the increment of the value of β, the length of the β–cut interval decreases. This decrease in the length results in a corresponding reduction in the coefficients of the fractional objective. Therefore, one can easily verify that with the increased value β, the expected values of each objective are decreased. The same can be easily seen in Figure 2, which comprises the obtained results for different β values with the bar diagram.

The solution to the real application of the three-product inventory management problem obtained using the proposed approach provides more accuracy and robustness to deal with the fuzzy random uncertainty present in practical situations.

The comparison between the proposed method and the existing approach is presented in Table 5. We can see that the range of the objective function given by this method is more adaptable and wide for decision-makers than the existing one. In the existing approach [24], the methodology only considered the uncertainty (fuzzy) and finds an acceptable range of objectives for different values of α and β. In the proposed methodology, both types of uncertainties (fuzzy and stochastic) have been considered simultaneously which allows decision-makers to adapt their wide ranges for acceptable objective values for different β-levels.

Table 5. Comparison of result of proposed method with Nayak and Ojha [24].

	Nayak and Ojha [24]	Proposed method
$f_1(x)$	[0.8519, 2.3265]	[1.0223, 4.5398]
$f_2(x)$	[0.7063, 1.4217]	[0.8627, 1.3295]
$f_3(x)$	[0.8240, 3.8214]	[1.4088, 8.7629]

8.2 Limitations of the study

As, we have restricted the denominator of the fractional objective function, it is taken as a nonzero positive fuzzy number so that level set properties have to be applicable. This is the limitation of the suggested approach. For mathematical feasibility and tractability to be guaranteed, this limitation is required. The denominator may change and even turn negative in some real-life situations. Due to this constraint, the model's capacity to handle situations where the denominator may change is limited, which affects how broadly the solution can be applied.

8.3 Future development and challenges

In the future, this limitation could be overcome by applying advanced methods that provide more flexibility in the denominator. One approach could be to reformulate the objective function or use robust optimization techniques that handle cases where the denominator might vary or become negative. Another approach is to include penalty terms or specific conditions to allow these fluctuations and make the model more applicable to real-life situations without imposing strict non-negativity restrictions.

9 Conclusions

Here, this study developed a novel approach for solving MOFSLFPP which involves FRV with respect to both the objectives and the chance constraint parameters. Using the expectation model for the fuzzy-stochastic objective function, the chance constraint approach and β-level set property of the fuzzy set are utilized to transform MOFSLFPP to its equivalent deterministic-crisp form. The proposed method is used to achieve the maximum compromise solution for the acceptable range of β- level according to the perspective of the decision maker. For validation of this method, we have taken the same numerical problem of Nayak et al. [24] with the changed environment, and find the solution for different values of α. The comparison of the results of the proposed method with the existing method has been placed in Table 5. For easy visualization of the comparison, an area graph has also been given in Figure 3, which clearly shows the superiority of the proposed method over the existing method. To illustrate the effectiveness of the proposed methodology, an inventory management problem has been mathematically formulated and analysed.

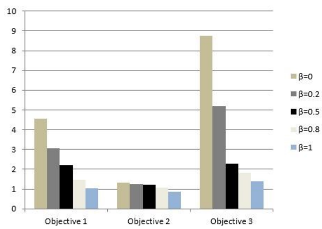

Figure 2. Comparison bar graph for different values of β.

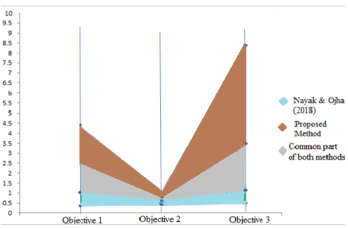

Figure 3. Results obtained from proposed method and existing method.

10 Future scope

In the future, we can apply this approach to solving multi-objective integer fractional decision-making problems in a fuzzy stochastic environment. It can also be applicable in real-life applications such as inventory control optimization, supply chain planning, transportation, assignment problems, etc. Also, the proposed work can be improved by taking nonlinear membership functions for the fuzzy numbers to model realistic problems more efficiently.

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