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Calderón-Zygmund estimates for Schrödinger equations revisited

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Abstract. We establish a global Calderón-Zygmund estimate for a quasilinear elliptic equation with a potential. If the potential has a reverse Hölder property, then the estimate was known in [6]. In this note, we observe that the estimate remains valid when the potential is merely Lebesgue integrable. Our proof is short and elementary.

Keywords: Calderón-Zygmund estimates; quasilinear Schrödinger equation.

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1 Introduction

This paper targets the equation

$$\begin{cases} -\mathrm{div}\mathbf{A}(x,\nabla u) + V\,|u|^{p-2}\,u = -\mathrm{div}(|F|^{p-2}\,F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in which the following *structural conditions* are imposed:

- $n \in \{2,3,4,\ldots\}$, $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ is an open bounded domain that is (δ_0, r_0)-Reifenberg flat and at the same time (δ_0, r_0)-vanishing for some small constants $\delta_0, r_0 > 0$. See Definitions 2 and 3 below.
- $\mathbf{A} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Carathédory function, in the sense that $\mathbf{A}$ is measurable in the first variable and differentiable in the second variable. Moreover, there exist constants $0 < \Lambda_0 \le \Lambda_1 < \infty$ such that

$$\nabla_\xi \mathbf{A}(x,\xi)\,\eta\cdot\eta \ge \Lambda_0\,|\xi|^{p-2}\,|\eta|^2$$

and

$$\big|\mathbf{A}(x,\xi)\big| + |\nabla_\xi \mathbf{A}(x,\xi)|\,|\xi| \le \Lambda_1\,|\xi|^{p-1}$$

for a.e. $x \in \mathbb{R}^n$ and for all $\xi, \eta \in \mathbb{R}^n$.

- $F \in L^q(\Omega, \mathbb{R}^n)$ for some $q > p$.
- $V \in L^\gamma(\Omega)$ with

$$\gamma \in \begin{cases} \left(\dfrac{n}{p}, n\right), & \text{if } p < n, \\ (1,n), & \text{if } p \ge n. \end{cases} \tag{1.2}$$

The aim is to derive a Calderón-Zygmund estimate for a weak solution to (1.1). A weak solution to (1.1) is understood as follows:

Definition 1. A function $u \in W_0^{1,p}(\Omega) \cap L^p(\Omega, V dx)$ is called a weak solution of (1.1) if

$$\int_\Omega \mathbf{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega V\,|u|^{p-2}\,u\,\varphi\,dx = \int_\Omega F\cdot\nabla\varphi\,dx \tag{1.3}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^p(\Omega, V dx)$, where

$$L^p(\Omega, V dx) := \left\{\text{measurable function } g : \Omega \longrightarrow \mathbb{R} : \int_\Omega |g|^p\,V\,dx < \infty\right\}.$$

If the potential V is non-negative and belongs to a reverse Hölder class B^γ, in the sense that

$$\sup \left(\fint_B V\,dx\right)^{-1} \left(\fint_B V^\gamma\,dx\right)^{\frac{1}{\gamma}} < \infty,$$

where γ is given by (1.2) and the supremum is taken over all balls $B \subset \mathbb{R}^n$, then [6, Corollary 2.6] established the global Calderón-Zygmund estimate

$$\|\nabla u\|_{L^q(\Omega)} + \mathbb{1}_{[q<\gamma p]}\,\|V^{\frac{1}{p}}\,u\|_{L^q(\Omega)} \lesssim \|F\|_{L^q(\Omega)} \tag{1.4}$$

for all $p < q < \gamma^*(p-1)$, where

$$\mathbb{1}_{[q<\gamma p]} := \begin{cases} 1, & \text{if } q < \gamma p, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \gamma^* := \begin{cases} \dfrac{n\,\gamma}{n-\gamma}, & \text{if } \gamma < n, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

In this note, we show that the condition $V \in B^\gamma$ can be removed, and yet (1.4) remains valid. In fact, our aforementioned structural conditions require $V \in L^\gamma(\Omega)$ only. Unlike [6], we make no use of the uniform estimate [6, Lemma 3.5] which is crucial in their consideration. Moreover, our proof is short and elementary.

The first regularity estimates of type (1.4) can be traced back to the work [8]. Specifically, [8, Corollary 0.10] asserts that a weak solution u to the Schrödinger equation

$$-\Delta u + V\,u = -\mathrm{div} F \quad \text{in } \mathbb{R}^n$$

satisfies

$$\|\nabla u\|_{L^q(\mathbb{R}^n)} + \mathbb{1}_{[q<2\gamma]}\,\|V^{\frac{1}{2}}\,u\|_{L^q(\mathbb{R}^n)} \lesssim \|F\|_{L^q(\mathbb{R}^n)} \quad \text{for all } q \in \big[(\gamma^*)', \gamma^*\big] \setminus \{\infty\},$$

where $V \in B^\gamma$ with $\gamma \geq \frac{n}{2}$. Moreover, this range for q is optimal (cf. [8, Section 7]).

Further extensions are available in [1,2,3,7] for elliptic equations with discontinuous coefficients and in [4,9,10,11] for parabolic Schrödinger equations.

Before stating our main result, we provide the notions of (δ_0, r_0)-Reifenberg flat and (δ_0, r_0)-vanishing domains required by the structural conditions.

Definition 2. Let $\delta_0 \in (0, \frac{1}{8})$ and $r_0 > 0$. Then Ω is called a (δ_0, r_0)-Reifenberg flat domain if for all $x \in \partial\Omega$ and $r \in (0, r_0]$, there exists a new coordinate system $\{y_1, \dots, y_n\}$ in which x is the origin and

$$B_r(0) \cap \{y_n > \delta_0\, r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta_0\, r\}.$$

Definition 3. Let $\delta_0, r_0 > 0$. We say that $\mathbf{A}$ is (δ_0, r_0)-vanishing if

$$\sup_{\substack{0<r\leq r_0 \\ x\in\mathbb{R}^n}} \fint_{B_r(x)} \Theta\big[\mathbf{A}, B_r(x)\big](y)\, dy \leq \delta_0$$

for all $r \in (0, r_0)$, where

$$\Theta\big[\mathbf{A}, B_r(x)\big](y) := \sup_{0\neq\xi\in\mathbb{R}^n} \frac{|\mathbf{A}(y,\xi) - \mathbf{A}_{B_r(x)}(\xi)|}{|\xi|^{p-1}},$$
$$\mathbf{A}_{B_r(x)}(\xi) := \fint_{B_r(x)} \mathbf{A}(y,\xi)\, dy.$$

With these in mind, the global Calderón-Zygmund estimate is formulated as follows.

Theorem 1. *Assume the structural conditions. Let u be a weak solution to* (1.1)*. Then there exists a constant $\delta_0 = \delta_0(n, \Lambda_0, \Lambda_1, p) > 0$ such that if Ω is (δ_0, r_0)-Reifenberg flat and $\mathbf{A}$ is (δ_0, r_0)-vanishing for some $r_0 \in (0,1)$, then,*

$$\begin{aligned} \|\nabla u\|_{L^q(\Omega)} + \mathbb{1}_{[q<p\gamma]}\,\|V^{\frac{1}{p}}\,u\|_{L^q(\Omega)} &\leq C\left(\frac{\operatorname{diam}(\Omega)}{r_0}\right)^{\frac{n}{p}-\frac{n}{q}} \\ &\quad\times \Big(\|\nabla u\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)}\Big) \end{aligned} \tag{1.6}$$

for all $q \in \big(p, \gamma^*\,(p-1)\big)$, where $C = C(n, \Lambda_0, \Lambda_1, \gamma, p, q, \|V\|_{L^\gamma(\Omega)}) > 0$ and γ^* is given by (1.5)*.*

Two remarks are immediate.

Remark 1. When $V \geq 0$ in Theorem 1, we further derive that

$$\|\nabla u\|_{L^p(\Omega)} \leq C(p, \Lambda_0)\, \|F\|_{L^p(\Omega)} \leq C(p, \Lambda_0)\, \operatorname{diam}(\Omega)^{\frac{n}{p}-\frac{n}{q}}\, \|F\|_{L^q(\Omega)}$$

by using u as a test function in (1.3). Consequently, (1.6) can be written more succinctly as

$$\|\nabla u\|_{L^q(\Omega)} + \mathbb{1}_{[q<p\gamma]} \|V^{\frac{1}{p}} u\|_{L^q(\mathbb{R}^n)} \le C \left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^{2\left(\frac{n}{p}-\frac{n}{q}\right)} \|F\|_{L^q(\Omega)},$$

where $C = C(n, \Lambda_0, \Lambda_1, \gamma, p, q, \|V\|_{L^\gamma(\Omega)}) > 0$.

Remark 2. If $V \in B^\gamma$ in Theorem 1, then the endpoint case $\gamma = \frac{n}{p}$ may also be included due to the self-improving property of this class.

2 Proof of Theorem 1

Given an exponent $q \in (1, \infty)$, we define

$$q_* := \frac{nq}{n+q} \quad \text{and} \quad q^* := \begin{cases} \frac{nq}{n-q}, & \text{if } 1 < q < n, \\ \infty, & \text{if } q \ge n, \end{cases}$$

whence

$$(q^*)_* = (q_*)^* = q, \qquad \text{for all } 1 < q < n.$$

Our proof of Theorem 1 rests upon the estimate from [5, Corollary 2.5].

Proposition 1. *The following statements hold.*

(a) *Let $p > 1$ and $s > \max\{p, n(p-1)/(n-1)\}$. Assume that $f \in L^{(s/(p-1))_*}(\Omega)$ and $F \in L^s(\Omega)$. Let u be a weak solution to (1.1). Then there exists a constant $\delta_0 = \delta_0(n, \Lambda_0, \Lambda_1, p) > 0$ such that if Ω is (δ_0, r_0)-Reifenberg flat and $\boldsymbol{A}$ is (δ_0, r_0)-vanishing for some $r_0 \in (0, 1)$, then,*

$$\|\nabla u\|_{L^s(\Omega)} \le C \left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^{\frac{n}{p}-\frac{n}{s}} \left(\|f\|^{\frac{1}{p-1}}_{L^{\left(\frac{s}{p-1}\right)_*}(\Omega)} + \|F\|_{L^s(\Omega)}\right),$$

where $C = C(n, \Lambda_0, \Lambda_1, p, s) > 0$.

(b) *Let $p > n$ and $p < s \le \frac{n(p-1)}{n-1}$ and $1 < w < n$. Assume that $f \in L^w(\Omega)$ and $F \in L^s(\Omega)$. Let u be a weak solution to (1.1). Then there exists a constant $\delta_0 = \delta_0(n, \Lambda_0, \Lambda_1, p) > 0$ such that if Ω is (δ_0, r_0)-Reifenberg flat and $\boldsymbol{A}$ is (δ_0, r_0)-vanishing for some $r_0 \in (0, 1)$, then,*

$$\|\nabla u\|_{L^s(\Omega)} \le C \left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^{\frac{n}{p}-\frac{n}{w^*(p-1)}} \left(\|f\|^{\frac{1}{p-1}}_{L^w(\Omega)} + \|F\|_{L^s(\Omega)}\right),$$

where $C = C(n, \Lambda_0, \Lambda_1, p, s, w) > 0$.

In Proposition 1, if $p < n$ then,

$$\max\{p, n(p-1)/(n-1)\} = p$$

and Part (a) asserts that the global Calderón-Zygmund estimate

$$\|\nabla u\|_{L^s(\Omega)} \lesssim \|f\|_{L^t(\Omega)}^{\frac{1}{p-1}} + \|F\|_{L^s(\Omega)} \tag{2.1}$$

is valid for all $s > p$ (and suitable t). Whereas, if $p \geq n$ then Parts (a) and (b) together ensure that (2.1) is again valid for all $s > p$ (and suitable t).

Hereafter, we always assume the structural conditions. The next observation is also crucial.

Lemma 1. *Let u be a weak solution to* (1.1).

(i) *Let $1 < p < n$. Suppose further that $|\nabla u| \in L^s(\Omega)$ for some $p \leq s < n$. Then $V|u|^{p-2}u \in L^{\left(\frac{s^\sharp}{p-1}\right)_*}(\Omega)$, where*

$$s^\sharp := \frac{n\gamma(p-1)s}{n(p-1)\gamma - (p\gamma - n)\,s} \in \big(s, \gamma^*\,(p-1)\big).$$

In particular, $s^\sharp$ is increasing as a function of s with

$$s^\sharp - s > h := \frac{(p\gamma - n)\,p^2}{n(p-1)\gamma} > 0, \qquad \lim_{s\to n^-} s^\sharp = \gamma^*\,(p-1).$$

Moreover, there exists a constant $C = C(n, \gamma, p, s) > 0$ such that

$$\left\|V|u|^{p-1}\right\|_{L^{\left(\frac{s^\sharp}{p-1}\right)_*}(\Omega)}^{\frac{1}{p-1}} \leq C\,\|V\|_{L^\gamma(\Omega)}^{\frac{1}{p-1}}\,\|\nabla u\|_{L^s(\Omega)}.$$

(ii) *Let $p \geq n$. Suppose further that $p < q < \gamma^*\,(p-1)$. Then $V|u|^{p-2}u \in L^{\left(\frac{q}{p-1}\right)_*}(\Omega)$.*
Moreover, there exists a constant $C = C(n, \gamma, p, q) > 0$ such that

$$\left\|V|u|^{p-1}\right\|_{L^{\left(\frac{q}{p-1}\right)_*}(\Omega)}^{\frac{1}{p-1}} \leq C\,\|V\|_{L^\gamma(\Omega)}^{\frac{1}{p-1}}\,\|\nabla u\|_{L^p(\Omega)}.$$

Proof. (i) One has

$$\frac{1}{s^\sharp} = \frac{1}{s} - \frac{p\gamma - n}{n(p-1)\gamma}. \tag{2.2}$$

It follows that $s^\sharp$ is increasing as a function of s. At the same time,

$$\frac{1}{s} > \frac{1}{s^\sharp} > \frac{1}{n} - \frac{p\gamma - n}{n(p-1)\gamma} = \frac{n-\gamma}{n(p-1)\gamma} = \frac{1}{\gamma^*\,(p-1)}.$$

Equivalently, $s < s^\sharp < \gamma^*\,(p-1)$. Still in view of (2.2),

$$s^\sharp - s = \frac{p\gamma - n}{n(p-1)\gamma}\,s\,s^\sharp > \frac{(p\gamma - n)\,p^2}{n(p-1)\gamma} > 0,$$

$$\lim_{s\to n^-} \frac{1}{s^\sharp} = \frac{1}{n} - \frac{p\gamma - n}{n(p-1)\gamma} = \frac{1}{\gamma^*\,(p-1)}.$$

Next,

$$\left(\frac{s^\sharp}{p-1}\right)_* = \frac{ns^\sharp}{n(p-1)+s^\sharp}$$

and our choice of $s^\sharp$ guarantees that

$$\frac{\gamma\,[n(p-1)+s^\sharp]}{ns^\sharp} > 1, \tag{2.3}$$

$$0 < \frac{n\gamma(p-1)s^\sharp}{n\gamma(p-1)-(n-\gamma)s^\sharp} = s^*, \tag{2.4}$$

while $u \in L^{s^*}(\Omega)$ due to Sobolev's embedding theorem. We have

$$\left\| V|u|^{p-1} \right\|_{L^{\left(\frac{s^\sharp}{p-1}\right)_*}(\Omega)}^{\left(\frac{s^\sharp}{p-1}\right)_*} = \int_\Omega |V|^{\frac{ns^\sharp}{n(p-1)+s^\sharp}} |u|^{\frac{n(p-1)s^\sharp}{n(p-1)+s^\sharp}}\,dx$$

$$\le \left(\int_\Omega |V|^\gamma\,dx\right)^{\frac{ns^\sharp}{\gamma\,[n(p-1)+s^\sharp]}} \left(\int_\Omega |u|^{s^*}\,dx\right)^{\frac{n\gamma(p-1)-(n-\gamma)s^\sharp}{n\gamma(p-1)+\gamma s^\sharp}}$$

$$\le C(n,\gamma,p,s) \left(\int_\Omega |V|^\gamma\,dx\right)^{\frac{ns^\sharp}{\gamma\,[n(p-1)+s^\sharp]}} \left(\int_\Omega |\nabla u|^{s}\,dx\right)^{\frac{s^*}{s}\cdot\frac{n\gamma(p-1)-(n-\gamma)s^\sharp}{n\gamma(p-1)+\gamma s^\sharp}}$$

$$= C(n,\gamma,p,s)\,\|V\|_{L^\gamma(\Omega)}^{\frac{ns^\sharp}{n(p-1)+s^\sharp}} \|\nabla u\|_{L^s(\Omega)}^{\frac{n(p-1)s^\sharp}{n(p-1)+s^\sharp}}$$

by Hölder's inequality and Sobolev's embedding theorem in the second and third steps respectively. The claim then follows from this estimate.

(ii) We repeat the arguments in (i) and replace $s^\sharp$ with q. The range for q ensures that (2.3) is still valid with q in place of $s^\sharp$, whereas (2.4) is replaced by

$$0 < \frac{n\gamma(p-1)q}{n\gamma(p-1)-(n-\gamma)q} < p^* := \infty.$$

Furthermore, $u \in L^t(\Omega)$ for all $t \in (1,\infty)$ by Sobolev's embedding theorem. These enable us to proceed with Hölder's inequality and arrive at the conclusion as required. □

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. Let $q \in (p, \gamma^*(p-1))$. We divide the proof into two steps.

Step 1: We show that

$$\|\nabla u\|_{L^q(\Omega)} \le C \left(\frac{\operatorname{diam}(\Omega)}{r_0}\right)^{\frac{n}{p}-\frac{n}{q}} \left(\|\nabla u\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)}\right), \tag{2.5}$$

where $C = C(n, \Lambda_0, \Lambda_1, \gamma, p, q, \|V\|_{L^\gamma(\Omega)}) > 0$. By virtue of Lemma 1 and Proposition 1, it suffices to show that (2.5) holds for $q < n$.

Let $q < n$. We consider two cases as follows.

Case 1: Suppose $1 < p < n$. By adjusting the step size h in Lemma 1(i) to a smaller value when necessary, we may assume that $q = p + kh$ for some $k \in \{1, 2, 3, \ldots\}$. Then the first application of Lemma 1(i) with $s = p$ yields that

$$\left\| V|u|^{p-1} \right\|^{\frac{1}{p-1}}_{L^{\left(\frac{p+h}{p-1}\right)_*}(\Omega)} \le C(n, \gamma, p) \, \|V\|^{\frac{1}{p-1}}_{L^\gamma(\Omega)} \, \|\nabla u\|_{L^p(\Omega)}.$$

In turn, Proposition 1(a) with $f = V|u|^{p-2} u$ gives

$$\begin{aligned} \|\nabla u\|_{L^{p+h}(\Omega)} &\le C \left(\frac{\operatorname{diam}(\Omega)}{r_0} \right)^{\frac{n}{p} - \frac{n}{p+h}} \left(\|V|u|^{p-1}\|^{\frac{1}{p-1}}_{L^{\left(\frac{p+h}{p-1}\right)_*}(\Omega)} + \|F\|_{L^q(\Omega)} \right) \\ &\le C \left(\frac{\operatorname{diam}(\Omega)}{r_0} \right)^{\frac{n}{p} - \frac{n}{p+h}} \max\left\{ 1, \|V\|^{\frac{1}{p-1}}_{L^\gamma(\Omega)} \right\} \\ &\quad \times \left(\|\nabla u\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \right), \end{aligned}$$

where $C = C(n, \Lambda_0, \Lambda_1, \gamma, p) > 0$. Iterating this last estimate $k = \frac{q-p}{h}$ times, we arrive at

$$\begin{aligned} \|\nabla u\|_{L^q(\Omega)} \le & C \left(\frac{\operatorname{diam}(\Omega)}{r_0} \right)^{\frac{n}{p} - \frac{n}{q}} \max\left\{ 1, \|V\|^{\frac{1}{p-1}}_{L^\gamma(\Omega)} \right\}^{\frac{q-p}{h}} \\ & \times \left(\|\nabla u\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \right), \end{aligned}$$

where $C = C(n, \Lambda_0, \Lambda_1, \gamma, p, q) > 0$.

Case 2: Suppose $p \ge n$. In this case, Lemma 1(ii) tells us that $V|u|^{p-2} u \in L^{\left(\frac{q}{p-1}\right)_*}(\Omega)$. It is straightforward to verify that

$$1 < \big(q/(p-1)\big)_* < n.$$

Hence applying Proposition 1(a) and (b) yields (2.5) immediately.

Step 2: We show that

$$\||V|^{\frac{1}{p}} u\|_{L^q(\Omega)} \le C \left(\frac{\operatorname{diam}(\Omega)}{r_0} \right)^{\frac{n}{q} - \frac{n}{p}} \left(\|\nabla u\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \right)$$

for all $q \in (p, \gamma p)$, where $C = C(n, \gamma, p, q, \|V\|_{L^\gamma(\Omega)}) > 0$.

To this end, it suffices to show that

$$\||V|^{\frac{1}{p}} u\|_{L^q(\Omega)} \le C(n, \gamma, p, q) \, \|V\|^{\frac{1}{p}}_{L^\gamma(\Omega)} \, \|\nabla u\|_{L^q(\Omega)}$$

for all $q \in (p, \gamma p)$. Let $q \in (p, \gamma p)$. Recall that $\frac{n}{p} < \gamma < n$. Therefore,

$$\gamma p q / (\gamma p - q) < q^*.$$

At the same time, $u \in L^{q^*}(\Omega)$ since $|\nabla u| \in L^q(\Omega)$ by Step 1. Consequently, Hölder's inequality and Sobolev's embedding theorem give

$$\int_\Omega \left(|V|^{\frac{1}{p}} u\right)^q dx \le \left(\int_\Omega |V|^\gamma dx\right)^{\frac{q}{p\gamma}} \left(\int_\Omega |u|^{\frac{\gamma pq}{\gamma p - q}}\right)^{\frac{\gamma p - q}{\gamma p}}$$
$$\le C(n,\gamma,p,q) \|V\|_{L^\gamma(\Omega)}^{\frac{q}{p}} \|\nabla u\|_{L^q(\Omega)}^q$$

as required. The theorem now follows by combining the estimates in Step 1 and Step 2 together. □

3 Concluding remark

Certain interest is also paid to the local version of (1.1) which is given by

$$\begin{cases} -\mathrm{div}\mathbf{A}(x,Du)+V\,|u|^{p-2}\,u=-\mathrm{div}(|F|^{p-2}F) & \text{in } \Omega_{2r}(y):=B_{2r}(y)\cap\Omega, \\ u=0 \qquad \text{on } B_{2r}(y)\cap\partial\Omega \quad \text{if } B_{2r}(y)\not\subset\Omega, \end{cases} \tag{3.1}$$

where $y \in \overline{\Omega}$ and $r > 0$. A weak solution to (3.1) is understood in the sense of Definition 1 with Ω being replaced by Ω_{2r}.

Using analogous arguments as the above, we may also obtain a Calderón-Zygmund estimate for a weak solution to (3.1). Indeed, the arguments used to prove Theorem 1 is almost independent of the global property therein, with the exception being Proposition 1. The local counterpart of Proposition 1 can be found in [6, Theorems 2.3 and 2.4]). With this in mind, the local Calderón-Zygmund estimate can be stated as follows.

Theorem 2. *Assume the structural conditions. Let u be a weak solution to* (3.1). *Then, there exists a constant $\delta_0 = \delta_0(n,\Lambda_0,\Lambda_1,p) > 0$ such that if Ω is (δ_0, r_0)-Reifenberg flat and $\boldsymbol{A}$ is (δ_0, r_0)-vanishing for some $r_0 \in (0,1)$, then,*

$$\|\nabla u\|_{L^q(\Omega_{r2^{1-q/p}}(y))} + \mathbb{1}_{[q<p\gamma]} \|V^{\frac{1}{p}} u\|_{L^q(\Omega_{r2^{1-q/p}}(y))}$$
$$\le C\left(\|\nabla u\|_{L^p(\Omega_{2r}(y))} + \|F\|_{L^q(\Omega_{2r}(y))}\right)$$

for all $q \in \left(p, \gamma^*(p-1)\right)$, $y \in \overline{\Omega}$ and $r \in (0, \frac{r_0}{2}]$, where

$$C = C(n,\Lambda_0,\Lambda_1,\gamma,p,q,\|V\|_{L^\gamma(\Omega)}) > 0$$

and γ^* is given by (1.5).

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