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Some properties of solutions of α-conformable differential equations with piecewise constant arguments: existence and uniqueness, asymptotic stability, oscillation and periodicity

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Abstract. Conformable differential equations, based on the recently introduced conformable derivative, represent a novel and increasingly popular class of differential equations. This framework offers significant advantages over traditional models, particularly due to its simplicity and enhanced flexibility in modeling diverse phenomena. In this paper, we examine conformable differential equations with piecewise constant arguments. We establish the existence and uniqueness of solutions for these equations and derive conditions for oscillatory behavior, convergence, and periodicity. Additionally, we provide numerical examples to support and illustrate the theoretical results.

Keywords: conformable derivative; piecewise constant arguments; oscillatory solution; convergency; periodic solution.

AMS Subject Classification: 34K11; 34K13; 34K20; 34K37.

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1 Introduction

Differential equations with piecewise constant arguments, abbreviated as DEPCA, emerge from efforts to extend the theory of functional differential equations with continuous arguments to accommodate equations with discontinuous arguments. The appeal of these equations lies in their ability to describe hybrid dynamic systems that integrate both continuous and discrete elements, thus combining characteristics of differential and difference equations. As a result, these equations exhibit greater complexity compared to standard differential equations without piecewise arguments and additionally, they are generally more challenging to analyze.

Research studies on DEPCA were first put forward by Shah and Wiener in 1983 [31]. Then Cooke and Wiener studied DEPCA with time delay in their work [8] and in 1993 a distinguished book was written by Wiener [34]. Numerous studies have addressed various aspects of these equations, including stability [4,33]; the existence of periodic solutions [5]; oscillation [3,23,35], etc.

It is well known that delay terms play a crucial role in mathematical modeling, capturing the inherent time lag between an action and its observed effect in real-world processes. The connection between piecewise constant and delay arguments is well established—while delay differential equations use continuous delays like $x(t-\tau)$ with $\tau > 0$, DEPCA employs piecewise constant arguments, such as $x([t])$ where $[t]$ is a stepwise function (e.g., the floor or ceiling of t), resulting in a hybrid discrete-continuous structure. That connection makes piecewise constant arguments one of the main tools of mathematical modelling. As shown in [7], DEPCA can approximate solutions of delay differential equations with discrete delays. This approach first replaces the delay differential equation with a DEPCA, which is then simplified into a difference equation. Researchers have studied several properties of DEPCA in view of this approach. For example, in [22], a phytoplankton-zooplankton system was modeled using DEPCA. After deriving theoretical results, the Neimark–Sacker bifurcation was analyzed to explain plankton bloom dynamics and determine threshold values for periodicity. Then in [18], the author studied a DEPCA model for bacterial population density in a microcosm and, using the center manifold theorem and bifurcation theory, demonstrated flip and Neimark–Sacker bifurcations. An early brain tumor growth modelled by DEPCA was studied in [19]. There exist also more applications in the literature using DEPCA such as population models [15]; epidemic diseases [27,37]; spring-mass systems [9] and economical models [6].

The conformable derivative has become an important mathematical tool that provides a generalized framework for analyzing dynamical systems while preserving many familiar properties of classical derivatives. It was introduced by Khalil et al. in 2014 [26]. A limit form resembling the classical derivative is incorporated in this unique definition. The conformable derivative also preserves many familiar properties of classical derivative such as linearity, the mean value theorem, Rolle's theorem, the product rule and the quotient rule. Abdeljawad further advanced this idea by creating conformable versions of the Gronwall inequality, chain rule, and partial integration formulas. Additionally, in [1], he extended the conformable derivative framework to encompass power series expansions and Laplace transforms. On the other hand, the geometric aspects of conformable cords and conformable orthogonal trajectories can be seen in [25]. This new type of derivative has attracted a great deal of interest from scientists and has been the focus of multiple papers [11,28,38] and the references therein. Its local nature and simplicity allow for easier computation and interpretation, making it an attractive choice for problems that require extensions of classical calculus while maintaining intuitive mathematical structure. The conformable derivative also offers flexibility in describing processes with varying rates of change by introducing a parameter that adjusts the sensitivity of the derivative. This adaptability makes the conformable derivative particularly useful for modeling systems with scale-dependent or non-uniform behaviors, such as those found in biology, physics, and engineering [14, 30, 32].

Recently, there has been a growing interest in conformable differential equations with piecewise arguments. While numerous studies have been conducted on this topic, it is not feasible to mention them all here. However, we can highlight some notable works as follows: [13] analyzed the stability analysis, Neimark–Sacker bifurcation and chaotic dynamics behavior of a conformable order Lotka–Volterra predator-prey model by using a piecewise constant approximation. Subsequently, in [21], the stability and some bifurcations of the corresponding difference equation of a conformable order DEPCA were examined. In [24], the population density of a species of bacteria in a microcosm was modeled using conformable order DEPCA. Similarly, [36] considered the conformable-type three-dimensional Lotka–Volterra model with piecewise constant arguments. Moving forward, [17] examined a conformable order Lotka–Volterra model for COVID-19 dynamics, transforming it into a difference equation using a piecewise constant approximation. Later, in 2023, Kartal analyzed a two-species predator-prey model for guava borers using Caputo and conformable derivatives, discretizing it with piecewise constant arguments to study stability, Neimark–Sacker bifurcation, and complex dynamics like quasi-periodicity and chaos [20]. In [30], an HIV/AIDS transmission model was developed using a conformable derivative, and the complex behaviors of this system were investigated. For further details, we also refer to [12, 16, 29], along with the references therein.

The paper [8] by Cooke and Wiener is foundational to DEPCA theory, introducing a novel approach for modeling systems with piecewise constant delays. It extends classical differential equations by addressing delays that change in discrete steps. This work has influenced fields like biology, engineering, and economics. Additionally, as wee see from above studies, the conformable derivative, combined with piecewise constant arguments, provides a powerful tool for modeling complex systems. In [8], the authors considered the equation

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t-1]). \tag{1.1}$$

They obtained the unique solution on $[0, \infty)$ and provided conditions just for asymptotic stability. Building on this work, the purpose of the present paper is to investigate the same equation under the framework of the conformable derivative, offering further insights into the dynamics of such systems.

In this paper, we consider

$$\mathcal{D}_0^\alpha x(t) = ax(t) + a_0x([t]) + a_1x([t-1]), \quad t \geq 0, \tag{1.2}$$

with the initial conditions

$$x(-1) = c_{-1}, \ x(0) = c_0, \tag{1.3}$$

where $\mathcal{D}_0^\alpha$ denotes the conformable derivative of $x(t)$ with $\alpha \in (0, 1]$; a, a_0 and a_1 are real constants and $[.]$ denotes the greatest integer function.

The next section presents the preliminaries, covering key definitions and theorems relevant to the paper. Following that, the main results section addresses the existence and uniqueness of the given equation, along with periodic, asymptotically stable, and oscillatory solutions. Then, several examples that illustrate the application of the theorems are given. Finally, the last section provides the conclusion part.

2 Preliminaries

This section contains main tools for the paper.

DEFINITION 1. ([26]) The α-order conformable derivative of $f:[0,\infty)\to\mathbb{R}$ is denoted by $\mathcal{D}_0^\alpha(f)(t)$ or $f^{(\alpha)}(t)$ and defined as

$$\mathcal{D}_0^\alpha(f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$$

for all $t>0$ and $\alpha\in(0,1]$. If f is α-order differentiable in some interval $(0,\delta)$, $\delta>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

On the other hand, the conformable integral of a function $f:[0,\infty)\to\mathbb{R}$ is defined by

$$I_0^\alpha(f)(t)=\int_0^t s^{\alpha-1}f(s)ds,$$

where $\alpha\in(0,1]$.

DEFINITION 2. ([1]) For a function $f:[\sigma,\infty)\to\mathbb{R}$ the α-order conformable derivative can be defined as

$$\mathcal{D}_\sigma^\alpha(f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon(t-\sigma)^{1-\alpha})-f(t)}{\epsilon},\quad t>\sigma,$$

where $\alpha\in(0,1]$. Also, the conformable integral of a function $f:[\sigma,\infty)\to\mathbb{R}$ is defined as

$$I_\sigma^\alpha(f)(t)=\int_\sigma^t (s-\sigma)^{\alpha-1}f(s)ds,$$

where $\alpha\in(0,1]$.

These two definitions coincide with each other when $\sigma=0$.
The following properties of conformable derivative are proved by Khalil et al. in Theorem 2.2, [26]:
If f_1 and f_2 are α-order differentiable, $\alpha\in(0,1]$, at a point $t>0$, then

1. $D_0^\alpha(af_1+bf_2)=aD_0^\alpha(f_1)+bD_0^\alpha(f_2)$ for all $a,b\in\mathbb{R}$;
2. $D_0^\alpha(f_1f_2)=f_1D_0^\alpha(f_2)+f_2D_0^\alpha(f_1)$;
3. $D_0^\alpha\left(\frac{f_1}{f_2}\right)=\frac{f_2D_0^\alpha(f_1)-f_1D_0^\alpha(f_2)}{f_2^2}$;
4. If f is differentiable, then $D_0^\alpha(f)(t)=t^{1-\alpha}f'(t)$;

5. If c is a constant, then $D_0^\alpha(c) = 0$.

Now, we give the following definitions:

Definition 3. A solution of the initial value problem (1.2)–(1.3) defined on $\{-1\} \cup [0,\infty)$ is a function $x(t)$ that satisfies the following conditions:

(i) $x(t)$ is continuous on $[0,\infty)$;

(ii) The α-conformable derivative of $x(t)$ exists on $[0,\infty)$ with the possible exception at the points $[t] \in [0,\infty)$ where one-sided derivatives exist;

(iii) $x(t)$ satisfies Equation (1.2) on the each interval $[n, n+1)$, $n = 0, 1, ...$;

(iv) $x(t)$ satisfies the initial conditions (1.3).

Definition 4. A nontrivial solution of (1.2), say $x(t)$, defined on the interval $[0,\infty)$ is said to be oscillatory about zero if there exist two real valued sequences (t_n), $(\tilde{t}_n) \subset [0,\infty)$ such that $t_n \to \infty$, $\tilde{t}_n \to \infty$ as $n \to \infty$ and $x(t_n)\, x(\tilde{t}_n) \leq 0$ for $n \geq N$ where N is sufficiently large. Otherwise it is called nonoscillatory. Equation (1.2) is said to be oscillatory if all nontrivial solutions are oscillatory; otherwise it is said to be nonoscillatory if its all nontrivial solutions are nonoscillatory.

Before giving the main theorems, we need to remind some well-known results about the second-order difference equation

$$x_{n+2} + p\, x_{n+1} + q\, x_n = 0, \; n = -1, 0, 1, 2, ..., \tag{2.1}$$

where p and $q \neq 0$ are real constants. Solutions of (2.1) are supposed as in the form λ^n, where λ is a complex number. Substituting this function into (2.1), we get

$$\lambda^2 + p\lambda + q = 0, \tag{2.2}$$

which is called the characteristic equation of (2.1). Roots of (2.2), say λ_1 and λ_2, are in the form

$$\lambda_1 = \frac{1}{2}\big(-p + \sqrt{p^2 - 4q}\big) \quad \text{and} \quad \lambda_2 = \frac{1}{2}\big(-p - \sqrt{p^2 - 4q}\big),$$

those are called characteristic roots. There are three cases to formulate the general solution of (2.1):

Case 1. λ_1 and λ_2 are real and distinct. Then the linear independent solutions of (2.1) are λ_1^n and λ_2^n. In this case, the general solution of (2.1) is $x_n = k_1\lambda_1^n + k_2\lambda_2^n$, where k_1 and k_2 are arbitrary constants.

Case 2. $\lambda_1 = \lambda_2 (= \lambda)$. Then the linear independent solutions of (2.1) are λ^n and $n\lambda^n$. The general solution of (2.1) is

$$x_n = (k_1 + k_2 n)\lambda^n.$$

Case 3. $\lambda_1 = u + iv$ and $\lambda_2 = u - iv$ (u and $v \neq 0$ are real numbers). Then the real independent solutions are $r^n \cos n\theta$ and $r^n \sin n\theta$ where $r = \sqrt{u^2 + v^2}$ and $\theta = \arctan \frac{v}{u}$. In this case, the general solution is

$$x_n = r^n(k_1 \cos n\theta + k_2 \sin n\theta) \quad \text{or} \quad x_n = Ar^n \cos(n\theta - B),$$

where A and B are arbitrary constants.

Theorem 1. *([10], p. 94) The following statements hold:*

(A) All solutions of Equation (2.1) *oscillate about zero if and only if the characteristic equation* (2.2) *has no positive real roots.*

(B) All solutions of Equation (2.1) *converge to zero (i.e., the zero solution is asymptotically stable) if and only if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$.

Because of Theorem 1, the following criterion can be stated in terms of the coefficients p and q of Equation (2.1):

Theorem 2. *Every solution of Equation (2.1) is oscillatory about zero if one of the following conditions is satisfied:*

(i) $p > 0$ *and* $q = p^2/4$,

(ii) $p > 0$ *and* $0 < q < p^2/4$, *(iii)* $q > p^2/4$.

Proof. It is clear that each case of (i), (ii) and (iii) implies that the characteristic equation (2.2) has no positive real roots and so due to Theorem 1 (A) all solutions of Equation (2.1) oscillate about zero. Indeed, for the roots λ_1 and λ_2 of Equation (2.2) we have the following findings, respectively:
$\lambda_1 = \lambda_2 < 0$ when (i) is true; λ_1 and λ_2 are distinct and negative real numbers when (ii) is true and finally λ_1 and λ_2 are complex conjugate when (iii) is satisfied. □

Theorem 3. *([10], p.247, Schur-Cohn criterion) The necessary and sufficient condition for the zero solution of Equation (2.1) to be asymptotically stable is*

$$|p| < 1 + q < 2. \tag{2.3}$$

3 Main Results

Our main results are given as follows:

Theorem 4. *The the initial value problem* (1.2)–(1.3) *has a unique solution* $x(t)$ *defined on* $\{-1\} \cup [0, \infty)$.

Proof. We apply the method of steps to show the existence and uniqueness of the solution of (1.2)–(1.3). Let $x_0(t) \equiv x(t)$ be a solution of (1.2)–(1.3) on the interval $0 \le t < 1$. Then Equation (1.2) reduces to

$$\mathcal{D}_0^\alpha x(t) = ax(t) + a_0 x(0) + a_1 x(-1), \quad 0 \le t < 1.$$

By the initial conditions (1.3), the equation above can take the form

$$D_0^\alpha x(t) = ax(t) + a_0 c_0 + a_1 c_{-1}, \quad 0 \le t < 1. \tag{3.1}$$

We can establish the general solution of Equation (3.1) for $a \neq 0$ and $a = 0$, separately. Firstly, consider the case $a \neq 0$. The method for the general solution of Equation (3.1) mentioned in [[26], page 69, Section 4] is almost the same as the method of finding general solution of a nonhomogeneous linear differential equation: Suppose λ is a number and that $x(t) = e^{\lambda t^{\alpha}}$ is a solution to the homogeneous equation $D_0^{\alpha} x(t) = ax(t)$. Since

$$D_0^{\alpha} x(t) = D_0^{\alpha} e^{\lambda t^{\alpha}} = t^{1-\alpha}\big(\lambda \alpha t^{\alpha-1}\big) e^{\lambda t^{\alpha}} = \lambda \alpha e^{\lambda t^{\alpha}},$$

we see that the equation $\lambda \alpha e^{\lambda t^{\alpha}} = ae^{\lambda t}$ or $(\lambda\alpha - a)\, e^{\lambda t^{\alpha}} = 0$ must be satisfied in order for $e^{\lambda t^{\alpha}}$ to be a solution. Thus if λ is any number satisfying $\lambda\alpha - a = 0$, then $x(t) = e^{\lambda t^{\alpha}}$ is a solution to the homogeneous equation. Therefore $x_h(t) = ce^{\frac{a}{\alpha} t^{\alpha}}$ is the general solution of the homogeneous equation, where c is an arbitrary constant. On the other hand, by the method of variation of parameters, a particular solution of Equation (3.1) can be found as $x_p(t) = -\frac{1}{a}(a_0 c_0 + a_1 c_{-1})$. Hence, we get the general solution of Equation (3.1) as

$$x(t) = ce^{\frac{a}{\alpha} t^{\alpha}} - (a_0 c_0 + a_1 c_{-1})/a, \quad 0 \le t < 1, \tag{3.2}$$

where c is an arbitrary constant. Applying the initial condition $x(0) = c_0$,

$$c = c_0 + (a_0 c_0 + a_1 c_{-1})/a.$$

Substituting this value of c into (3.2), we have the solution

$$x(t) \equiv x_0(t) = \Big(\frac{a_0 c_0 + a_1 c_{-1}}{a}\Big)(-1 + e^{a\frac{t^{\alpha}}{\alpha}}) + c_0 e^{a\frac{t^{\alpha}}{\alpha}}, \quad 0 \le t < 1. \tag{3.3}$$

Now, let $x(t) \equiv x_1(t)$ be a solution of (1.2)–(1.3) for $t \in [1, 2)$. Then, Equation (1.2) reduces to the equation

$$\mathcal{D}_1^{\alpha} x(t) = ax(t) + a_0 x(1) + a_1 x(0)$$

or

$$\mathcal{D}_1^{\alpha} x(t) = ax(t) + a_0 c_1 + a_1 c_0, \tag{3.4}$$

where $c_1 = x(1)$. Again, by the method explained above, we have a general solution for Equation (3.4) as

$$x(t) = ce^{\frac{a}{\alpha}(t-1)^{\alpha}} - (a_0 c_1 + a_1 c_0)/a, \quad 1 \le t < 2, \tag{3.5}$$

where c is again an arbitrary constant. Applying $c_1 = x(1)$ to (3.5), we obtain

$$c = c_1 + (a_0 c_1 + a_1 c_0)/a.$$

Putting this value of c into (3.5), we have

$$x(t) \equiv x_1(t) = \Big(\frac{a_0 c_1 + a_1 c_0}{a}\Big)\Big(-1 + e^{a\frac{(t-1)^{\alpha}}{\alpha}}\Big) + c_1 e^{a\frac{(t-1)^{\alpha}}{\alpha}}, \quad 1 \le t < 2. \tag{3.6}$$

By the continuity of $x(t)$ at $t = 1$, from (3.3) and (3.6), we obtain

$$c_1 = \Big(-\frac{a_0}{a} + \frac{a_0}{a} e^{\frac{a}{\alpha}} + e^{\frac{a}{\alpha}}\Big) c_0 + \frac{a_1}{a}(-1 + e^{\frac{a}{\alpha}}) c_{-1}.$$

Now, let $x(t) \equiv x_n(t)$ denote the solution of (1.2)–(1.3) on the interval $n \le t < n+1$. Then, Equation (1.2) reduces to

$$\mathcal{D}_n^{\alpha} x(t) = ax(t) + a_0 x(n) + a_1 x(n-1)$$

or

$$D_n^{\alpha} x(t) = ax(t) + a_0 c_n + a_1 c_{n-1}, \tag{3.7}$$

where $c_n = x(n)$ and $c_{n-1} = x(n-1)$.
Following the same way above, the solution of Equation (3.7) with $c_n = x(n)$ can be found as

$$x(t) \equiv x_n(t) = \left(\frac{a_0 c_n + a_1 c_{n-1}}{a}\right)\left(-1 + e^{a\frac{(t-n)^{\alpha}}{\alpha}}\right) + c_n e^{a\frac{(t-n)^{\alpha}}{\alpha}}, \quad n \le t < n+1. \tag{3.8}$$

Similarly, on the interval $n+1 \le t < n+2$, the solution of (1.2)–(1.3) has the form

$$x(t) \equiv x_{n+1}(t) = \left(\frac{a_0 c_{n+1} + a_1 c_n}{a}\right)\left(-1 + e^{a\frac{(t-n-1)^{\alpha}}{\alpha}}\right) + c_{n+1} e^{a\frac{(t-n-1)^{\alpha}}{\alpha}}. \tag{3.9}$$

Because of the continuity of $x(t)$ at $t = n+1$, it can be written

$$\lim_{t \to (n+1)^+} x_{n+1}(t) = \lim_{t \to (n+1)^-} x_n(t).$$

Therefore, from (3.8) and (3.9), we get the difference equation

$$c_{n+2} + pc_{n+1} + qc_n = 0, \quad n = -1, 0, 1, 2, ... \tag{3.10}$$

with the initial conditions

$$c_{-1} = x(-1),\ c_0 = x(0), \tag{3.11}$$

where

$$p = \frac{a_0}{a} - \left(1 + \frac{a_0}{a}\right)e^{a/\alpha} \text{ and } q = \frac{a_1}{a}(1 - e^{a/\alpha}). \tag{3.12}$$

We note that Equation (3.10) is a kind of linear homogeneous difference equation with constant coefficients. If $a_1 \neq 0$, then Equation (3.10) is a second order difference equation. So, Equation (3.10) together with the initial conditions (3.11) has a unique solution c_n. If $a_1 = 0$ and $a_0 \neq \frac{ae^{a/\alpha}}{1-e^{a/\alpha}}$, then Equation (3.10) reduces to the first order difference equation

$$c_{n+1} + pc_n = 0, \quad n = 0, 1, 2, \ldots.$$

This equation with the initial condition $c_0 = x(0)$ has the unique solution

$$c_n = (-p)^n c_0, \quad n = 0, 1, 2, \ldots.$$

Also, if $a_1 = 0$ and $a_0 = \frac{ae^{a/\alpha}}{1-e^{a/\alpha}}$, then we have the unique solution $c_n = 0, n = 1, 2, \ldots$. Therefore, the solution $x(t)$ of (1.2)–(1.3) defined by (3.8) is unique on the interval $n \le t < n+1, n = 0, 1, 2, \ldots$. Hence the initial value problem (1.2)–(1.3) has a unique solution defined on $\{-1\} \cup [0, \infty)$ provided that $a \neq 0$.

Now, consider the case $a = 0$. In this case Equation (1.2) reduces to

$$\mathcal{D}_0^\alpha x(t) = a_0 x([t]) + a_1 x([t-1]), \quad t \geq 0. \tag{3.13}$$

Applying the above procedure, we can easily see that Equation (3.13) with the initial conditions (1.3) has only one solution as

$$x(t) = x_n(t) = \Big(1 + a_0 \frac{(t-n)^\alpha}{\alpha}\Big) c_n + a_1 \frac{(t-n)^\alpha}{\alpha} c_{n-1}, \quad t \in [n, n+1),$$
$$n = 0, 1, 2, ..., \tag{3.14}$$

where c_n is the unique solution to the difference equation

$$c_{n+2} - \Big(1 + \frac{a_0}{\alpha}\Big) c_{n+1} - \frac{a_1}{\alpha} c_n = 0, \quad n = -1, 0, 1, 2, ..., \tag{3.15}$$

with the initial conditions $c_{-1} = x(-1)$ and $c_0 = x(0)$. Hence the proof is completed. □

Remark 1. It is noted that the solution (3.8) can be expressed for $t \in \{-1\} \cup [0, \infty)$ as

$$x(t) = \Big(-\frac{a_0}{a} + \frac{a_0}{a} e^{a\frac{(t-[t])^\alpha}{\alpha}} + e^{a\frac{(t-[t])^\alpha}{\alpha}}\Big) c_{[t]} + \Big(-\frac{a_1}{a} + \frac{a_1}{a} e^{a\frac{(t-[t])^\alpha}{\alpha}}\Big) c_{[t-1]}, \tag{3.16}$$

where $c_{[t]}$ is the unique solution of the initial value problem

$$c_{[t+2]} + p c_{[t+1]} + q c_{[t]} = 0, \quad c_{-1} = x(-1),\ c_0 = x(0), \tag{3.17}$$

where p and q are shown in (3.12). Similarly, the solution (3.14) can be expressed for $t \in \{-1\} \cup [0, \infty)$ as

$$x(t) = \Big(1 + a_0 \frac{(t-[t])^\alpha}{\alpha}\Big) c_{[t]} + a_1 \frac{(t-[t])^\alpha}{\alpha} c_{[t-1]}, \tag{3.18}$$

where $c_{[t]}$ is the unique solution of

$$c_{[t+2]} - \Big(1 + \frac{a_0}{\alpha}\Big) c_{[t+1]} - \frac{a_1}{\alpha} c_{[t]} = 0, \quad c_{-1} = x(-1),\ c_0 = x(0). \tag{3.19}$$

Remark 2. Since the conformable derivative of a constant function is zero, the equilibrium equation of (1.2) is

$$(a + a_0 + a_1)\, x_e = 0.$$

If $a + a_0 + a_1 \neq 0$, then $x_e = 0$ is the unique equilibrium point of (1.2) and so $x(t) = 0$ is the only constant solution of (1.2)–(1.3) with $c_0 = c_{-1} = 0$. On the other hand, if $a + a_0 + a_1 = 0$, then every real number, namely $x_e = c \in \mathbb{R}$, is an equilibrium point of (1.2) and so $x(t) = c$ is a constant solution of (1.2)–(1.3) with $c_0 = c_{-1} = c$.

Due to this remark, henceforth we will assume that $a + a_0 + a_1 \neq 0$, that is the only constant solution of (1.2) is $x(t) = 0$.

Remark 3. If $\alpha = 1$, then the conformable derivative becomes classical derivative. Hence, taking $\alpha = 1$ in the solution (3.16) of (1.2)–(1.3) gives the solution of Equation (1.1) in [8] with the initial conditions $x(-1) = c_{-1}$, $x(0) = c_0$.

Theorem 5. *Assume that $a \neq 0$ and $a_1 \neq 0$. If one of the following conditions is true, then every solution of (1.2) oscillates:*

(i_1) $a > 0,\quad a_0 < \frac{ae^{\frac{a}{\alpha}}}{1-e^{\frac{a}{\alpha}}}$ *and* $a_1 = \frac{a}{4(1-e^{\frac{a}{\alpha}})}\left(\frac{a_0}{a} - \frac{a_0}{a}e^{\frac{a}{\alpha}} - e^{\frac{a}{\alpha}}\right)^2$;

(i_2) $a > 0,\quad a_0 < \frac{ae^{\frac{a}{\alpha}}}{1-e^{\frac{a}{\alpha}}}$ *and* $0 > a_1 > \frac{a}{4(1-e^{\frac{a}{\alpha}})}\left(\frac{a_0}{a} - \frac{a_0}{a}e^{\frac{a}{\alpha}} - e^{\frac{a}{\alpha}}\right)^2$;

(i_3) $a < 0$ *and* $a_1 < \frac{a}{4(1-e^{\frac{a}{\alpha}})}\left(\frac{a_0}{a} - \frac{a_0}{a}e^{\frac{a}{\alpha}} - e^{\frac{a}{\alpha}}\right)^2$.

Proof. For $a \neq 0$, a solution $x(t)$ of (1.2) is given by (3.16). It is trivial that the characteristic equation of the difference equation (3.10) is equal to (2.2) with the coefficients p and q are defined by (3.12). So, by Theorem 1-(A), Equation (3.10) is oscillatory if and only if the characteristic equation (2.2) has no positive real roots. Obviously, assumption (i_1) fulfill the condition (i) of Theorem 2 (Indeed, if $a > 0$, then $1 - e^{a/\alpha} < 0$ and therefore by the conditions in (i_1), it follows that $p > 0$ and $q = \frac{1}{4}p^2$). So, due to assumption (i_1), every solution c_n of Equation (3.10) is oscillatory. Since $x(n) = c_n$, every solution $x(t)$ of Equation (1.2) is oscillatory when (i_1) is satisfied. Similarly, assumptions (i_2) and (i_3), respectively, verify the conditions (ii) and (iii) of Theorem 2. Indeed, according to the first condition in (i_2), it is again $1 - e^{a/\alpha} < 0$. Therefore, by (i_2), it can be seen that $p > 0$ and $0 < q < \frac{1}{4}p^2$. Moreover, if $a < 0$, then $1 - e^{a/\alpha} > 0$ and hence by the conditions in (i_3), it takes out $q > \frac{1}{4}p^2$). So, each of these two assumptions also provides every solution of difference equation (3.10) and as well as that every solution of Equation (1.2) is oscillatory. Hence, the proof is completed. □

Corollary 1. Assume that $a = 0$ and $a_1 \neq 0$. If one of the following conditions is true, then every solution of (3.13) oscillates:

(i_1) $a_0 + \alpha < 0$ and $4\alpha a_1 + (\alpha + a_0)^2 = 0$;

(i_2) $a_0 + \alpha < 0,\quad a_1 < 0$ and $4\alpha a_1 + (\alpha + a_0)^2 > 0$;

(i_3) $4\alpha a_1 + (\alpha + a_0)^2 < 0$.

Proof. Equation (1.2) with $a = 0$ points out Equation (3.13). A solution $x(t)$ of (3.13) is shown by (3.18). The rest of the proof can be easily performed by applying the process in the proof of Theorem 5 and so it is omitted. □

Theorem 6. *Let us assume that $a \neq 0$ and $a_1 \neq 0$. The solution $x(t)$ of (1.2)–(1.3) goes to zero as $t \to +\infty$ if and only if*

$$|a_0 - (a_0 + a)e^{a/\alpha}| < |a| - a_1|1 - e^{a/\alpha}| < 2|a| \tag{3.20}$$

is satisfied.

Proof. Since $a \neq 0$, the solution $x(t)$ of (1.2)–(1.3) is given by (3.16). It is clear that the two brackets in (3.16) are bounded because $t - [t] \in [0, 1)$ for all $t \in \{-1\} \cup [0, \infty)$. Now, if $x(t) \to 0$ as $t \to +\infty$, then also $x([t]) \to 0$ as $[t] \to \infty$, that is $x([t]) = x(n) = c_n \to 0$ as $n \to \infty$, where c_n is given by (3.10)–(3.11). This means that the zero solution of the difference equation (3.10) is asymptotically stable. This requires that the condition (2.3) must be satisfied for the coefficients of (3.10) p and q that are determined by (3.12). Indeed, if we put these p and q into (2.3), then the condition (3.20) immediately emerges. So, this completes the proof of necessity.

For the sufficiency, let the condition (3.20) be verified. Clearly, this condition implies the condition (2.3). Therefore, Theorem 3 can be applied to the difference equation (3.10) and so it is seen that $x(n) = c_n \to 0$ as $n \to \infty$. Hence, from (3.16), $x(t) \to 0$ as $t \to +\infty$. □

Corollary 2. Assume that $a = 0$ and $a_1 \neq 0$. Then the solution of (3.13), (1.3) goes to zero as $t \to +\infty$ if and only if

$$|a_0 + \alpha| < \alpha - a_1 < 2\alpha. \tag{3.21}$$

Proof. Since $a = 0$, Equation (1.2) is reduced to Equation (3.13) and the solution of this equation with the initial conditions (1.3) is denoted by (3.18)–(3.19). The proof is completed by following the same procedure in the proof of Theorem 6 and so it is omitted. □

Theorem 7. *Let $a_1 \neq 0$. The nontrivial solution $x(t)$ of (1.2)–(1.3) is periodic of period k if and only if $x(k) = x(0)$ and $x(k-1) = x(-1)$, where k is a positive integer.*

Proof. For $a \neq 0$, the solution $x(t)$ of (1.2) is given by (3.8). Let $x(k) = x(0)$ and $x(k-1) = x(-1)$ be true, that is $c_k = c_0$ and $c_{k-1} = c_{-1}$. Then, we need to show that

$$x_n(t-k) = x_{n+k}(t), \quad t \in [n+k, n+k+1),\ n = 0, 1, 2, \ldots. \tag{3.22}$$

Clearly, Equation (3.22) implies that the solution $x(t)$ of (1.2) is periodic with period k, that is $x(t+k) = x(t)$ for $t \in \{-1\} \cup [0, \infty)$.

From (3.8), for $n = 0$ and $n = k$, we get, respectively,

$$\begin{aligned} x_0(t) &= \Big(\frac{a_0 c_0 + a_1 c_{-1}}{a}\Big)\Big(-1 + e^{a\frac{t^\alpha}{\alpha}}\Big) + c_0 e^{a\frac{t^\alpha}{\alpha}}, \quad 0 \le t < 1, \\ x_k(t) &= \Big(\frac{a_0 c_k + a_1 c_{k-1}}{a}\Big)\Big(-1 + e^{a\frac{(t-k)^\alpha}{\alpha}}\Big) + c_k e^{a\frac{(t-k)^\alpha}{\alpha}}, k \le t < k+1. \end{aligned}$$

Since $c_k = c_0$ and $c_{k-1} = c_{-1}$, we get

$$x_0(t-k) = x_k(t), \quad k \le t < k+1. \tag{3.23}$$

Considering Equation (3.23) and the continuity of the solution $x(t)$ along the interval $[0, \infty)$, it is emerged the equality $c_1 = c_{1+k}$. Subsequently, from (3.8), for $n=1$ and $n=1+k$, we obtain, respectively,

$$x_1(t)=\Big(\frac{a_0c_1+a_1c_0}{a}\Big)\Big(-1+e^{a\frac{(t-1)^\alpha}{\alpha}}\Big)+c_1e^{a\frac{(t-1)^\alpha}{\alpha}},\quad 1\le t<2,$$
$$x_{1+k}(t)=\Big(\frac{a_0c_{1+k}+a_1c_k}{a}\Big)\Big(-1+e^{a\frac{(t-1-k)^\alpha}{\alpha}}\Big)+c_{1+k}e^{a\frac{(t-1-k)^\alpha}{\alpha}},1+k\le t<2+k.$$

It is clear that $x_1(t-k)=x_{1+k}(t)$ for $t\in[1+k,2+k)$. In this way, we can show that $x_n(t-k)=x_{n+k}(t)$ for $t\in[n+k,n+k+1), n=0,1,2,\dots$.
Now, if $x(t+k)=x(t)$, that is the solution $x(t)$ is periodic of period k, then we openly have $c_k=c_0$ and $c_{k-1}=c_{-1}$.
Finally, for the case $a=0$, the solution $x(t)$ of (3.13) with the initial conditions (1.3) is given by (3.14) where c_n is the unique solution of (3.15). Following the above method, we arrive the same result if and only if $c_k=c_0$ and $c_{k-1}=c_{-1}$. So, the proof is completed. □

Remark 4. If $\alpha=1, a_0=0, a_1=-b$ and a replaced by $-a$, then Equation (1.2) reduces to Equation (14) in [2]. In this case, our result Theorem 7 and Lemma 2 in [2] coincide with one another.

Corollary 3. Assume that $a\neq 0$ and $a_1\neq 0$.

(1) If
$$a_0=a_1+a\left((1+e^{a/\alpha})/(1-e^{a/\alpha})\right), \tag{3.24}$$
then the solution $x(t)$ of (1.2)–(1.3) is periodic of period 2;

(2) If
$$\left[a_0-a_1-(a_0-a_1+a)\,e^{a/\alpha}\right]^2+\left[a-a_0+(a+a_0)\,e^{a/\alpha}\right]\times\left[a-a_1+a_1e^{a/\alpha}\right]=0,$$
then the solution $x(t)$ of (1.2)–(1.3) is periodic of period 3.

Proof. Here, we only give the proof of (1) because the proof of (2) can be done, similarly. Theorem 7 implies that the solution $x(t)$ of (1.2)–(1.3) given by (3.16)–(3.17) is periodic of period 2 whenever

$$c_2=c_0,\quad c_1=c_{-1}. \tag{3.25}$$

From (3.17), for $t=-1$ and $t=0$ (or equivalently from (3.10), for $n=-1$ and $n=0$) we get, respectively,

$$c_1=-pc_0-qc_{-1},\quad c_2=(p^2-q)c_0+pqc_{-1}, \tag{3.26}$$

where p and q are denoted by (3.12). Substituting (3.26) in (3.25), we have the system

$$\begin{cases}\left(p^2-q-1\right)c_0+pqc_{-1}=0,\\ (-p)c_0-(q+1)\,c_{-1}=0.\end{cases}$$

Since we are interested in nontrivial solutions, we have

$$\Delta = \begin{vmatrix} \left(p^2 - q - 1\right) & pq, \\ -p & -\left(q+1\right) \end{vmatrix} = 0,$$

which yields (3.24) and hence the proof of (1) is completed. □

Corollary 4. Assume that $a = 0$ and $a_1 \neq 0$.

(1) If

$$a_0 - a_1 + 2\alpha = 0,$$

then the solution $x(t)$ of (3.13), (1.3) is periodic of period 2;

(2) If

$$\left(a_0 - a_1 + \alpha\right)^2 + \left(a_0 + 2\alpha\right)\left(a_1 - \alpha\right) = 0,$$

then the solution $x(t)$ of (3.13), (1.3) is periodic of period 3.

Proof. The proof is similar to the proof of Corollary 3. □

4 Examples

This section provides several examples to verify our main results.

Example 1. Let us consider Equation (1.2) with $\alpha = 0.5$, $a = 0.5$, $a_0 = -1$ and $a_1 = -0.02$. In this case, all conditions in hypothesis (i_2) of Theorem 5 are satisfied. So, every solution of

$$D_0^{0.5} x(t) = 0.5x(t) - x([t]) - 0.02x([t-1]) \tag{4.1}$$

is oscillatory. In addition to this property, also every solution of (4.1) goes to zero as $t \to +\infty$ because all conditions in Theorem 6 are satisfied for (4.1). Indeed, these two properties can be seen by obtaining the solution $x(t)$ of (4.1) with the initial conditions $x(-1) = 0$ and $x(0) = 1$ from (3.16)–(3.17) (see Figure 1).

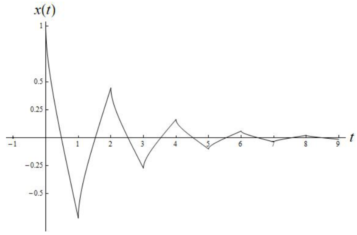

Figure 1. Graph of the solution of (4.1) with $x(-1) = 0$ and $x(0) = 1$.

Example 2. Let us consider the initial value problem

$$\begin{cases} D_0^{0.25}x(t) = x(t) + \frac{e^4}{1-e^4}x([t]) - \frac{1}{1-e^4}x([t-1]), \\ x(-1) = 1,\ x(0) = 2, \end{cases} \tag{4.2}$$

for $t \geq 0$. Here, $\alpha = 0.25$, $a = 1$, $a_0 = \frac{e^4}{1-e^4}$, $a_1 = -\frac{1}{1-e^4}$. Clearly, the case (1) of Corollary 3 is verified and so the solution $x(t)$ of (4.2) is periodic of period 2. Indeed, from (3.16), the solution $x(t)$ of (4.2) is obtained as

$$x(t) = \frac{1}{1-e^4}\left[\left(-e^4 + e^{4\left(t-[t]\right)^{0.25}}\right)c_{[t]} + \left(1 - e^{4(t-[t])^{0.25}}\right)c_{[t-1]}\right], \tag{4.3}$$

where $c_{[t]}$ is calculated by (3.17) as

$$c_{[t+2]} = c_{[t]}. \tag{4.4}$$

From (4.3), it follows

$$x(t+2) = \frac{1}{1-e^4}\left[\left(-e^4+e^{4\left(t-[t]\right)^{0.25}}\right)c_{[t+2]} + \left(1 - e^{4(t-[t])^{0.25}}\right)c_{[t+1]}\right]. \tag{4.5}$$

By (4.4), $c_{[t+1]} = c_{[t-1]}$ and so (4.5) is equal to (4.3), that is $x(t+2) = x(t)$. This means that the solution $x(t)$ is periodic with period 2 (see Figure 2).

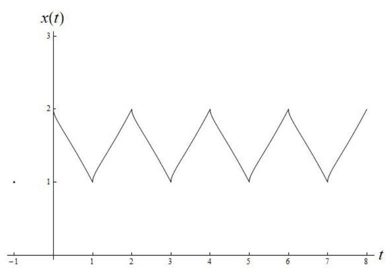

Figure 2. 2- periodic solution $x(t)$ of (4.2).

Example 3. Consider the conformable differential equation

$$D_0^{0.25}x(t) = x(t) + \frac{1+e^4}{1-e^4}x([t]) + \frac{1}{1-e^4}x([t-1]) \tag{4.6}$$

and the initial conditions

$$x(-1) = -1, \quad x(0) = 1. \tag{4.7}$$

The solution $x(t)$ of (4.6)–(4.7) is periodic with period 3 because this time the case (2) of Corollary 3 is satisfied for the values $\alpha = 0.25$, $a = 1$, $a_0 = \frac{1+e^4}{1-e^4}$ and $a_1 = \frac{1}{1-e^4}$ (see Figure 3).

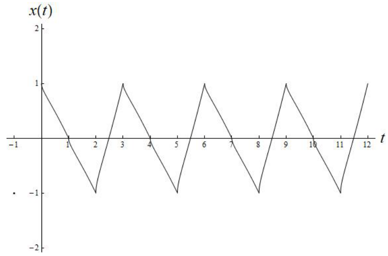

Figure 3. 3 periodic solution of (4.6)–(4.7).

The following examples are considered for the case $a = 0$. Especially, it is noted that parameter α in Equation (4.9) can be regarded as a bifurcation parameter because the stability type changes at a point of α.

Example 4. For the parameters $a_0 = -\frac{1}{4}$, $a_1 = -\frac{1}{36}$ and $\alpha \in (0, 1]$, the asymptotic stability condition given by (3.21) reduces to

$$\left|-1/4 + \alpha\right| < \alpha + 1/36 < 2\alpha. \tag{4.8}$$

We emphasize that (4.8) is not satisfied for $\alpha \in (0, \frac{1}{9}]$ and so the zero solution of

$$D_0^\alpha x(t) = -\frac{1}{4}x([t]) - \frac{1}{36}x([t-1]) \tag{4.9}$$

is not asymptotically stable when $\alpha \in (0, \frac{1}{9}]$. However, (4.8) holds for $\alpha \in (\frac{1}{9}, 1]$ and so the zero solution of (4.9) is asymptotically stable when $\alpha \in (\frac{1}{9}, 1]$. According to these facts, there is a bifurcation at $\alpha = \frac{1}{9}$. So, the value $\alpha = \frac{1}{9}$ is a bifurcation point of Equation (4.9).

Specifically, for $\alpha = 0.1 \in (0, \frac{1}{9}]$ the zero solution of (4.9) is not asymptotically stable (see Figure 4) but for $\alpha = 0.5 \in (\frac{1}{9}, 1]$ the zero solution asymptotically stable (see Figure 5).

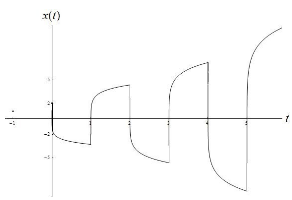

Figure 4. The solution $x(t)$ of (4.9) with $x(-1) = 1$ and $x(0) = 2$.

Example 5. The condition in (2) of Corollary 4 is verified for the following equation:

$$D_0^{0.8}x(t) = -1.6x([t]) - 0.8x([t-1]). \tag{4.10}$$

So, the solution of Equation (4.10) with the initial conditions $x(-1) = 0$ and $x(0) = 1$ is periodic of period 3 (see Figure 6).

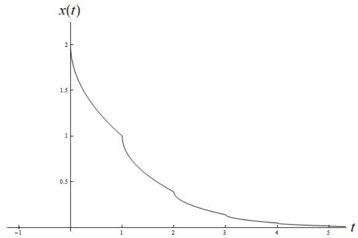

Figure 5. The solution $x(t)$ of (4.9) with $x(-1) = 0$ and $x(0) = 2$.

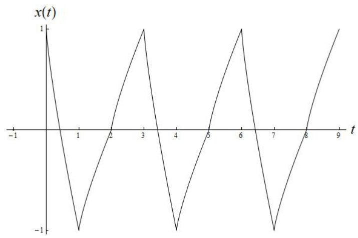

Figure 6. 3 periodic solution of (4.10).

5 Conclusions

In this paper, we have studied conformable differential equations with piecewise constant arguments, establishing existence and uniqueness results, as well as conditions for oscillatory behavior, convergence, and periodicity. Numerical examples validated the theoretical findings, highlighting their applicability in modeling systems with discontinuous dynamics. Future research could explore stability and bifurcation in more complex conformable systems with piecewise constant arguments. Additionally, extending the analysis to multi-dimensional systems may provide deeper insights into real-world applications.

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