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About Regularization of Severely Ill-Posed Problems by Standard Tikhonov's Method with the Balancing Principle

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Abstract. In the present paper for a stable solution of severely ill-posed problems with perturbed input data, the standard Tikhonov method is applied, and the regularization parameter is chosen according to balancing principle. We establish that the approach provides the order of accuracy $O((\underbrace{\ln\ldots\ln}_{K\text{-times}}(1/(h+\delta)))^{-p})$ on the class of problems under consideration.

Keywords: severely ill-posed problems, standard Tikhonov method, balancing principle, approximate accuracy.

AMS Subject Classification: 47A52; 65R30.

1 Statement of Problem

In the present paper we consider the issue of approximate solving severely ill-posed problems represented by operator equation of the first kind

$$Ax = y, \tag{1.1}$$

where $A : X \to Y$ is a linear compact injective operator acting between Hilbert spaces X and Y. The set Range(A) is not closed in Y. Let us denote inner products in these spaces by $(\cdot,\cdot)$ and corresponding norms by $\|\cdot\|$. The symbol $\|\cdot\|$ stands also for a standard operator norm. It will become clear from the context which exactly space or norm is under consideration. Moreover, suppose also that an available perturbation $y_\delta \in Y$: $\|y - y_\delta\| \le \delta$, $\delta > 0$, is given instead of the right-hand side y and a perturbed operator A_h: $\|A - A_h\| \le h$, $h > 0$, is known instead of A. Here $A_h : X \to Y$ is a linear compact operator.

The equation (1.1) is generally referred to as a severely ill-posed problem if its solution $x_0 = A^{-1}y$ has a finite "smoothness" in some sense, but A is an infinitely smoothing operator.

Notice that a distinguishing characteristic of such kind of problems is the fact that x_0 belongs to some subspace V continuously embedded in X, the singular values of the canonical embedding operator J_V from V into X tend to zero with the polynomial rate, while the singular values $\{\sigma_l\}_{l=1}^{\infty}$ of the operator A tend to zero exponentially.

Following [2, 7] suppose that x_0 belongs to the set

$$M_{p,\rho}^{K}(A) := \big\{x\colon x = \big(\underbrace{\ln\ldots\ln}_{K\text{-times}}(A^*A)^{-1}\big)^{-p}v,\ \|v\| \le \rho\big\} \tag{1.2}$$

for some given parameters $\rho > 0$, $K = 1, 2, \ldots$ and unknown parameter $p > 0$, where the operator function $(\underbrace{\ln\ldots\ln}_{K\text{-times}}(A^*A)^{-1})^{-p}$ is well-defined by the spectral decomposition of the operator

$$A^*A = \sum_{k=1}^{\infty} \sigma_k^2(\Psi_k, \cdot)\Psi_k$$

as follows

$$\big(\underbrace{\ln\ldots\ln}_{K\text{-times}}(A^*A)^{-1}\big)^{-p}v = \sum_{l=1}^{\infty}\big(\underbrace{\ln\ldots\ln}_{K\text{-times}}(\sigma_l^{-2})\big)^{-p}(\Psi_l, v)\Psi_l.$$

Further, without loss of generality we assume that

$$\|A\| \le M_K, \quad M_K = m_K^{1/2}, \quad m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, \ldots, K, \end{cases}$$

i.e.

$$\sigma_l \le m_K, \quad l = 1, 2, \ldots.$$

Note that a lot of inverse problems of satellite gradiometry, acoustic scattering, the potentional theory and etc. belong to the severely ill-posed problems. The detailed description of different examples of severely ill-posed problems one can find, for instance in [3, 5, 6, 15].

As far as the history of the question, we note that intensive study of the problem of finding a stable solution of severely ill-posed problems was initiated by the work [8]. Here for regularization of the problems under consideration the standard Tikhonov method was applied. Later in the work [6] suggested a general class of regularization methods (according to Bakushinski; see, e.g., [1]) for solving both linear and non-linear severely ill-posed problems (1.1) with perturbed input data; for choosing regularization parameter, a modification from [14] was employed. Among the works devoted to the research of approximate methods of solving severely ill-posed problems we should mention [2, 7, 15, 17]. For instance, in [15] an approach for solving severely ill-posed problems (1.1) with solutions from $M_{p,\rho}^{1}(A)$ and exact given operators was proposed. It suggests a combination of the standard Tikhonov regularization with the Morozov discrepancy principle. The indicated strategy allows to achieve an order optimal accuracy (in the logarithmic scale) of finding approximate solutions from the set $M^1_{p,\rho}(A)$ for any $p > p_0$, where $p_0 > 0$ is given. In [17] for solving the same problems, the standard Tikhonov method was employed again. Herewith, for the stop rule the balancing principle was considered (about balancing principle see also [4, 9, 10, 13, 16]). In [17] was established that described approach also allows to attain the order-optimal accuracy of recovering solutions, but only for $0 < p \le 1$. Moreover, studies initiated in [15] were extended in the series of works, among which we should mentioned [11, 12, 18]. In particular, in [18] the more wide, than in [15], class of ill-posed problems (1.1) with solutions (1.2) for arbitrary $K = 1, 2, \dots$ and $p > p_0 > 0$ were considered. The order-optimal accuracy of recovering solutions $O((\underbrace{\ln\dots\ln}_{K\text{-times}}\frac{1}{\delta})^{-p})$ was obtained in the case of exact given operators.

In the present paper for the study of severely ill-posed problems (1.1) with perturbed input data A_h, y_δ and solutions from the set (1.2) for any $K = 1, 2, \dots,$ the standard Tikhonov method will be employed. A regularization parameter will be chosen according to the balancing principle. We will establish that suggested approach provides the order accuracy $O((\underbrace{\ln\dots\ln}_{K\text{-times}}\frac{1}{h+\delta})^{-p})$ on the set of solutions $M^K_{p,\rho}(A)$. As opposite to the works mentioned above, our method does not require any additional information about smoothness of the desired solution.

2 The Finite-Dimensional Analogue of Tikhonov Method

We recall, that within the framework of the standard Tikhonov method, the regularized solution $x^{h,\delta}_\alpha$ is determined as a solution of the variation problem

$$I^h_\alpha(x) := \|A_h x - y_\delta\|^2 + \alpha\|x\|^2 \to \min. \tag{2.1}$$

Since any numerical realization of Tikhonov's regularization schema requires to carry out all computations with a finite-dimensional input data, then the variation problem (2.1) is replaced by its finite-dimensional analogue

$$I^h_{\alpha,n}(x) := \|A_{h,n} x - y_\delta\|^2 + \alpha\|x\|^2 \to \min.$$

Here $A_{h,n}$ is some finite-dimensional approximation of A_h, with rank$(A_{h,n})=n$.

In order to find approximate solution in this case, we have to solve the linear operator equation

$$\alpha x + A^*_{h,n}A_{h,n}x = A^*_{h,n}y_\delta,$$

put it another way, we are looking for approximate solution of the form

$$x^{h,\delta}_{\alpha,n} = (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y_\delta.$$

The finite-dimensional approximation $A_{h,n}$ acting in Y is such that the condition

$$\|A_h - A_{h,n}\| \le \varepsilon, \quad \text{where } \varepsilon = \begin{cases} \delta\rho^{-1}, & 0 < h \le \delta, \\ h, & h > \delta \end{cases} \tag{2.2}$$

is valid.

Examples of sufficiently close approximation of operators in severely ill-posed problems one can find in the work [15].

3 Auxiliary Statements

In this section we formulate some definitions and facts, and also the series of auxiliary assertions which shall later need.

Previously T. Hohage (see [6, Lemma 3.13]) has proved that for all $p > 0$ and some constant $C_1 = C_1(p)$ the following inequality

$$\frac{\alpha}{\alpha+\lambda} \ln^{-p} \frac{1}{\lambda} \leq C_1 \ln^{-p} \frac{1}{\alpha}, \quad \alpha, \lambda \in \big(0; e^{-1}\big] \tag{3.1}$$

holds true.

First, we specify the constant C_1 in the inequality (3.1). We consider two cases.

1) Let $0 < \lambda \leq \alpha$. Then, due to monotonicity of ln, we have

$$\frac{\alpha}{\alpha+\lambda} \ln^{-p} \frac{1}{\lambda} \leq \ln^{-p} \frac{1}{\alpha}.$$

2) Let $\lambda \geq \alpha$. We recall (see [6, Lemma 3.13]) that in this case the inequality (3.1) directly follows from the next result: if put $q := \frac{\alpha}{\lambda}$, then the inequality

$$\left(\ln \frac{1}{q} + 1\right)^p \leq C_1, \quad 0 < q \leq 1 \tag{3.2}$$

holds true. Thus, we have to find the constant C_1 for which the inequality (3.2) is valid. For that reason we consider auxiliary function

$$f(q) := q\left(\ln \frac{1}{q} + 1\right)^p, \quad 0 < q \leq 1$$

and determine the largest value of this function on the interval $(0; 1]$. First, we find critical points of $f(q)$. It is easy to show that

$$f'(q) = \left(\ln \frac{1}{q} + 1\right)^{p-1} \left[\ln \frac{1}{q} + 1 - p\right].$$

Obviously, $f'(q) = 0$ when $q = e^{1-p}$. And again we distinguish two cases.

a) Let $0 < p \leq 1$, then $q = e^{1-p} \notin (0; 1]$ and, $f'(q) > 0$ on the interval $(0; 1]$. This in turn means that the function $f(q)$ is monotonously increasing on $(0; 1]$. Hence, it reaches the largest value when $q = 1$, i.e. $\max\limits_{(0;1]} f(q) = f(1) = 1$.

b) Let $p \geq 1$. In this case $q = e^{1-p} \in (0; 1]$. It is well known that a function reaches the largest value either at the critical points or on the ends of the interval. Let us compute these values.

Since the function $f(q)$ is non-define when $q = 0$, then, we find its boundary value when $q \to 0$. As $p \geq 1$, then we can represent p on the form $p = k + \gamma$, where $k \in \mathbb{N}$, $0 \le \gamma < 1$. Applying k-times the Lopital rule to compute of $\lim\limits_{q\to 0} f(q)$ we get

$$\lim_{q\to 0} f(q) = p(p-1)\cdots(p-k+1)\lim_{q\to 0}\frac{(\ln(1/q)+1)^{\gamma}}{1/q}$$
$$= p(p-1)\cdots(p-k)\lim_{q\to 0}\frac{(\ln(1/q)+1)^{\gamma-1}q(-1/q^2)}{-1/q^2} = 0.$$

Moreover, $f(1) = 1$ and $f(e^{1-p}) = e^{1-p}p^p$. Comparing values computed above allows to be sure that for any $p \ge 1$

$$\max_{(0;1]} f(q) = f\left(e^{1-p}\right) = e^{1-p}p^p.$$

Thus, combing found estimates we obtain that in relation (3.1) the constant C_1 has the form

$$C_1 = \begin{cases} 1, & 0 < p \le 1, \\ e\left(\frac{p}{e}\right)^p, & p > 1. \end{cases} \tag{3.3}$$

The constant C_1 computed above will be used repeatedly in our further statements. In addition, we can refine one of the Hohage results we shall need in our discussions later. Namely, let us rephrase Proposition 3. 12 [6] in our notations.

Assume $x_0 = A^{-1}y \in M^1_{p,\rho}(A)$ and (3.1) is fulfilled. Then, the inequality

$$\|x_0 - x_\alpha\| \le C_1\rho\ln^{-p}(1/\alpha)$$

holds true, where $x_\alpha = (\alpha I + A^*A)^{-1}A^*y$ and C_1, as we have made sure above, satisfies (3.3).

We extend this result in the case of arbitrary $K \in \mathbb{N}$.

Lemma 1. *Let*

$$\|A\| \le M_K, \quad M_K = m_K^{1/2}, \quad m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2,\dots,K \end{cases}$$

and $x_0 = A^{-1}y \in M^K_{p,\rho}(A)$, $p > 0$, $K = 1, 2, \dots$. *Then, the following estimate*

$$\|x_0 - x_\alpha\| \le C_1\rho\Bigg(\underbrace{\ln\dots\ln}_{K\text{-times}}\frac{1}{\alpha}\Bigg)^{-p} \tag{3.4}$$

holds true, where $x_\alpha = (\alpha I + A^*A)^{-1}A^*y$, *the constant* C_1, *as we have made sure above, satisfies* (3.3).

Proof. First, let us estimate the norm

$$\begin{aligned}\|x_0 - x_\alpha\| = \Big\|\big((\underbrace{\ln\ldots\ln}_{K\text{-times}}(A^*A)^{-1}\big)^{-p} v \\ - \big(\alpha I + A^*A\big)^{-1} A^*A\big((\underbrace{\ln\ldots\ln}_{K\text{-times}}(A^*A)^{-1}\big)^{-p} v\Big\| \\ \le \rho \sup_{0<\lambda\le m_k} \left| \Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} - \frac{\lambda}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} \right| \\ = \rho \sup_{0<\lambda\le m_k} \left| \frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} \right|. \qquad (3.5)\end{aligned}$$

Now we estimate the expression standing under the supremum sign

$$\frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p}. \qquad (3.6)$$

Assume in the beginning $0 < \lambda \le \alpha$. Obviously, the function $(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda})^{-p}$ is monotonously increasing by λ. Then, from $\lambda \le \alpha$ it follows

$$\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} \le \Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\alpha}\Big)^{-p}.$$

Eventually, we have

$$\frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} \le \Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\alpha}\Big)^{-p}.$$

Assume now that $0 < \alpha \le \lambda \le m_k$. In turn, let us consider two cases.

a) $0 < p \le 1$. The expression (3.6) is transformed as follows

$$\frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p} = \frac{\alpha\lambda}{\alpha+\lambda}\hat{h}(\lambda),$$

where

$$\hat{h}(\lambda) = \frac{1}{\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\Big)^{-p}, \quad \lambda \in (0; m_k].$$

It is easy to see that

$$\hat{h}'(\lambda) = \frac{(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda})^{-p-1}}{\lambda^2 \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \frac{1}{\lambda}\cdots\ln\frac{1}{\lambda}} \left[p - \underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\lambda}\ \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \frac{1}{\lambda}\cdots\ln\frac{1}{\lambda}\right].$$

Obviously, for any $\lambda \in (0; m_k)$, we have $\hat{h}'(\lambda) < 0$. It means that the function $\hat{h}(\lambda)$ is monotonously decreasing on $(0; m_k]$. Then, $\hat{h}(\lambda) \le \hat{h}(\alpha)$ for $\lambda \ge \alpha$. Thus, we establish

$$\frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\lambda}\Big)^{-p} = \frac{\alpha\lambda}{\alpha+\lambda}\hat{h}(\lambda) \le \frac{\alpha\lambda}{\alpha+\lambda}\frac{1}{\alpha}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\alpha}\Big)^{-p}$$
$$= \frac{\lambda}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\alpha}\Big)^{-p} \le \Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\alpha}\Big)^{-p}.$$

b) It is remained to consider the case $p \ge 1$. Recall, we would like to establish validity of the following inequality

$$\frac{\alpha}{\alpha+\lambda}\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\lambda}\Big)^{-p} \le C_1\Big(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{\alpha}\Big)^{-p}, \tag{3.7}$$
$$\lambda \in (0; m_k],\ m_k = \begin{cases} e^{-1}, & k=1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2,\ldots,K \end{cases}$$

for arbitrary $K = 2, 3, 4, \ldots$.

First, let's consider the case $K = 2$, i.e. we will establish that inequality

$$\frac{\alpha}{\alpha+\lambda}\Big(\ln\ln\frac{1}{\lambda}\Big)^{-p} \le C_1\Big(\ln\ln\frac{1}{\alpha}\Big)^{-p}$$

is valid. Denote as

$$h(\lambda) := \frac{\alpha}{\alpha+\lambda}\Big(\ln\ln\frac{1}{\lambda}\Big)^{-p}, \quad \alpha \le \lambda \le e^{-e}.$$

Further, we rewrite $h(\lambda)$ as follows

$$h(\lambda) = \frac{\alpha}{\alpha+\lambda}\ln^{-p}\frac{1}{\lambda}\ln^{p}\frac{1}{\lambda}\Big(\ln\ln\frac{1}{\lambda}\Big)^{-p}$$

and consider the auxiliary function

$$v(\lambda) := \ln^{p}\frac{1}{\lambda}\Big(\ln\ln\frac{1}{\lambda}\Big)^{-p}, \quad \lambda \in \big(0; e^{-e}\big].$$

It is easy to show, that

$$v'(\lambda) = \frac{p\ln^{p-1}\frac{1}{\lambda}(\ln\ln\frac{1}{\lambda})^{-p-1}}{\lambda}\Big[1 - \ln\ln\frac{1}{\lambda}\Big]$$

and clearly, $v'(\lambda) < 0$ for any $\lambda \in (0; e^{-e})$. Hence, the function $v(\lambda)$ is monotonously decreasing on $(0; e^{-e}]$. In turn, it leads that for $\lambda \ge \alpha$ the inequality

$$v(\lambda) \le v(\alpha) \tag{3.8}$$

holds. Since $(0; e^{-e}] \subset (0;1]$, then due to (3.1) and (3.8) we have

$$h(\lambda) \le C_1 \ln^{-p} \frac{1}{\alpha} \ln^{p} \frac{1}{\alpha} \left(\ln \ln \frac{1}{\alpha} \right)^{-p} = C_1 \left(\ln \ln \frac{1}{\alpha} \right)^{-p},$$

where the constant C_1 is determined by (3.3).

To prove the inequality (3.7) in the case of arbitrary K, we will use the math induction method. Thus, for $K = 2$ the inequality (3.7) was established above. Further, assume that the inequality (3.7) is valid for some arbitrary $K-1 \ge 2$, i.e. the inequality

$$\frac{\alpha}{\alpha+\lambda} \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\lambda} \Bigg)^{-p} \le C_1 \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\alpha} \Bigg)^{-p}, \tag{3.9}$$

$$\lambda \in (0; m_{k-1}], \ m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, \ldots, K-1 \end{cases}$$

is fulfilled. It remains to prove (3.7) for K. Let's consider the function

$$\begin{aligned} \varkappa(\lambda) &:= \frac{\alpha}{\alpha+\lambda} \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda} \Bigg)^{-p} \\ &= \frac{\alpha}{\alpha+\lambda} \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\lambda} \Bigg)^{-p} \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\lambda} \Bigg)^{p} \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda} \Bigg)^{-p}, \\ &\lambda \in (0; m_k], \ K > 2. \end{aligned}$$

Denote by

$$\hat{\varkappa}(\lambda) = \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\lambda} \Bigg)^{p} \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda} \Bigg)^{-p}, \quad \lambda \in (0; m_k].$$

Now we obtain

$$\hat{\varkappa}'(\lambda) = \frac{p(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\lambda})^{p-1} (\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda})^{-p-1}}{\lambda \underbrace{\ln \ldots \ln}_{(K-2)\text{-times}} \frac{1}{\lambda} \cdots \ln \frac{1}{\lambda}} \Bigg[1 - \underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda} \Bigg].$$

Obviously, $\hat{\varkappa}'(\lambda) < 0$ for any $\lambda \in (0; m_k)$. It means that the function $\hat{\varkappa}(\lambda)$ is monotonously decreasing on interval $(0; m_{k-1}]$, hence, $\hat{\varkappa}(\lambda) \le \hat{\varkappa}(\alpha)$ when $\lambda \ge \alpha$. Since intervals $(0; m_k]$ make the sequence of embedded intervals , i.e. $(0; m_k] \subset (0; m_{k-1}]$, $k = 2, \ldots, K$, then under assumption (3.9) the inequality

$$\hat{\varkappa}(\lambda) \le C_1 \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\alpha} \Bigg)^{-p} \Bigg(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{1}{\alpha} \Bigg)^{p} \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\alpha} \Bigg)^{-p} = C_1 \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\alpha} \Bigg)^{-p}$$

holds.

Further combining all found estimates, we obtain (3.7). As a result of substitution of (3.7) into the estimation of the norm (3.5), we finally obtain

$$\|x_0 - x_\alpha\| \le C_1 \rho \Big(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\alpha} \Big)^{-p},$$

where C_1 is determined by (3.3). Thus, Lemma is completely proved. □

Remark 1. As we have already noted, for $K = 1$ this result, originally, was obtained by T. Hohage (see [6]). Further the generalization of the estimation (3.3) for arbitrary $K = 1, 2, \ldots$ was presented in monograph [2, p.38, Lemma 2] for self-adjoined non-negative operators. Thus, Lemma 1 extends result under discussion from [2] for arbitrary linear compact injective operators. Moreover, we have computed the constant C_1, as it turned out, is independent of K.

Recall (see, for instance, [19]) that for any bounded linear operator B

$$B(\alpha I + B^*B)^{-1} = (\alpha I + BB^*)^{-1}B, \quad \big\|B(\alpha I + B^*B)^{-1}B^*\big\| \le 1, \tag{3.10}$$
$$\big\|(\alpha I + B^*B)^{-1}\big\| \le \alpha^{-1}, \quad \big\|(\alpha I + B^*B)^{-1}B^*\big\| \le \frac{1}{2\sqrt{\alpha}},$$

hold true.

Lemma 2. *Let*

$$\|A\| \le M_K, \quad M_K = m_K^{1/2}, \quad m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, \ldots, K \end{cases}$$

and $x_0 = A^{-1}y \in M^K_{p,\rho}(A)$, $K = 1, 2, \ldots$. *Then, the following estimate*

$$\big\|x_0 - x^h_{\alpha,n}\big\| \le (h + \varepsilon)\rho/\sqrt{\alpha}$$

holds, where $x^h_{\alpha,n} = (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y$.

Proof. First, we note that

$$\|x_0\| = \Big\|\big(\underbrace{\ln \ldots \ln}_{K\text{-times}}(A^*A)^{-1}\big)^{-p} v\Big\| \le \rho \sup_{0<\lambda\le e^{-1}} \Big(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\lambda^2} \Big)^{-p} \le \rho,$$
$$\|A - A_{h,n}\| \le \|A - A_h\| + \|A_h - A_{h,n}\| \le h + \varepsilon.$$

Further let's estimate the norm

$$\big\|x_0 - x^h_{\alpha,n}\big\| = \big\|(\alpha I + A^*A)^{-1}A^*y - (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y\big\|.$$

For that reason we transform the last expression standing under the norm sign:

$$\begin{aligned}
&(\alpha I + A^*A)^{-1}A^*y - (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y \\
&= A^*(\alpha I + AA^*)^{-1}y - (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y \\
&= \big[A^* - (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}(\alpha I + AA^*)\big](\alpha I + AA^*)^{-1}y \\
&= (\alpha I + A^*_{h,n}A_{h,n})^{-1}\big[(\alpha I + A^*_{h,n}A_{h,n})A^* - A^*_{h,n}(\alpha I + AA^*)\big](\alpha I + AA^*)^{-1}y \\
&= (\alpha I + A^*_{h,n}A_{h,n})^{-1}\big[\alpha(A^* - A^*_{h,n}) + A^*_{h,n}(A_{h,n} - A)A^*\big](\alpha I + AA^*)^{-1}Ax_0.
\end{aligned}$$

Thus,

$$\|x_0 - x^h_{\alpha,n}\| = \|\alpha(\alpha I + A^*_{h,n}A_{h,n})^{-1}(A^* - A^*_{h,n})(\alpha I + AA^*)^{-1}Ax_0 + (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}(A_{h,n} - A)A^*(\alpha I + AA^*)^{-1}Ax_0\| \le I_1 + I_2,$$

where

$$I_1 := \alpha\|(\alpha I + A^*_{h,n}A_{h,n})^{-1}(A^* - A^*_{h,n})(\alpha I + AA^*)^{-1}Ax_0\|,$$
$$I_2 := \|(\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}(A_{h,n} - A)A^*(\alpha I + AA^*)^{-1}Ax_0\|.$$

To estimate each of the terms I_1 and I_2, we apply (2.2) and (3.10). Thus, we have

$$I_1 \le \|A^* - A^*_{h,n}\|\frac{1}{2\sqrt{\alpha}}\rho \le \frac{(h+\varepsilon)\rho}{2\sqrt{\alpha}},$$
$$I_2 \le \frac{1}{2\sqrt{\alpha}}\|A - A_{h,n}\|\rho \le \frac{(h+\varepsilon)\rho}{2\sqrt{\alpha}}.$$

Combining estimations for I_1 and I_2, we finally find

$$\|x_0 - x^h_{\alpha,n}\| \le (h+\varepsilon)\rho/\sqrt{\alpha}.$$

Thus, Lemma is proved. □

Theorem 1. *Let*

$$\|A\| \le M_K, \quad M_K = m_K^{1/2}, \quad m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, \dots, K \end{cases}$$

and $x_0 = A^{-1}y \in M^K_{p,\rho}(A)$, $K = 1, 2, \dots$. *Then, the estimate*

$$\|x_0 - x^{h,\delta}_{\alpha,n}\| \le \rho C_1 \Big(\underbrace{\ln \dots \ln}_{K\text{-times}} \frac{1}{\alpha}\Big)^{-p} + \frac{2(h+\varepsilon)\rho + \delta}{2\sqrt{\alpha}} \tag{3.11}$$

holds, where $x^{h,\delta}_{\alpha,n} = (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y_\delta$ *and* C_1 *is determined by* (3.3).

Proof. Using the triangle rule we obtain

$$\|x_0 - x^{h,\delta}_{\alpha,n}\| \le \|x_0 - x_\alpha\| + \|x_\alpha - x^h_{\alpha,n}\| + \|x^h_{\alpha,n} - x^{h,\delta}_{\alpha,n}\|$$

and estimate the last term

$$\|x^h_{\alpha,n} - x^{h,\delta}_{\alpha,n}\| = \|(\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y - (\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}y_\delta\| \le \|(\alpha I + A^*_{h,n}A_{h,n})^{-1}A^*_{h,n}(y - y_\delta)\|.$$

Thus by (3.10) we have

$$\|x^h_{\alpha,n} - x^{h,\delta}_{\alpha,n}\| \le \frac{\delta}{2\sqrt{\alpha}}. \tag{3.12}$$

Due to Lemmas 1, 2 and the relation (3.12) we finally obtain the assertion of Theorem. □

4 The Balancing Principle

We will minimizing the right-hand side of (3.11), choosing α in accordance with the balancing principle. The balancing principle consists in choosing regularization parameter α such that to balance two functions which give accuracy estimation. In our case, these functions are represented by (see (3.11))

$$\Phi(\alpha) := \rho C_1 \Big(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\alpha}\Big)^{-p}, \quad \Psi(\alpha) := \frac{2(h+\varepsilon)\rho + \delta}{2\sqrt{\alpha}}.$$

Taking into account (see (2.2)) that

$$\varepsilon = \begin{cases} \delta\rho^{-1}, & 0 < h \le \delta, \\ h, & h > \delta, \end{cases}$$

we can represent the function $\Psi(\alpha)$ as follows

$$\Psi(\alpha) = (c_1 h\rho + c_2\delta)/(2\sqrt{\alpha}),$$

where

$$c_1 = \begin{cases} 2, & h < \delta, \\ 4, & h \ge \delta, \end{cases} \qquad c_2 = \begin{cases} 3, & h < \delta, \\ 1, & h \ge \delta. \end{cases}$$

Now we rewrite (3.11) as follows

$$\big\|x_0 - x_{\alpha,n}^{h,\delta}\big\| \le \Phi(\alpha) + \Psi(\alpha). \tag{4.1}$$

Since $\varphi(t) = (\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{t})^{-p}$ is monotonously increasing function, then for increasing α the value $\Phi(\alpha)$ increases. On the other hand, the function $\Psi(\alpha)$ is monotonously decreasing. According to behavior of functions Φ and Ψ (namely, their monotonicity and concavity) choosing a value of regularization parameter $\alpha = \hat{\alpha}$ minimizing right-hand side of (3.11), will balance the values $\Phi(\alpha)$ and $\Psi(\alpha)$, i.e. $\Phi(\hat{\alpha}) = \Psi(\hat{\alpha})$. Hence,

$$\big\|x_0 - x_{\hat{\alpha},n}^{h,\delta}\big\| \le 2\Phi(\hat{\alpha}).$$

Since function Φ is unknown (namely, parameter p is unknown), then such a priori choice of the best value $\hat{\alpha}$ is practically impossible. Therefore, in considering case we need to make use of some a posteriori choice of α. For further studying we choose the balancing principle as such rule. Let us describe this principle according to our problem.

Consider discrete set of possible values of the regularization parameter

$$\triangle_N = \{\alpha_i = \big(q^2\big)^i \alpha_0,\ i = 1, 2, \ldots, N\}, \quad q > 1.$$

Here $\alpha_0 = n(h+\delta)^2$, N: $\alpha_{N+1} > m_k$, $k = 1, \ldots, K$.

Following [13] we construct the set

$$M^+(\triangle_N) = \big\{\alpha_i \in \triangle_N : \big\|x_{\alpha_i,n}^{h,\delta} - x_{\alpha_j,n}^{h,\delta}\big\| \le 4\Psi(\alpha_j),\ j = 1, \ldots, i\big\}.$$

This choice allows to realize the balancing principle. As the value of the regularization parameter we take

$$\alpha = \alpha_+ := \max\big\{\alpha \in M^+(\triangle_N)\big\}. \tag{4.2}$$

Moreover, consider the auxiliary set

$$M(\triangle_N) := \big\{\alpha_i \in \triangle_N : \Phi(\alpha_i) \le \Psi(\alpha_i),\ i = 1, 2, \dots, N\big\}$$

and the auxiliary value

$$\alpha_* := \max\big\{\alpha \in M(\triangle_N)\big\}.$$

Without loss of generality we assume that

$$M(\triangle_N) \ne \emptyset, \qquad \triangle_N \diagdown M(\triangle_N) \ne \emptyset.$$

Finally, we can estimate closeness of exact and approximate solutions for the regularization parameter $\alpha = \alpha_+$.

5 The Main Results

Theorem 2. *Assume that the regularization parameter $\alpha = \alpha_+$ is chosen according to* (4.2). *Then, for any $x_0 \in M^K_{p,\rho}(A)$ the following estimate*

$$\big\|x_0 - x^{h,\delta}_{\alpha_+,n}\big\| \le 6q\rho C_1 \Big(\underbrace{\ln \dots \ln}_{K\text{-times}} \frac{1}{\hat\alpha}\Big)^{-p}$$

is valid, where C_1 is determined by (3.3).

Proof. First, we show that $\alpha_* \le \alpha_+$. Due to (4.1), behavior of functions $\Phi(\alpha)$, $\Psi(\alpha)$ and definition of the set $M(\triangle_N)$, for any $\alpha_j < \alpha_*$, we have

$$\begin{aligned}\big\|x^{h,\delta}_{\alpha_*,n} - x^{h,\delta}_{\alpha_j,n}\big\| &\le \big\|x_0 - x^{h,\delta}_{\alpha_*,n}\big\| + \big\|x_0 - x^{h,\delta}_{\alpha_j,n}\big\| \\ &\le \Phi(\alpha_*) + \Psi(\alpha_*) + \Phi(\alpha_j) + \Psi(\alpha_j) \\ &\le 2\Phi(\alpha_*) + \Psi(\alpha_*) + \Psi(\alpha_j) \le 3\Psi(\alpha_*) + \Psi(\alpha_j) \le 4\Psi(\alpha_j).\end{aligned}$$

Thus, $\alpha_* \in M^+(\varDelta_N)$ and, hence, the inequality $\alpha_* \le \alpha_+$ is valid. Further according to (5.4), when $\alpha = \alpha_*$, and also definition of the sets $M^+(\triangle_N)$ and $M(\triangle_N)$, we obtain

$$\big\|x_0 - x^{h,\delta}_{\alpha_+,n}\big\| \le \big\|x_0 - x^{h,\delta}_{\alpha_*,n}\big\| + \big\|x^{h,\delta}_{\alpha_*,n} - x^{h,\delta}_{\alpha_+,n}\big\| \le 6\Psi(\alpha_*). \tag{5.1}$$

It is easy to see that from definition of the function Ψ it follows

$$\Psi\big(q^2\alpha_*\big) = \frac{\rho c_1 h + c_2\delta}{2\sqrt{q^2\alpha_*}} = \frac{1}{q}\cdot\frac{\rho c_1 h + c_2\delta}{2\sqrt{\alpha_*}} = \frac{1}{q}\Psi(\alpha_*). \tag{5.2}$$

On the other hand, obviously $\alpha_* \le \hat\alpha \le q^2\alpha_*$. Due to (5.1) and (5.2) we obtain

$$\big\|x_0 - x^{h,\delta}_{\alpha_+,n}\big\| \le 6q\Psi\big(q^2\alpha_*\big) \le 6q\Psi(\hat\alpha) = 6q\Phi(\hat\alpha) = 6q\rho C_1 \Big(\underbrace{\ln \dots \ln}_{K\text{-times}} \frac{1}{\hat\alpha}\Big)^{-p}.$$

Proof of Theorem is completed. □

Theorem 3. *Let $x_0 \in M_{p,\rho}^K(A)$, $K = 1, 2, \ldots$ and the condition of Theorem 2 is satisfied. Then, for sufficiently small $h, \delta > 0$ the following estimate*

$$\left\|x_0 - x_{\alpha_+,n}^{h,\delta}\right\| \le \tilde{C}_p \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-}times} \frac{1}{h+\delta}\Bigg)^{-p}$$

holds, where $\tilde{C}_p$ depends only on q, ρ, p and K.

Proof. On account of the equality $\Phi(\hat{\alpha}) = \Psi(\hat{\alpha})$ we have

$$\rho C_1 \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\hat{\alpha}}\Bigg)^{-p} = \frac{\rho c_1 h + c_2 \delta}{2\sqrt{\hat{\alpha}}}$$

or

$$\hat{\alpha} = \left(\frac{\rho c_1 h + c_2 \delta}{2\rho C_1}\right)^2 \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\hat{\alpha}}\Bigg)^{2p}.$$

Since for any

$$x > \begin{cases} 0, & K = 1, \\ 1, & K = 2, \\ \underbrace{\exp(\ldots(\exp}_{(K-2)\text{-times}}(1))), & K = 3, 4, \ldots, \end{cases}$$

the inequality $\underbrace{\ln \ldots \ln}_{K\text{-times}} x < x$ holds true, then

$$\hat{\alpha} \le \left(\frac{\rho c_1 h + c_2 \delta}{2\rho C_1}\right)^2 \left(\frac{1}{\hat{\alpha}}\right)^{2p}, \quad \hat{\alpha} \le \left(\frac{\rho c_1 h + c_2 \delta}{2\rho C_1}\right)^{\frac{2}{2p+1}}.$$

Hence, by Theorem 2 we obtain

$$\left\|x_0 - x_{\alpha_+,n}^{h,\delta}\right\| \le 6q\rho C_1 \Bigg(\underbrace{\ln \ldots \ln}_{K\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta}\right)^{\frac{2}{2p+1}}\Bigg)^{-p}, \tag{5.3}$$

where h and δ such that the condition

$$\left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta}\right)^{\frac{2}{2p+1}} \ge m_k^{-1}, \quad m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, \ldots, K \end{cases} \tag{5.4}$$

is fulfilled. First, let us estimate the accuracy of the suggested method when $K = 1$:

$$\begin{aligned} \left\|x_0 - x_{\alpha_+,n}^{h,\delta}\right\| &\le 6q\rho C_1 \left(\ln \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta}\right)^{\frac{2}{2p+1}}\right)^{-p} \\ &= 6q\rho C_1 \left(\frac{2p+1}{2}\right)^p \left(\ln \frac{2\rho C_1}{\rho c_1 h + c_2 \delta}\right)^{-p} \le \tilde{C}_1 \ln^{-p} \frac{1}{h+\delta}, \end{aligned}$$

where $\tilde{C}_1$ depends only on p, q and ρ. Then, let's show that for arbitrary $K = 2, 3, \dots$ the inequality

$$\underbrace{\ln \dots \ln}_{K\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right)^{\frac{2}{2p+1}} \geq \hat{C}_K \underbrace{\ln \dots \ln}_{K\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right) \tag{5.5}$$

is valid, where

$$\hat{C}_K = \begin{cases} 1, & 0 < p \leq \frac{1}{2}, \\ 1/\big(1 + \ln(1 + \ln(1 + \dots + \ln(1 + \ln \frac{2p+1}{2})))\big), & p \geq \frac{1}{2} \end{cases}$$

and ln is repeated $(K-1)$-times.

First, consider the case $0 < p \leq \frac{1}{2}$. Let us show that for any $K = 2, 3, \dots$ the inequality

$$\underbrace{\ln \dots \ln}_{K\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right)^{\frac{2}{2p+1}} \geq \underbrace{\ln \dots \ln}_{K\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right) \tag{5.6}$$

holds. Since for $0 < p \leq \frac{1}{2}$ the inequality

$$\left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right)^{\frac{2}{2p+1}} \geq \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right)$$

is valid, then from monotonicity of ln the inequality (5.4) is proved.

Thus, it remains to consider the case $p \geq \frac{1}{2}$. Assume at the beginning that $K = 2$, i.e. we have to establish the validity of the following inequality

$$\ln \ln \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right)^{\frac{2}{2p+1}} / \ln \ln \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right) \geq \hat{C}_2, \tag{5.7}$$

where $\hat{C}_2 = \frac{1}{1 + \ln \frac{2p+1}{2}}$. Denote by

$$t = \ln \left(\frac{2\rho C_1}{\rho c_1 h + c_2 \delta} \right).$$

Since h, δ satisfy the condition (5.4), then $t \geq \frac{2p+1}{2} e$. Further consider the function

$$u_1(t) := \frac{\ln \frac{2}{2p+1} t}{\ln t} = 1 + \frac{\ln \frac{2}{2p+1}}{\ln t}, \quad t \geq \frac{2p+1}{2} e$$

and examine its on the least value on the interval $[\frac{2p+1}{2} e; \infty)$.

Since $u_1'(t) = \ln \frac{2}{2p+1} (-\frac{1}{\ln^2 t}) \frac{1}{t} > 0$ for any $t \geq \frac{2p+1}{2} e$, then the function $u_1(t)$ is monotonously increasing by t and, hence, the least value is in the point $t = \frac{2p+1}{2} e$. It means that

$$\min_{\frac{2p+1}{2} e \leq t < \infty} u_1(t) = u_1 \left(\frac{2p+1}{2} e \right) = \frac{1}{\ln(\frac{2p+1}{2} e)} = \frac{1}{1 + \ln \frac{2p+1}{2}}.$$

Denoting by $\hat{C}_2 = \frac{1}{1+\ln\frac{2p+1}{2}}$ we obtain the inequality (5.7).

To prove the inequality (5.5) for arbitrary K, we will use the math induction method. For $K=2$ the inequality (5.5) holds (see proof of (5.7)).

Assume that the inequality (5.5) is valid for arbitrary $K-1>2$, i.e.

$$\underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} \geq \hat{C}_{K-1} \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right), \tag{5.8}$$

when $\hat{C}_{K-1} = \frac{1}{1+\ln(1+\ln(1+\cdots+\ln(1+\ln\frac{2p+1}{2})))}$, where ln is repeated $(K-2)$-times.

Let's prove (5.5). Note

$$\underbrace{\ln\ldots\ln}_{K\text{-times}}\left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} = \ln\Bigg[\underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\Bigg].$$

By monotonicity of ln and the inequality (5.8) we get

$$\ln\Bigg[\underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\Bigg] \geq \ln\Bigg[\hat{C}_{K-1} \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)\Bigg].$$

It remains to show that

$$\frac{\ln\big[\hat{C}_{K-1} \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \big(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\big)\big]}{\underbrace{\ln\ldots\ln}_{K\text{-times}}\big(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\big)} \geq \hat{C}_K.$$

Further we will use the same arguments that in the case $K=2$ and denote by

$$t = \underbrace{\ln\ldots\ln}_{(K-1)\text{-times}} \left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right).$$

Since h,δ satisfy the condition (5.4), we have $t \geq \frac{e}{\hat{C}_{K-1}}$. Now, consider the function

$$u_2(t) := \frac{\ln(\hat{C}_{K-1}t)}{\ln t} = 1 + \frac{\ln(\hat{C}_{K-1})}{\ln t}, \quad t \geq \frac{e}{\hat{C}_{K-1}}$$

and examine its on the least value on the interval $[\frac{e}{\hat{C}_{K-1}};\infty)$. Since $\ln(\hat{C}_{K-1}) < 0$, then $u_2'(t) = \ln(\hat{C}_{K-1})(-\frac{1}{\ln^2 t})\frac{1}{t} > 0$ for any $t \geq \frac{e}{\hat{C}_{K-1}}$, i.e. the function $u_2(t)$ is monotonously increasing on the interval $[\frac{e}{\hat{C}_{K-1}};\infty)$. Hence, the least value is in the point $t = \frac{e}{\hat{C}_{K-1}}$, i.e.

$$\begin{aligned}\min_{\frac{e}{\hat{C}_{K-1}} \leq t < \infty} u_2(t) = u_2\left(\frac{e}{\hat{C}_{K-1}}\right) &= \frac{1}{\ln(\frac{e}{\hat{C}_{K-1}})} \\ &= \big(1+\ln(1+\ln(1+\cdots+\ln(1+\ln\frac{2p+1}{2}))))\big)^{-1}.\end{aligned}$$

Denoting by $\hat{C}_K = \bigl(1+\ln(1+\ln(1+\cdots+\ln(1+\ln\frac{2p+1}{2})))\bigr)^{-1}$, where ln is repeated $(K-1)$-times, we obtain the inequality (5.5). By substitution of the inequality (5.5) into the estimation of the norm (5.3), we finally get

$$\begin{aligned}\left\|x_0 - x^{h,\delta}_{\alpha_+,n}\right\| &\le 6q\rho C_1\left(\hat{C}_K\underbrace{\ln\ldots\ln}_{K\text{-times}}\left(\frac{2\rho C_1}{\rho c_1 h + c_2\delta}\right)\right)^{-p} \\ &\le \tilde{C}_p\left(\underbrace{\ln\ldots\ln}_{K\text{-times}}\frac{1}{h+\delta}\right)^{-p}. \qquad (5.9)\end{aligned}$$

Thus, Theorem is proved. □

Remark 2. Notice that for $0 < p \le 1$ and $h = 0$ the result of Theorem 3 previously was obtained in [17]. In other words, Theorem 3 extends corresponding result [17] to the case of the arbitrary $0 < p < \infty$ and $h > 0$.

Remark 3. Earlier in [15] and [18], to solve equations (1.1) with $x_0 \in M^K_{p,\rho}(A)$, the standard Tikhonov method was also applied, but the stop rule, the Morozov discrepancy principle was considered. Comparing results from works under discussion with those from Theorems 2, 3 of the present paper shows that the orders of accuracy of all compared methods coincides. However, in the same time our approach has substantial advantage. Namely, by realization of the algorithm from [15, 18] the lower bound of possible values p ($p > p_0 > 0$) was used, and the value $p_0 > 0$ supposed to be given. In the present paper this restriction is removed and all results are valid for any $p > 0$.

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