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On Asymptotics of Some Fractional Differential Equations

Łukasz Płociniczak

Wroclaw University of Technology
ul. Janiszewskiego 14a, 50-372 Wroclaw, Poland
E-mail: lukasz.plociniczak@pwr.wroc.pl

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Abstract. In this paper we study the large-argument asymptotic behaviour of certain fractional differential equations with Caputo derivatives. We obtain exponential and algebraic asymptotic solutions. The latter, decaying asymptotics differ significantly from the integer-order derivative equations. We verify our theorems numerically and find that our formulas are accurate even for small values of the argument. We analyze the zeros of fractional oscillations and find the approximate formulas for their distribution. Our methods can be used in studying many other fractional equations.

Keywords: approximation, numerical method, asymptotic solution.

AMS Subject Classification: 26A33; 34E10; 45D05.

1 Introduction

Recently a number of researchers have observed that certain phenomena cannot be described accurately by usual ordinary differential equations. As it turned out, methods of fractional calculus made a successful attempt on explaining those anomalous phenomena. To name only a few, these include: anomalous diffusion [13, 16], bacterial growth [19, 39], viscoelasticity [40], fluid dynamics [2, 23], nuclear magnetic resonance (NMR) [26] and bioengineering [27]. A vast number of applications have encouraged a rapid growth of knowledge and development of methods in fractional calculus. Moreover, this field proved to be very interesting theoretically as well as it introduced some new concepts in mathematical analysis and special function theory [22]. The main objects being investigated in the field of fractional calculus are fractional differential equations [30]. Many authors still develop new methods and find new analytical results concerning fractional equations (see for example [10,20,24]). Besides the pure theoretical results, numerical methods are also being efficiently applied in solving fractional differential problems [4, 31, 41].

In this paper we investigate the asymptotic behaviour of solutions of the equations with Caputo derivative in the form

$$^{C}D^{\alpha}\big(y(x)\big) = \lambda q(x)y(x), \quad 0 < \alpha < 1, \text{ or } 1 < \alpha < 2, \tag{1.1}$$

where q is asymptotically polynomial as $x \to \infty$. The formulas we obtain are generalizations of the classical Green–Liouville or WKB approximation of the second order ordinary differential equations. Some results on fractional WKB method were also obtained in [38]. In the series of papers [3, 5] the authors dealt with power-function asymptotic behaviour of some linear and nonlinear fractional differential equations with different variants of fractional derivatives. These results have been generalized in [6]. The case of linear asymptotic behaviour was also treated in [29].

In our work we look at algebraically decaying and exponentially large–argument behaviour of (1.1). All proofs are based on the assumption of polynomial growth of $q(x)$ and the Laplace method for asymptotic integrals. When α is an integer, our results reduce to the classical formulas known from asymptotic theory. As a side-result we state two lemmas which show the large-argument behaviour of fractional integrals of power and exponential functions. In further work these lemmas can be used for proving similar asymptotic theorems concerning different classes of fractional differential equations. These results are important since they give both qualitative and quantitative characterization of equations governing fractional dynamics. We also indicate a different than ordinary behaviour of solutions of fractional equations like algebraic rather than exponential decay in some cases. Many of our findings were known before only for the case of Mittag-Leffler functions. We give a more general description of solutions to fractional equations.

When $1 < \alpha < 2$ and $\lambda < 0$ the equation (1.1) admits a behaviour known as fractional oscillations, which was observed before in the case of Mittag-Leffler functions (see for example [14]). These oscillations possess a finite number of periods which makes this case different from the classical result. A more general case of fractional oscillator and the corresponding Lagrangian was investigated in [7], where the exact series solutions were obtained. Recently, a study of a general class of non-homogeneous fractional equations has been initiated in [17], where authors gave conditions for oscillations to occur. A thorough investigation of a forced fractional oscillator was undertaken in [33]. In that work, authors described physical characteristics of an oscillator given by the equation (1.1) with $q(x) = 1$ and non-homogeneous forcing term driving the oscillations. Our results are generalizations for free oscillations with $q(x)$ being polynomial.

To verify the asymptotic results we use numerical analysis and find that obtained approximations are accurate even for small values of argument. This suggests that our asymptotic solutions can be used in applications for example as an approximation to the fractional Schrödinger equation or bacterial population dynamics (see for example [25,32]). We also analyze distribution of zeros of fractional oscillations described by equation (1.1). Some authors obtained similar results for the case of Mittag-Leffler functions (for example [1,15,18,28,37]).

This paper is structured as follows. First, we review some preliminary definitions and facts from fractional analysis. Then we prove our main results concerning the asymptotics of the fractional equations. Next, we provide some examples and numerical analysis of the stated solutions. We conclude with an analysis of zeros of the fractional oscillations. The results presented in this paper generalize some well-known facts concerning asymptotic solutions to the second order ordinary differential equations. We obtain new formulas that characterize large-argument behaviour for a general class of fractional oscillations. We believe that our results might be useful in describing many natural phenomena governed by fractional dynamics.

2 Fractional Calculus Preliminaries

In this section we summarize some basic facts from the fractional analysis. A very detailed and thorough expositions can be found in [21, 36].

The definition of a fractional integral is a straightforward generalization of the classical formula for a multiple integral. We define the fractional integral of order α by

$$I_a^\alpha\big(y(x)\big) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}y(t)\,dt. \tag{2.1}$$

In what follows we will always choose the lower terminal $a=0$ and suppress it from the notation, i.e. we will always write I^α. As it will turn out, our results are independent from the choice of a.

Riemann–Liouville fractional derivative of order α is defined on a suitably chosen function space by the formula

$$D^\alpha\big(y(x)\big) = \frac{d^n}{dx^n}I^{n-\alpha}\big(y(x)\big) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}y(t)\,dt, \tag{2.2}$$

where $n=[\alpha]+1$. As it turns out, this definition introduces some issues in defining suitable initial conditions for physical processes, thus we will work with the Caputo fractional derivative

$${}^C D^\alpha\big(y(x)\big) = I^{n-\alpha}\big(y^{(n)}(x)\big) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}y^{(n)}(t)\,dt, \tag{2.3}$$

where $n=[\alpha]+1$. By a straightforward calculation we see that when α is an integer, the fractional derivatives coincide with the classical ones. The fractional integral acts on power functions in the following, natural way

$$I^\alpha(x^\beta) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha)}x^{\beta+\alpha}, \quad \beta>-1. \tag{2.4}$$

In what follows, we will also need a formula for a composition of Caputo fractional derivative and the fractional integral

$$I^\alpha\big({}^C D^\alpha\big(y(x)\big)\big) = y(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}x^k. \tag{2.5}$$

3 Analytical Results

In this section we will state our main results concerning asymptotics of fractional differential equations. First, we prove two technical lemmas giving asymptotic behaviour of fractional integrals of exponential and algebraic functions. Later, we will use these results in the analysis of fractional equations.

It will always be understood that we investigate the asymptotics when $x \to \infty$ even when it is not indicated explicitly.

Lemma 1. *Let $p(x)$ and $q(x)$ be positive functions with asymptotic polynomial growth, say $p(x) \sim C_1 x^\mu$ and $q(x) \sim C_2 x^\nu$ as $x \to \infty$ and $\nu, \mu, C_i > 0$. Then the following asymptotic behaviour occurs*

$$I^\alpha\big(p(x)e^{q(x)}\big) \sim q'(x)^{-\alpha} e^{q(x)} \Bigg(p(x) - \alpha p'(x) q'(x)^{-1} + \frac{\alpha(\alpha+1)}{2} p(x) q'(x)^{-2} q''(x) + o\big(x^{\mu-\nu}\big)\Bigg). \tag{3.1}$$

Proof. We will obtain an asymptotic expansion of the fractional integral by the Laplace method (see [8, 34]). Using formula (2.1) we have

$$I^\alpha\big(p(x)e^{q(x)}\big) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} p(t) e^{q(t)}\, dt. \tag{3.2}$$

With a change of variable $t = x(1-s)$ we fix the integration limits

$$I^\alpha\big(p(x)e^{q(x)}\big) = \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 s^{\alpha-1} p\big(x(1-s)\big) e^{q(x(1-s))}\, ds. \tag{3.3}$$

Fix an arbitrarily small cutoff parameter $0 < \delta < 1$ and separate the integral into two parts

$$I^\alpha(p(x)e^{q(x)}) = \int_0^\delta + \int_\delta^1 = J_1 + J_2, \tag{3.4}$$

where integrands are the same as in (3.3). Since $q(x)$ is monotone increasing for $x \to \infty$, the exponential in the integrand of (3.3) has a maximum for $s = 0$. That is, the mass of the integral is concentrated near the origin. Guided by this remark we claim that the integral over $(\delta, 1)$ is exponentially small with respect to $p(x)\exp q(x)$. To see this, note that p is monotone increasing when $x \to \infty$ and

$$\frac{\Gamma(\alpha)}{x^\alpha} J_2 = \int_\delta^1 s^{\alpha-1} p\big(x(1-s)\big) e^{q(x(1-s))}\, ds \le e^{q(x(1-\delta))} p\big(x(1-\delta)\big) \int_\delta^1 s^{\alpha-1}\, ds \le \frac{1}{\alpha} e^{q(x(1-\delta))} p\big(x(1-\delta)\big) = o\big(p(x)e^{q(x)}\big). \tag{3.5}$$

Thus, we note that the behaviour of (3.3) is determined by the integral over $(0, \delta)$. Now, expanding $p(x(1-s))$ and $q(x(1-s))$ into Taylor series in the neighborhood of $s = 0$ we get

$$p\big(x(1-s)\big) = p(x) - xp'(x)s + \frac{1}{2}x^2 p''(\xi_1) s^2,$$

$$q\big(x(1-s)\big) = q(x) - xq'(x)s + \frac{1}{2}x^2q''(x)s^2 - \frac{1}{6}x^3q'''(\xi_2)s^3. \tag{3.6}$$

Similarly, we have

$$e^{q(x(1-s))} = e^{q(x)}e^{-xq'(x)s}\left(1 + \frac{1}{2}x^2q''(x)s^2 + \frac{1}{8}x^4q''(x)^2s^4e^{\xi_3}\right)e^{-\frac{1}{6}x^3q'''(\xi_2)s^3}. \tag{3.7}$$

Since δ is close to zero, all of these Taylor expansions are valid on $(0, \delta)$. Since $q'(x) > 0$ (by monotonicity) we can change the variable $t = xq'(x)s$ and observe that by combining (3.6) and (3.7) we get

$$\begin{aligned} J_1 \sim \frac{q'(x)^{-\alpha}}{\Gamma(\alpha)}e^{q(x)} \int_0^{xq'(x)\delta} t^{\alpha-1}e^{-t}\Big(p(x) - p'(x)q'(x)^{-1}t \\ + \frac{1}{2}p(x)q'(x)^{-2}q''(x)t^2 + o\big(x^{\mu-\nu}\big)\Big)\, dt. \end{aligned} \tag{3.8}$$

In (3.8) we recognize the formula for the Incomplete Gamma function. As it can easily be shown by integration by parts, it has the following asymptotic behaviour

$$\int_0^{xq'(x)\delta} t^{\alpha-1}e^{-t}\, dt \sim \Gamma(\alpha), \quad \text{as } x \to \infty. \tag{3.9}$$

Taking this into account we obtain the asymptotic expansion for J_1 and thus by (3.5) for I^α. This concludes the proof. □

The next lemma shows the asymptotic behaviour of fractional integral of a function which falls algebraically at infinity.

Lemma 2. *If $y(x)$ is an algebraically decaying function, say $y(x) \sim Cx^{-\beta}$ when $x \to \infty$, then for $0 < \alpha < 1$ we have*

$$I^\alpha\big(y(x)\big) \sim \begin{cases} \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)}x^\alpha y(x), & \beta < 1, \\ C\frac{x^{\alpha-1}\log(1+x)}{\Gamma(\alpha)}, & \beta = 1, \\ \frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(t)\, dt, & \beta > 1, \end{cases} \tag{3.10}$$

as $x \to \infty$.

Proof. When $y(x) \sim Cx^{-\beta}$ with $\beta < 1$ by (2.4) we obtain

$$I^\alpha\big(y(x)\big) \sim CI^\alpha\big(x^{-\beta}\big) = C\frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)}x^{\alpha-\beta} \sim \frac{\Gamma(1-\beta)}{\Gamma(1-\beta+\alpha)}x^\alpha y(x) \tag{3.11}$$

since $x^{-\beta}$ is integrable. For $\beta > 1$ we observe that

$$I^\alpha\big(y(x)\big) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_0^x \left(1 - \frac{t}{x}\right)^{\alpha-1} y(t)\, dt \sim \frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(t)\, dt \tag{3.12}$$

by monotone convergence theorem and the fact that $y(x)$ is integrable at infinity. For $\beta = 1$ we have $y(x) \sim Cx^{-1} \sim C/(1+x)$ and similarly

$$I^{\alpha}(y(x)) \sim C\frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_0^x \frac{dt}{1+t} = C\frac{x^{\alpha-1}\log(1+x)}{\Gamma(\alpha)}. \tag{3.13}$$

Thus, we have proved our claims. $\square$

Now, we prove our main result concerning an asymptotic behaviour of a solution to the particular fractional differential equations. This result is a generalization of classical Green–Liouville approximation. Also, WKB perturbation method applied for Schrödinger equation gives a similar result as for ordinary differential equations. In the fractional setting we cannot rely on simple formulas for the derivative of the product and composite function used in those methods. Thus we have to proceed differently. The most striking property of solutions to differential equations of fractional order is that they admit an algebraic, rather than exponential decay at infinity. This result was previously obtained for Mittag-Leffler functions, which are the solutions to a particular case of equations we consider. A necessary information about existence and uniqueness of solutions of fractional differential equations is given in [9, 21]. In the proofs we use the fact that homogeneous, linear equation of order $n-1 < \alpha \leq n$ has n linearly independent solutions.

Theorem 1. *Let $y(x)$ be a solution of the following fractional differential equations of order α*

$${}^{C}D^{\alpha}\big(y(x)\big) = \lambda q(x)y(x), \quad 0 < \alpha < 1 \tag{3.14}$$

with $q(x) \sim C_q x^{\mu} > 0$, $\mu > 0$. Then for $\lambda > 0$ it has the following behaviour as $x \to \infty$

$$y(x) \sim C_1 \exp\biggl(\lambda^{\frac{1}{\alpha}} \int q(x)^{\frac{1}{\nu}}\, dx\biggr), \tag{3.15}$$

while for $\lambda < 0$ we have

$$y(x) \sim \frac{y(0)}{1 - \lambda\Gamma(1-\alpha)q(x)x^{\alpha}}, \tag{3.16}$$

where the α-root lie on the principal branch and C_1 is some constant.

Proof. Assume that $\lambda > 0$, and seek for exponentially growing solutions. Using the formula for a composition of Caputo fractional derivative and fractional integral (2.5) we rewrite (3.14) as

$$y(x) = y(0) + \lambda I^{\alpha}\big(q(x)y(x)\big). \tag{3.17}$$

We see that as $x \to \infty$ the constant term on the right-hand side is subdominant with respect to the exponential behaviour, thus only the second term is significant for the asymptotics

$$y(x) \sim \lambda I^{\alpha}\big(q(x)y(x)\big). \tag{3.18}$$

Now, we want to find an exponential asymptotic solution to (3.18), thus we look for

$$y(x) = e^{r(x)}. \tag{3.19}$$

This form of $y(x)$ will simplify the reasoning. The exponential behaviour at infinity is often seen in linear equations (for a thorough analysis of asymptotic behaviours see for example [8]). We plug this form of $y(x)$ into (3.18) and use Lemma 1 retrieving only the first term. We obtain

$$e^{r(x)} \sim \lambda q(x) r'(x)^{-\alpha} e^{r(x)}. \tag{3.20}$$

We cancel the exponentials and set $r'(x) = (\lambda q(x))^{1/\alpha}$ which makes (3.20) satisfied and together with (3.19) gives us the first part of the assertion. Notice that the assumption of $\lambda > 0$ is crucial for the $\lambda^{1/\alpha}$ to be meaningful as $\alpha \to 0$.

Now, assume that $\lambda < 0$ and look for solutions which decay algebraically, say $y(x) \sim Cx^{-\beta}$. Applying Lemma 2 (with $q(x)y(x)$ instead of $y(x)$) to (3.17) and recalling that $q(x) \sim C_q x^{\mu}$, for $\beta - \mu < 1$ we get

$$y(x) \sim y(0) + \lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} q(x) x^{\alpha} y(x). \tag{3.21}$$

Notice that the cases $\beta - \mu \geq 1$ in Lemma 2 introduce contradiction with the assumed form of $y(x)$. We rearrange terms in (3.21) and obtain

$$y(x) \sim \frac{y(0)}{1 - \lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} q(x) x^{\alpha}}.$$

This gives us $\beta = \alpha + \mu$ which is consistent with assumption that $\beta - \mu < 1$. The theorem is proved. □

Theorem 2. *If $y(x)$ is a solution of the fractional differential equation*

$$^{C}D^{1+\alpha}\big(y(x)\big) = \lambda q(x) y(x), \quad 0 < \alpha < 1 \tag{3.22}$$

with $q(x) \sim C_q x^{\mu} > 0$, $\mu > 0$, then for $\lambda > 0$ it has the following asymptotic behaviour as $x \to \infty$

$$y(x) \sim C_1 q(x)^{-\frac{\alpha}{2(1+\alpha)}} \exp\big(\lambda^{\frac{1}{1+\alpha}} r_{1+\alpha}(x)\big). \tag{3.23}$$

For $\lambda < 0$ the fractional oscillations occur:

$$\begin{aligned} y(x) \sim\ & C_2 q(x)^{-\frac{\alpha}{2(1+\alpha)}} e^{|\lambda|^{\frac{1}{1+\alpha}} \cos(\frac{\pi}{1+\alpha}) r_{1+\alpha}(x)} \sin\left(|\lambda|^{\frac{1}{1+\alpha}} \sin\left(\frac{\pi}{1+\alpha}\right) r_{1+\alpha}(x)\right) \\ & + C_3 q(x)^{-\frac{\alpha}{2(1+\alpha)}} e^{|\lambda|^{\frac{1}{1+\alpha}} \cos(\frac{\pi}{1+\alpha}) r_{1+\alpha}(x)} \cos\left(|\lambda|^{\frac{1}{1+\alpha}} \sin\left(\frac{\pi}{1+\alpha}\right) r_{1+\alpha}(x)\right) \\ & - \frac{y'(0)}{\lambda \Gamma(1-\alpha) q(x) x^{\alpha}} \end{aligned} \tag{3.24}$$

where all $(1+\alpha)$-roots are on the principal branch, C_i are some constants and we have denoted $r_{\nu}(x) := \int q(x)^{\frac{1}{\nu}}\, dx$.

Proof. Let $\lambda > 0$, and similarly as before, using (2.5) we obtain

$$y'(x) \sim \lambda I^{\alpha}\big(q(x)y(x)\big). \tag{3.25}$$

First, we will find an exponential behaviour for $y(x)$ using the same approach as in the proof of Theorem 1. After that we will include more terms in the expansion from Lemma 1 to find the factor that stands next to the exponent. To this end, let $y(x)$ be as follows $y(x) = e^{r(x)}$. Lemma 1 applied to (3.25) gives us

$$r'(x)e^{r(x)} \sim \lambda q(x) r'(x)^{-\alpha} e^{r(x)}.$$

We cancel exponentials and in order to fulfill this statement set

$$r(x) := \int \big(\lambda q(x)\big)^{\frac{1}{1+\alpha}}\, dx. \tag{3.26}$$

This is the leading-order exponential behaviour for $y(x)$.

Now, we turn to the analysis of a non-exponential factor. For large x, we seek for $y(x)$ which has the following form $y(x) = f(x)e^{r(x)}$, where f is the correction to the previously found exponential (3.26). We plug $y(x)$ into (3.25) and obtain

$$\big(f'(x) + f(x)r'(x)\big)e^{r(x)} \sim \lambda I_0^{\alpha}\big(q(x)f(x)e^{r(x)}\big). \tag{3.27}$$

After applying Lemma 1 with all the terms retained, (3.27) transforms into

$$\begin{aligned} f'(x) + f(x)r'(x) \sim \lambda r'(x)^{-\alpha}\Big(&f(x)q(x) - \alpha\big(f(x)q(x)\big)' r'(x)^{-1} \\ &+ \frac{\alpha(\alpha+1)}{2} f(x)q(x)r'(x)^{-2}r''(x)\Big), \end{aligned} \tag{3.28}$$

where the exponentials have been cancelled. All the terms in the expansion are necessary to obtain the whole contribution to the behaviour of f. This can be verified by a simple computation. By (3.26) we have $r'(x)^{-\alpha} = \lambda r'(x)/q(x)$ and (3.28) reduces to

$$\begin{aligned} (1+\alpha)f'(x) &\sim \Big(\frac{\alpha(\alpha+1)}{2} r'(x)^{-1}r''(x) - \alpha q'(x)q(x)^{-1}\Big) f(x) \\ &= -\Big(\frac{\alpha}{2}\big(\log q(x)\big)' + \alpha\big(\log q(x)\big)'\Big) f(x) \end{aligned} \tag{3.29}$$

for $(\log(\lambda))' = 0$. Finally, we have

$$(1+\alpha)f'(x) \sim -\frac{\alpha}{2}\big(\log q(x)\big)' f(x). \tag{3.30}$$

We see that $f(x) = C_1 q(x)^{-\frac{\alpha}{2(1+\alpha)}}$ which proves the exponential case of $\lambda > 0$. For the equation with $\lambda < 0$ we observe, that

$$\lambda^{\frac{1}{1+\alpha}} = |\lambda|^{\frac{1}{1+\alpha}}(-1)^{\frac{1}{1+\alpha}} = |\lambda|^{\frac{1}{1+\alpha}} e^{i\frac{\pi}{1+\alpha}}. \tag{3.31}$$

Since $1 < 1+\alpha < 2$ the root of negative number always lies on principal branch. By the known uniqueness results for fractional equations we do not include roots with different argument. Taking real and imaginary parts of the right-hand side of (3.31) we obtain an oscillatory form of $y(x)$.

The last to find is the algebraically decaying behaviour when $\lambda < 0$. Similarly as before, we use the formula for a composition of Caputo derivative and fractional integral (2.5) and then from (3.22) obtain

$$y'(x) = y'(0) + \lambda I^{\alpha}\big(q(x)y(x)\big). \tag{3.32}$$

Assume that $y(x) \sim Cx^{-\beta}$ as $x \to \infty$. Now, use Lemma 2 to obtain the asymptotic relation for the fractional integral of $q(x)y(x)$. Recall that $q(x) \sim C_q x^{\mu}$. We immediately see from Lemma 2 and (3.32) that the case of $\beta - \mu \geq 1$ would lead to a contradiction with the boundedness of $y(x)$ as $x \to \infty$. We thus assume that $\beta - \mu < 1$, what gives us

$$y'(x) \sim y'(0) + \lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q x^{\alpha+\mu} y(x), \tag{3.33}$$

since $q(x) \sim C_q x^{\mu}$. This is an asymptotic differential relation, which has the solution

$$y(x) \sim e^{\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q \frac{x^{1+\alpha+\mu}}{1+\alpha+\mu}} \left(y(0) + y'(0) \int_0^x e^{-\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q \frac{t^{1+\alpha+\mu}}{1+\alpha+\mu}} \, dt \right). \tag{3.34}$$

Since $\lambda < 0$, only the second term on the right-hand side gives contribution to the leading behaviour of $y(x)$, that is

$$\begin{aligned} y(x) &\sim y'(0) e^{\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q \frac{x^{1+\alpha+\mu}}{1+\alpha+\mu}} \int_0^x e^{-\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q \frac{t^{1+\alpha+\mu}}{1+\alpha+\mu}} \, dt \\ &\sim -\frac{y'(0)}{\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} C_q x^{\alpha+\mu}} \sim -\frac{y'(0)}{\lambda \frac{\Gamma(1+\mu-\beta)}{\Gamma(1+\mu-\beta+\alpha)} q(x) x^{\alpha}}. \end{aligned} \tag{3.35}$$

The last asymptotic equivalence can be easily verified by integration by parts. We see that eventually we have $\beta = \alpha + \mu$, which is indeed consistent with the assumption that $\beta - \mu < 1$. This ends the proof of the theorem. □

We immediately see many differences between the solutions of ordinary and fractional differential equations. In the ordinary differential equation $y'(x) = \lambda q(x) y(x)$ we have an asymptotically exponential decaying solution for $\lambda < 0$. This is not the case of fractional equation, where exponential behaviour is replaced by an algebraic one. Also, notice that the formula for exponential behaviour in Theorem 1 becomes exact if $\alpha = 1$. The case of an equation of order $1+\alpha$ is even more interesting. Here, we see that in $\lambda < 0$ case our equation always admits exponentially decaying solutions as well as the algebraic ones. For ordinary differential equations we observe only oscillatory solutions. This phenomenon is known as fractional oscillations and has been investigated by a number of authors in a model case of Mittag-Leffler function.

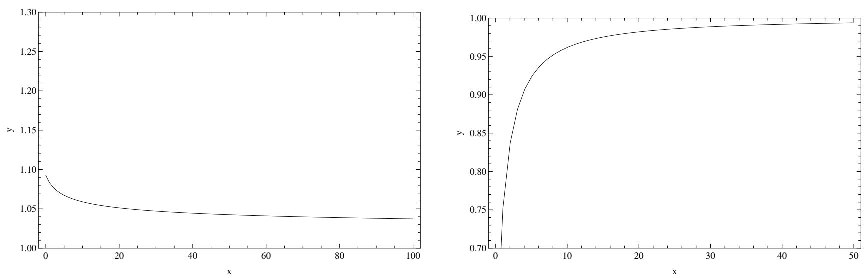
Note also that as it is indicated in [36] the behaviour of fractional derivative for $x \to \infty$ does not depend on the lower terminal (here: 0). Hence, our result is valid for ${}^C D^{\alpha}_a$ for any a. We also remark that as in perturbation theory, asymptotic formulas (3.15) and (3.16) should become more accurate if $|\lambda|^{-1} =: \epsilon \ll 1$. We will see in the next section that this is indeed true.

4 Examples and Numerical Analysis

All the numerical analysis of fractional differential equations presented in this section was done using MATLAB code for predictor-corrector PECE method. The code is obtained from the Web page MATLAB Central File Exchange [12]. It uses Adams–Bashforth–Moulton method which was described in [11]. The numerical examples of fractional integrals were computed using MATHEMATICA. In the numerical examples we use different values of α to show various behaviours of the solutions. This shows how the fractional equation is different from the classical.

4.1 Asymptotics of fractional integral

Lemma 2 gives the asymptotic behaviour for algebraically decaying functions such as $y(x) \sim x^{-\beta}$. As the behaviour of the case when $\beta < 1$ is obvious the other cases are more interesting. As an example we use $y(x) = (1 + x^{\beta})^{-1}$. Plots of $I^{\alpha}(y)/y_{asym}$ are shown on Fig. 1.



a) b)

Figure 1. Plots of $I^{\alpha}(y)/y_{asym}$ as in Lemma 2 for $y(x) = (1 + x^{\beta})^{-1}$ with a) $\beta = 1$, b) $\beta = 2$.

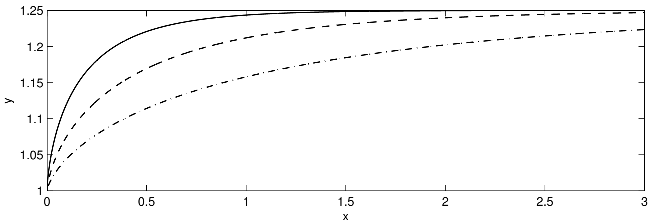
We have also used elementary integral $\int_0^{\infty} (1 + t^{\beta})^{-1}\, dt = \pi/(\beta \sin(\pi/\beta))$. We can see that the asymptotic accuracy for $\beta = 2$ is much faster than when $\beta = 1$, as might have been expected by the slowness of logarithmic convergence. For example, when $\beta = 1$ we get the accuracy of one decimal number just as when $x > 0.5$ but the accuracy for the second decimal number is obtained only for $x > 10^8$.

4.2 Basic fractional equations

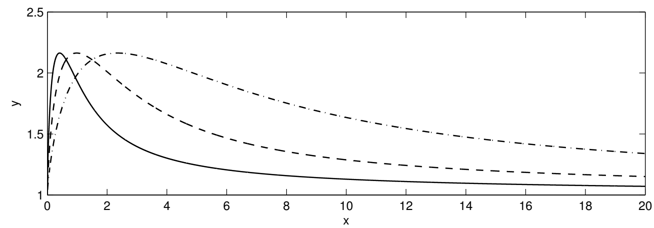
As an illustration of the asymptotics of the solutions of fractional differential equations we present several numerical examples. If we consider an equation

$$^{C}D^{\alpha} y(x) = \lambda y(x), \quad 0 < \alpha < 1, \tag{4.1}$$

then from Theorem 1 we obtain a simple result that $y(x) \sim C \exp(\lambda^{\frac{1}{\alpha}} x)$ for $\lambda > 0$ and $y(x) \sim y(0)/(1 - \lambda \Gamma(1 - \alpha) x^{\alpha})$ when $\lambda < 0$. It is known that a solution to (4.1) can be expressed with the aid of Mittag-Leffler functions and our theorem gives their correct asymptotic expansion up to a constant (see [35]). The particular case of $y(0) = 1$, $\lambda = \pm 0.5, \pm 1, \pm 2$ and $\alpha = 0.8$ is depicted on Fig. 2.



a)



b)

Figure 2. Plots of y_{exact}/y_{asym} as in (4.1) for $C = y(0)$, $\alpha = 0.8$, $\lambda = 0.5$ (dot-dash) a) 1 (dash) and 2 (solid line), b) -1 (dash) and -2 (solid line) .

We see that the greater $|\lambda|$ the better the approximation. Also notice that for exponential behaviour a good approximation is obtained for x of order of unity. We also see that the approximations of algebraically decaying solutions converge more slowly than in the exponential case.

For the next example consider the case of $q(x) = x^{\mu}$. We have

$$r_{\nu}(x) = \int q(x)^{\frac{1}{\nu}}\, dx = \frac{1}{1 + \mu/\nu} x^{1+\frac{\mu}{\nu}}.$$

Consider the equation

$$^{C}D^{1+\alpha} y(x) = \lambda x^{\mu} y(x), \quad 0 < \alpha < 1. \tag{4.2}$$

Due to Theorem 1 we see that an asymptotic form of solution to (4.2) (up to a constant) will have the form $x^{-\frac{\alpha\mu}{2(1+\alpha)}} \exp(\lambda^{\frac{1}{1+\alpha}} \frac{1}{1+\frac{\mu}{1+\alpha}} x^{1+\frac{\mu}{1+\alpha}})$ when $\lambda > 0$ and with appropriate changes when $\lambda < 0$. For example, when $\mu = 1$ equation (4.2) generalizes the well-known Airy equation and for its solution the asymptotic behaviour of our formula coincides with classical as $\alpha \to 1$ (see for example [8]). The plot of this case with $\lambda > 0$ is presented on Fig. 3. Once again, we see that the greater $|\lambda|$ the better the approximation.

4.3 Zeros of fractional oscillations

As was indicated before, when $\lambda < 0$ equation (3.22) models the so-called fractional oscillations. These oscillations eventually diminish so that only monotone

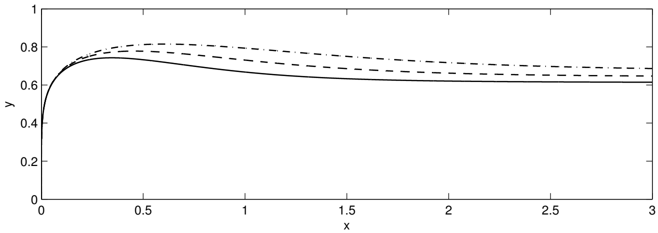


Figure 3. Plot of y_{exact}/y_{asym} as in (4.2) for $\mu = 1$, $\alpha = 0.5$, $\lambda = 0.5$ (dot-dash), 1 (dash) and 2 (solid line).

decreasing behaviour is present. That kind of oscillatory asymptotic behaviour is described by equation (3.24). It is very interesting to find the number of oscillations and the furthest point in which the last zero occurs. Recently, in [17] a study of fractional oscillations has been published. According to these results, since our equation (1.1) is homogeneous it can admit oscillations with finite number of zeros. The frequency of the oscillations in approximation (3.24) is the same as in [33] (with the time-scale λ).

As $x \to \infty$ in (3.24) there is a moment at which the algebraic decay starts to dominate. This gives us a clue how to proceed since after that moment no zero should occur. As an example, let us try to find the largest zero of (3.22) with $y(0) = 0$ and $q(x) = x^\mu$. By the initial conditions, in the asymptotic formula (3.24) we leave only the term with sine function. We can approximate the furthest zero by the solution to the algebraic equation

$$x_0^{-\frac{\mu\alpha}{2(1+\alpha)}} \exp\left(\cos\left(\frac{\pi}{1+\alpha}\right)\frac{1}{1+\frac{\mu}{1+\alpha}} x_0^{1+\frac{\mu}{1+\alpha}}\right) \approx \frac{y'(0)}{-\lambda\Gamma(1-\alpha)x_0^{\alpha+\mu}},$$

that is, x_0 occurs when the algebraic behaviour starts to dominate over the exponential. We have also taken the multiplicative constant in front of the exponent to be equal to 1. When we turn to the oscillatory part of our approximation we can determine not only the number but also the approximate location of zeros of the fractional oscillations. Since when we know x_0 we are not interested in the oscillations' amplitude but rather in their period. If the algebraically decaying part was absent the zeros would occur approximately when

$$\sin\left(|\lambda|^{\frac{1}{1+\alpha}} \sin\left(\frac{\pi}{1+\alpha}\right)\frac{1}{1+\frac{\mu}{1+\alpha}} x^{1+\frac{\mu}{1+\alpha}}\right) = 0,$$

that is, for

$$x_k := \left(\left(1+\frac{\mu}{1+\alpha}\right)\frac{k\pi}{|\lambda|^{\frac{1}{1+\alpha}}\sin(\frac{\pi}{1+\alpha})}\right)^{\frac{1}{1+\frac{\mu}{1+\alpha}}}. \tag{4.3}$$

What we are looking for are the zeros of the equation

$$\sin\left(|\lambda|^{\frac{1}{1+\alpha}} \sin\left(\frac{\pi}{1+\alpha}\right)\frac{1}{1+\frac{\mu}{1+\alpha}} x^{1+\frac{\mu}{1+\alpha}}\right) - \frac{y'(0)}{\lambda\Gamma(1-\alpha)x^{\alpha+\mu}} = 0. \tag{4.4}$$

To simplify the notation, we will write (4.4) as $f(x)+g(x)=0$. Using techniques similar to those in perturbation theory we look for approximate solutions to (4.4) in the form $\widetilde{x_k}=x_k+\xi_k$, where ξ_k is a small correction to the (4.3). If x_k is a solution to $f(x)=0$ we want to have $f(\widetilde{x_k})+g(\widetilde{x_k})=0$, which to the first approximation is

$$f(x_k)+f'(x_k)\xi_k+g(x_k)+g'(x_k)\xi_k=0 \quad \Rightarrow \quad \xi_k=-\frac{g(x_k)}{f'(x_k)+g'(x_k)}. \tag{4.5}$$

This approximation of the zeros of (4.4) becomes more accurate if g is smaller, that is when the largest zero is far from the origin. By (3.24) this happens when α is close to 1. The results of our approximation for Mittag-Leffler function ($q(x)=1$) are shown in Table 1.

Table 1. Zeros of fractional oscillations as in (3.14) with $y(0)=0$, $y'(0)=1$, $\alpha=0.75$, $\lambda=-2$ and $q(x)=1$ (Mittag-Leffler function). Zeros in rows from the top: exact solution, approximate solution (3.16) and approximate formulas (4.4) and (4.5).

zero	1	2	3	4
y	2.450	4.326	6.949	8.406
y_{asym}	2.270	4.211	6.711	8.364
$f+g$	2.221	4.305	6.528	8.655

In the Table 2 we can see the values of calculated zeros for generalized Airy function ($q(x)=x$).

Table 2. Zeros of fractional oscillations as in (3.14) with $y(0)=0$, $y'(0)=1$, $\alpha=0.75$, $\lambda=-2$ and $q(x)=x$. Zeros in rows from the top: exact solution, approximate solution (3.16) and approximate formulas (4.4) and (4.5).

zero	1	2	3	4	5	6
y	2.130	3.307	4.367	5.181	6.102	6.678
y_{asym}	2.218	3.361	4.426	5.222	6.158	6.706
$f+g$	2.215	3.372	4.403	5.261	6.085	6.816

Also, on the Fig. 4 a plot of this approximation is depicted. All calculations were done using $\lambda=2$, $\alpha=0.75$ and $\mu=1$. Again we see that even for small arguments our asymptotic approximation is very good and that the number of zeros differs between the two analyzed cases.

5 Conclusions

In this paper we presented an analysis of the large-argument asymptotic behaviour of some fractional differential equations with Caputo derivatives. We found the exponential asymptotic behaviours and the algebraically decaying ones. These decaying solutions are the characteristic aspects of fractional oscillations, which are not present in ordinary differential equations. Numerical

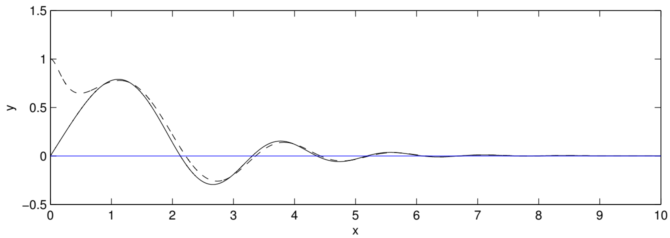


Figure 4. Fractional oscillations as in (3.14) with $y(0) = 0$, $y'(0) = 1$, $\alpha = 0.75$, $\lambda = -2$ and $q(x) = x$ with exact solution (solid line) and asymptotic approximation (dash).

analysis has verified that the asymptotics are converging to the exact solutions very quickly. Our results concerning exponential behaviours are a generalization of the WKB or Green–Liouville approximations of the solutions of second order ordinary differential equations. These fractional approximations may find theirs application for example in fractional quantum mechanics, biological sciences and many different fields where fractional dynamics is used to model natural phenomena.

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