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γ-Agregation Operators and Some Aspects of Generalized Aggregation Problem*

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Abstract. We explore questions related to the aggregation operators and aggregation of fuzzy sets. No preliminary knowledge of the aggregation operators theory and of the fuzzy sets theory are required, because all necessary information is given in Section 2. Later we introduce a new class of γ-aggregation operators, which "ignore" arguments less than γ. Due to this property γ-aggregation operators simplify the aggregation process and extend the area of possible applications. The second part of the paper is devoted to the generalized aggregation problem. We use the definition of generalized aggregation operator, introduced by A. Takaci in [7], and study the pointwise extension of a γ-agop.

Keywords: aggregation operator, γ-aggregation operator, pointwise extension, generalized aggregation, order relation.

AMS Subject Classification: 15A09; 65F05.

1 Introduction

We explore questions related to the aggregation operators and aggregation of fuzzy sets, i.e. elements of the class $F(\mathbb{R}) = \{P | P : \mathbb{R} \to [0, 1]\}$. No preliminary knowledge of the aggregation operators theory and of the fuzzy sets theory are required, because all necessary information is given in Section 2. Further the paper is organized in the following way: Section 3 is devoted to a new class of aggregation operators, so called γ-aggregation operators; Section 4 deals with the aggregation of fuzzy sets w.r.t. different order relations and pointwise extension is studied in details; Section 5 provides some guidelines on practical application and we give conclusions in Section 6.

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2 Preliminaries

We provide basic knowledge of the aggregation operators (hereinafter for the brevity–agop) and fuzzy sets theory necessary for understanding the materials of the paper.

2.1 Fundamentals on aggregation operators

Aggregation of several input values into a single output value is an indispensable tool not only of mathematics or physics, but also when studying different problems in engineering, economics and other fields of science. The problems of aggregation are very broad in general. We give only definition, examples and the main properties of agops which are needed for our work. For more information an interested reader can refer to e.g. [8].

Definition 1. A mapping $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an agop on the unit interval if for every $n \in \mathbb{N}$ the following conditions hold:

$$(A1) \quad A(0,\dots,0) = 0, \quad (A2) \quad A(1,\dots,1) = 1,$$
$$(A3) \quad (\forall i = \overline{1,n})\ (x_i \le y_i) \Rightarrow A(x_1, x_2, \dots, x_n) \le A(y_1, y_2, \dots, y_n).$$

Conditions (A1) and (A2) are called boundary conditions, and they ensure that aggregation of completely bad (good) results will give the completely bad (good) output. Condition (A3) resembles the monotonicity property of A.

In general, the number of the input values to be aggregated is unknown, and therefore an agop can be presented as a family $A = (A_{(n)})_{n\in\mathbb{N}}$, where $A_{(n)} = A|_{[0,1]^n}$. Operators $A_{(n)}$ and $A_{(m)}$ for different n and m need not be related, i.e. given an operator $A_{(n)}$ not always we can restore the form of $A_{(m)}$ for an arbitrary $n, m \in \mathbb{N}$.

Definition 2. An element $x \in [0;1]$ is called A-idempotent element whenever $A_{(n)}(x,\dots,x) = x, \forall n \in \mathbb{N}$. A is called an idempotent agop if each $x \in [0;1]$ is an idempotent element of A.

0 and 1 are trivial A-idempotent elements for an arbitrary agop.

Definition 3. An agop A is called a symmetric agop if

$$\forall n \in \mathbb{N}, \quad \forall x_1, \dots, x_n \in [0;1] : A(x_1, \dots, x_n) = A(x_{\pi(1)}, \dots, x_{\pi(n)})$$

for all permutations $\pi = \big(\pi(1), \dots, \pi(n)\big)$ of $(1, \dots, n)$.

A weighted mean $A(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_i$, where $\sum_{i=1}^n w_i = 1$ is an example of a nonsymmetric agop.

Definition 4. An agop A is associative if

$$\forall n, m \in \mathbb{N}, \forall x_1, \dots, x_n,\ y_1, \dots, y_m \in [0;1] :$$
$$A(x_1, \dots, x_n, y_1, \dots, y_m) = A(A(x_1, \dots, x_n), A(y_1, \dots, y_n)).$$

The associativity of an agop allows to aggregate first some subsystems of all inputs, and then the partial outputs. For practical purposes we can start with aggregation procedure before knowing all inputs to be aggregated.

Definition 5. An agop A is bisymmetric if $\forall n, m \in \mathbb{N}$, $\forall x_{11}, \dots, x_{mn} \in [0;1]$:

$$\begin{aligned} A_{(mn)}(x_{11},\dots,x_{mn}) &= A_{(m)}(A_{(n)}(x_{11},\dots,x_{1n}),\dots,A_{(n)}(x_{m1},\dots,x_{mn})) \\ &= A_{(n)}(A_{(m)}(x_{11},\dots,x_{m1}),\dots,A_{(m)}(x_{1n},\dots,x_{mn})). \end{aligned}$$

The bisymmetry allows to aggregate first rows and then partial outputs or first columns and then partial outputs if information is stored in the form of the matrix. Bisymmetry is implied by associativity and symmetry.

Definition 6. An element $e \in [0;1]$ is called a neutral element of A

if $\forall n \in \mathbb{N},\ \forall x_1, \dots, x_n, \in [0;1]$
if $x_i{=}e$ for some $i{\in}\{1,\dots,n\}$ then $A(x_1,\dots,x_n){=}A(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$.

So the neutral element can be omitted from aggregation inputs without influencing the final output.

Definition 7. An element $a \in [0;1]$ is called an absorbing element or annihilator of A if

$$\forall n \in \mathbb{N}, \forall x_1,\dots,x_n, \in [0;1] : a \in \{x_1,\dots,x_n\} \Rightarrow A(x_1,\dots,x_n) = a.$$

As the most popular examples of aggregation operators there can be mentioned the following:

1. The Arithmetic Mean: $A(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a symmetric, bisymmetric and idempotent agop with no neutral and absorbing elements and no associativity.

2. The Weighted Mean: $A(x_1,\dots,x_n) = \sum_{i=1}^{n} w_i x_i$, where $\forall i\ w_i \in [0;1]$, $\sum_{i=1}^{n} w_i = 1$ is a bisymmetric and idempotent agop with no neutral and absorbing elements and having no associativity and symmetry properties.

3. The Minimum: $A(x_1,\dots,x_n) = \min(x_1,\dots,x_n)$ is a symmetric, associative, bisymmetric and idempotent agop with 1 being a neutral element and 0 being an absorbing element.

2.2 Fuzzy set theory: basic notions

The concept of the fuzzy sets was introduced by Zadeh in [10]. Now there are published many fundamental works on fuzzy sets both on the theoretical aspects and applications. An interested reader can find information on different aspects of the fuzzy sets theory e.g. in monographs [1, 4] and others. Here we restrict ourselves to the notions, which are needed for our work.

Definition 8. A mapping $P : X{\to}L$, where L is an arbitrary complete lattice is called a fuzzy subset of a set X.

Further in the paper we will take the interval $[0,1]$ with natural order as the lattice L and the set of real numbers $\mathbb{R}$ in the role of X. $F(\mathbb{R})$ denotes the set of all fuzzy subsets of $\mathbb{R}$. Let's fix $\alpha \in [0,1]$:

Definition 9. The set $P_\alpha = \{x : P(x) \geq \alpha\}$ is called the α-cut of a fuzzy set P.

Definition 10. The set $P^\alpha = \{x : P(x) > \alpha\}$ is called the strict α-cut of the fuzzy set P.

Definition 11. The set P^0 is called the support of the fuzzy set P.

Although fuzzy sets conceptually are interpreted as generalized sets, P is often called a membership function. The concept of a fuzzy number is very important in the theory of fuzzy sets.

Definition 12. A fuzzy set $P : \mathbb{R} \to [0,1]$ is called a fuzzy number if the following holds:

1. There exists exactly one point $x_P \in \mathbb{R}$ such that $P(x_P) = 1$ (x_P is called the vertex of P).
2. All α-cuts $\alpha > 0$ are closed, bounded intervals.
3. Function P is nondecreasing on the interval $(-\infty; x_P]$ and non-increasing on the interval $(x_P; +\infty)$.

We define a triangle fuzzy number as a fuzzy number with the following membership function:

$$P(x) = \begin{cases} 0, \quad \text{if} \quad x < p_1, \\ (x - p_1)/(p_2 - p_1), \quad x \in [p_1; p_2], \\ (x - p_3)/(p_2 - p_3), \quad x \in [p_2; p_3], \\ 0, \quad \text{if} \quad x > p_3 \end{cases}$$

for some $p_1 \leq p_2 \leq p_3$ and:

$$P(p_1) = \frac{p_2 - p_1}{p_2 - p_1} = 1, \text{ when } p_1 = p_2 < p_3,$$
$$P(p_3) = \frac{p_3 - p_2}{p_3 - p_2} = 1, \text{ when } p_1 < p_2 = p_3,$$
$$P(p_1) = \frac{p_2 - p_1}{p_2 - p_1} = 1, \text{ when } p_1 = p_2 = p_3.$$

Fuzzy numbers P_1, P_2 are ordered by inclusion

$$P_1 \leq_F P_2 \text{ if } P_1(x) \leq P_2(x), \ \forall x \in \mathbb{R}.$$

Other order relations used in this paper are specified later in the corresponding subsections.

3 New Class of Aggregation Operators: γ-agops

3.1 Definition of γ-agop

We introduce the notion of γ-agop by means of additional property (A_γ). Let $\gamma \in [0;1]$ and $\varphi_\gamma : [0,1] \to \{0\} \cup [\gamma,1]$ is defined in the following way:

$$\varphi_\gamma(x) = \begin{cases} 0, & \text{if } x < \gamma, \\ x, & \text{if } x \geq \gamma. \end{cases}$$

Definition 13. $A : \cup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an γ-agop on the unit interval if the following conditions hold:

$(A1)\quad A(0,\ldots,0) = 0, \quad (A2)\quad A(1,\ldots,1) = 1$

$(A_\gamma)\ (\forall i = \overline{1,n}, \gamma \in [0,1])\ (\varphi_\gamma(x_i) \leq \varphi_\gamma(y_i)) \Rightarrow A(x_1,\ldots x_n) \leq A(y_1,\ldots,y_n).$

Remark. In case $\gamma = 0$ and $\varphi_0(x) = x$ condition (A_γ) reduces to condition (A3) in the Definition 1.

Proposition 1. *If A satisfies (A_γ) and $\gamma > \gamma^{'}$ then A satisfies $(A_{\gamma'})$.*

Proof. Let's take arbitrary $(x_1,\ldots,x_n),(y_1,\ldots,y_n)$ satisfying the inequality $\varphi_{\gamma'}(x_i) \leq \varphi_{\gamma'}(y_i)$, $\forall i = \overline{1,n}$. Since $\varphi_{\gamma'}(x_i) \leq \varphi_{\gamma'}(y_i)$ and $\gamma > \gamma^{'}$ from the definition of φ_γ it follows that $\varphi_\gamma(x_i) \leq \varphi_\gamma(y_i)$. Therefore by condition (A_γ):

$$A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$$

and thus $(A_{\gamma'})$ is satisfied. □

Corollary 1. Each γ-agop A satisfies properties (A1)–(A3), and hence is an agop.

Assertion of Proposition 1 is not working in the opposite direction, i.e. when $\gamma^{'} > \gamma$. Examples of γ-agops given in the next section show this.

3.2 Examples of γ-agops

It is intuitively clear that the formula of γ-agop should neutralize all arguments less than γ. Otherwise the left part of implication (A_γ) will be true, but due to the monotonicity we will not receive the right part of implication.

Here are examples of γ-agops:

Example 1.

$$A_1(x_1,\ldots,x_n) = \sum_{i=1}^{n} w_i x_i,$$

where

$$w_i = \begin{cases} 0, & \text{if } x_i < \gamma, \\ 1/n, & \text{if } x_i \geq \gamma. \end{cases}$$

Example 2.

$$A_2(x_1, \dots, x_n) = \min(w_1 x_1, \dots, w_n x_n),$$

where

$$w_i = \begin{cases} 0, & \text{if } x_i < \gamma, \\ 1, & \text{if } x_i \geq \gamma. \end{cases}$$

The class of γ-agops is wider than the class of agops. As an example we can mention usual arithmetic mean, which does not satisfy (A_γ) if $\gamma > 0$.

3.3 Equivalence relation induced by φ_γ

Let's introduce relation $\equiv_{\varphi_\gamma}$ on $[0,1]^n$ in the following way:

$$(x_1, \dots, x_n) \equiv_{\varphi_\gamma} (y_1, \dots, y_n) \Leftrightarrow (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), \dots, \varphi_\gamma(y_n)).$$

Further we show that $\equiv_{\varphi_\gamma}$ is an equivalence relation:

Reflexivity:

$$(\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) = (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n) \Rightarrow (x_1, \dots, x_n) \equiv_{\varphi_\gamma} (x_1, \dots, x_n).$$

Symmetry:

$$(x_1, \dots, x_n) \equiv_{\varphi_\gamma} (y_1, \dots, y_n) \Rightarrow (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), \dots, \varphi_\gamma(y_n))$$
$$\Rightarrow (\varphi_\gamma(y_1), \dots, \varphi_\gamma(y_n)) = (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) \Rightarrow (y_1, \dots, y_n) \equiv_{\varphi_\gamma} (x_1, \dots, x_n).$$

Transitivity:

$$\begin{aligned} (x_1, \dots, x_n) &\equiv_{\varphi_\gamma} (y_1, \dots, y_n) \text{ and } (y_1, \dots, y_n) \equiv_{\varphi_\gamma} (z_1, \dots, z_n) \\ &\Rightarrow (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) = (\varphi_\gamma(y_1), \dots, \varphi_\gamma(y_n)) \text{ and} \\ (\varphi_\gamma(y_1), \dots, \varphi_\gamma(y_n)) &= (\varphi_\gamma(z_1), \dots, \varphi_\gamma(z_n)) \Rightarrow (\varphi_\gamma(x_1), \dots, \varphi_\gamma(x_n)) \\ &= (\varphi_\gamma(z_1), \dots, \varphi_\gamma(z_n)) \Rightarrow (x_1, \dots, x_n) \equiv_{\varphi_\gamma} (z_1, \dots, z_n). \end{aligned}$$

When $\gamma < 1$ the number of equivalence classes s is infinite. In the particular case, when $\gamma = 1$, s is finite. We will denote equivalence classes X_k, $k = 1, 2, \dots, s$

Proposition 2. *If* $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X_k$, *$A$ is a γ-agop then*

$$A(x_1, \dots, x_n) = A(y_1, \dots, y_n)$$

Proof. Let's take $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X_k$, then according to the definition of $\equiv_\gamma$ we can write: $\varphi_\gamma(x_1, \dots, x_n) = \varphi_\gamma(y_1, \dots, y_n)$. Let's assume that $A(x_1, \dots, x_n) \neq A(y_1, \dots, y_n)$. Then (A_γ) implies:

$$A(x_1, \dots, x_n) < A(y_1, \dots, y_n).$$

The same reasoning will lead us to the inequality:

$$A(y_1, \dots, y_n) < A(x_1, \dots, x_n).$$

The above derived inequalities cannot be true at the same time, and this means that our assumption on $A(x_1,\dots,x_n) \neq A(y_1,\dots,y_n)$ is not true. □

As a corollary from Proposition 2 we obtain the following result:

Corollary 2. γ-agops for $\gamma > 0$ are not idempotent agops.

Proof. Proof immediately follows from the result of Proposition 2 and the definition of $\equiv_\gamma$:

$$\forall\gamma \in (0;1],\ \forall(x_k,\dots,x_k):\ x_k < \gamma, \quad A(x_k,\dots,x_k) = A(0,\dots,0) = 0 \neq x_k.$$

□

4 Generalized Aggregation

The problem of aggregation can be generalized if we use fuzzy subsets as input information. Functions are aggregated in this case. We are further developing this approach, which is initiated by Takaci in [7]. Other interesting, conceptually different methods of generalization can be found in literature, e.g. in [5, 6, 9] and others.

4.1 Definition of generalized agop

We give the definition of a generalized agop ([7]). This notion is the base of our further considerations.

Let $F(\mathbb{R}) = \{P | P : \mathbb{R} \to [0,1]\}$ and $\prec$ is some order relation on $F(\mathbb{R})$ with minimal element $\tilde{0} \in F(\mathbb{R})$ and maximal element $\tilde{1} \in F(\mathbb{R})$ then:

Definition 14. A mapping $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ is called a generalized agop w.r.t. the order relation $\prec$, if the following conditions hold:

$$(\tilde{A1})\ \tilde{A}(\tilde{0},\dots,\tilde{0}) = \tilde{0}, \quad (\tilde{A2})\ \tilde{A}(\tilde{1},\dots,\tilde{1}) = \tilde{1},$$
$$(\tilde{A3})\ (\forall i = \overline{1,n})\ (P_i \prec Q_i) \Rightarrow \tilde{A}(P_1,\dots,P_n) \prec \tilde{A}(Q_1,\dots,Q_n),$$
$$\text{where } P_1,\dots,P_n,Q_1,\dots,Q_n \in F(\mathbb{R}).$$

Many questions related to generalized aggregation arise. The most important of them are touched in the sequel.

4.2 Pointwise extension

We recall the definition of a pointwise extension:

Definition 15. Let $P_1,\dots,P_n \in F(\mathbb{R})$, $\tilde{A} : \cup_{n\in\mathbb{N}} F(\mathbb{R})^n \to F(\mathbb{R})$ and A be an ordinary agop on the unit interval, then $\tilde{A}$ is a pointwise extension of A if the following holds:

$$(\tilde{E1}) : (\forall x \in \mathbb{R}, \tilde{A}(P_1,\dots,P_n)(x) = A(P_1(x),\dots,P_n(x)),$$

where $\tilde{A}(P_1,\dots,P_n)$ is a resulting fuzzy set obtained by means of application the operator $\tilde{A}$ to the fuzzy subsets $P_1,\dots,P_n$.

We start the study of a pointwise extension in the next subsections. At the beginning we review its properties and later we consider a pointwise extension of a γ-agop.

4.2.1 Properties of pointwise extension of an agop

We consider properties of a pointwise extension of an arbitrary agop A in this subsection.

Proposition 3. *Let $\tilde{A}$ be a pointwise extension of A, then the following statements are true:*

$$\begin{aligned} &(1) \quad \textit{if } A \textit{ is symmetric then } \tilde{A} \textit{ is symmetric},\\ &(2) \quad \textit{if } A \textit{ is associative then } \tilde{A} \textit{ is associative},\\ &(3) \quad \textit{if } A \textit{ is bisymmetric then } \tilde{A} \textit{ is bisymmetric},\\ &(4) \quad \textit{if } A \textit{ is idempotent then } \tilde{A} \textit{ is idempotent}.\end{aligned}$$

Proof. In order to prove (1) we need to show that

$$\forall P_1,\ldots,P_n \in F(\mathbb{R}), \forall x{\in}\mathbb{R}, \forall n{\in}\mathbb{N} : \tilde{A}(P_1,\ldots,P_n)(x) = \tilde{A}(P_{\pi(1)},\ldots,P_{\pi(n)})(x)$$

for all permutations $\pi = (\pi(1),\ldots,\pi(n))$ of $(1,\ldots,n)$. Due to definition of $\tilde{A}$ and symmetry of A we can write:

$$\tilde{A}(P_1,\ldots,P_n)(x) = A(P_1(x),\ldots,P_n(x)) = A(P_{\pi(1)}(x),\ldots,P_{\pi(n)}(x)) \qquad (4.1)$$

for an arbitrary permutation $\pi = (\pi(1),\ldots,\pi(n))$ of $(1,\ldots,n)$.

We recall definition of $\tilde{A}$ again and continue (4.1):

$$A(P_{\pi(1)}(x),\ldots,P_{\pi(n)}(x)) = \tilde{A}(P_{\pi(1)},\ldots,P_{\pi(n)})(x).$$

Proof of (2) and (3) is analogous to (1). In order to prove (4) we need to show that

$$\forall P_i \in F(\mathbb{R}),\ \forall x \in \mathbb{R},\ \forall n \in \mathbb{N} : \tilde{A}_{(n)}(P_i,\ldots,P_i)(x) = P_i(x).$$

It immediately follows from the definition of $\tilde{A}$ and idempotence of A:

$$\tilde{A}_{(n)}(P_i,\ldots,P_i)(x) = A_{(n)}(P_i(x),\ldots,P_i(x)) = P_i(x).$$

□

Proposition 4. *If $\tilde{A}$ is a pointwise extension of A, a and e are correspondingly an absorbing and a neutral elements of A, then the following statements are true:*

(1) element $P^*(x) \in F(\mathbb{R})$ defined by the equality $P^*(x) = a$, $\forall x \in \mathbb{R}$ is an absorbing element of $\tilde{A}$,

(2) element $Q^*(x) \in F(\mathbb{R})$ defined by the equality $Q^*(x) = e$, $\forall x \in \mathbb{R}$ is a neutral element of $\tilde{A}$.

Proof. In order to prove (1) we need to show that $\forall P_1,\ldots,P_n \in F(\mathbb{R})$, $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$ we have that $\tilde{A}(P_1,\ldots,P_n)(x) = P^*(x)$, if $P^* \in \{P_1,\ldots,P_n\}$. Let's consider an arbitrary $x \in \mathbb{R}$ then due to the definition of $\tilde{A}$ we can write:

$$\tilde{A}(P_1,\ldots,P_n)(x) = A(P_1(x),\ldots,P_n(x)). \tag{4.2}$$

$P^* \in \{P_1,\ldots,P_n\}$ and $P^*(x) = a, \forall x \in \mathbb{R}$, therefore one of $A(P_1(x),\ldots,P_n(x))$ arguments is a. Since a is an absorbing element of A, the equality (4.2) can be continued in the following way

$$A(P_1(x),\ldots,P_n(x)) = a$$

for an arbitrary chosen $x \in \mathbb{R}$. Thus, we have received that

$$\tilde{A}(P_1,\ldots,P_n)(x) = a,\ \forall P_1,\ldots,P_n \in F(\mathbb{R}),\ \forall x \in \mathbb{R}.$$

If we assume that $P^{**}(x) \neq P^*(x)$ for some $x \in \mathbb{R}$ and $P^{**}(x)$ is also an absorbing element of $\tilde{A}$ then $\tilde{A}(P_1,...,P_n)(x)$ where vector $(P_1,\ldots,P_n)$ is such that $P^*, P^{**} \in \{P_1,\ldots,P_n\}$ on the one hand is equal to $P^*(x)$, but on the other hand it is equal to $P^{**}(x)$. Thus we see that inequality is impossible, therefore our assumption on existence of P^{**} different from P^* is not true and P^* is the only absorbing element or equivalently all absorbing elements of $\tilde{A}$ are equal. In order to prove (2) we need to show that $\forall P_1,\ldots,P_n \in F(\mathbb{R}), \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$

$$\tilde{A}(P_1,\ldots,P_i,\ldots,P_n)(x) = \tilde{A}(P_1,\ldots,P_{i-1},P_{i+1},\ldots,P_n)(x)$$

if $P_i = Q^*$ for some $i \in \{1,\ldots,n\}$. We consider an arbitrary $x \in \mathbb{R}$ and assume that $P_i(x) = Q^*(x) = e, \forall x$ for some $i \in \{1,\ldots,n\}$, then exploiting $\tilde{A}$ definition we can write:

$$\tilde{A}(P_1,\ldots,P_i,\ldots,P_n)(x) = A(P_1(x),\ldots,P_i(x),\ldots,P_n(x)) \tag{4.3}$$

i-th argument of $A(P_1(x),\ldots,P_i(x),\ldots,P_n(x))$ is e and can be omitted as it is a neutral element of A. Thus we continue (4.3) in the following way:

$$\begin{aligned} A(P_1(x),\ldots,P_i(x),\ldots,P_n(x)) &= A(P_1(x),\ldots,P_{i-1}(x),P_{i+1}(x),\ldots,P_n(x)) \\ &= \tilde{A}(P_1,\ldots,P_{i-1},P_{i+1},\ldots,P_n)(x). \end{aligned}$$

We assume that a neutral element is not unique, i.e. $\exists Q^{**}(x) \neq Q^*(x)$ for some $x \in \mathbb{R}$ and $Q^{**}(x)$ is also a neutral element of $\tilde{A}$ and we consider a vector $(P_1,\ldots,P_n)$ s.t. $P_1 = Q^*$ and $P_i = Q^{**}, i = 2,\ldots,n$. Using neutrality of $\tilde{A}$ versus Q^* we obtain:

$$\tilde{A}(P_1,\ldots,P_n) = \tilde{A}_{(n-1)}(Q^{**},\ldots,Q^{**}) \tag{4.4}$$

now we apply neutrality versus Q^{**}, property $\tilde{A}_{(1)}(P) = P$ and continue (4.4):

$$\tilde{A}_{(n-1)}(Q^{**},\ldots,Q^{**}) = \tilde{A}_{(n-2)}(Q^{**},\ldots,Q^{**}) = \cdots = \tilde{A}_{(1)}(Q^{**}) = Q^{**}.$$

In the same way first employing neutrality of Q^{**} and then property $\tilde{A}_{(1)}(P) = P$ we obtain:

$$\tilde{A}(P_1,\ldots,P_n) = \tilde{A}_{(1)}(Q^*) = Q^*.$$

Thus we have

$$\tilde{A}(P_1, \ldots, P_n) = Q^* \text{ and } \tilde{A}(P_1, \ldots, P_n) = Q^{**}$$

but $Q^* \neq Q^{**}$ and so, we have obtained contradiction. □

4.2.2 Pointwise extension of γ- agop w.r.t. order relation $\subseteq^{\alpha}_{F_1}$

The main subject of this subsection is a γ-agop, and we study its pointwise extension w.r.t. to order relation $\subseteq^{\alpha}_{F_1}$. If one considers order $\subseteq_{F1}$ defined on $F(\mathbb{R})$ in the following way:

$$P, Q \in F(\mathbb{R}), P \subseteq_{F_1} Q \Leftrightarrow (\forall x \in \mathbb{R})(P(x) \leq Q(x)),$$

then in this case the minimal element $\tilde{0}$ and the maximal element $\tilde{1}$ are given respectively by :

$$\tilde{0}(x) = 0, \quad \tilde{1}(x) = 1, \quad \forall x \in \mathbb{R}.$$

Then it is proved (see [7]) that pointwise extension is a generalized agop w.r.t. order $\subseteq_{F1}$. Given $\alpha \in [0, 1]$ we introduce an order $\subseteq^{\alpha}_{F1}$ on $F(\mathbb{R})$ in the following way:

$$P, Q \in F(\mathbb{R}), P \subseteq^{\alpha}_{F1} Q \Leftrightarrow (\forall x \in \mathbb{R})(P(x) \geq \alpha \Rightarrow P(x) \leq Q(x)).$$

The minimal element $\tilde{0}$ and the maximal element $\tilde{1}$ are defined correspondingly:

$$\tilde{0}(x) = \alpha_x \leq \alpha, \quad \tilde{1}(x) = 1, \quad \forall x \in \mathbb{R}.$$

α_x depends on x and it is clear that $\tilde{0}$ defined in this way is not unique. E.g.

$$\tilde{0}(x) = \alpha/2, \forall x \text{ and } \tilde{0} = \begin{cases} 0, & \text{if } x < x_0, \\ \alpha/3, & \text{if } x \geq x_0, \end{cases}$$

both represent the class of the minimal elements. Unless it is specified differently, we consider that $\tilde{0}$ is defined in the general way. As one can see $\tilde{1}$ for both orders $\subseteq_{F1}$ and $\subseteq^{\alpha}_{F1}$ are defined in the same manner. And $\tilde{0}$ for $\subseteq_{F1}$ is one particular case of $\tilde{0}$ for $\subseteq^{\alpha}_{F1}$. It immediately follows from order definitions that $\subseteq_{F1}$ is a particular case of $\subseteq^{\alpha}_{F1}$ (when $\alpha = 0$). The result formulated below, shows that under a special condition pointwise extension of a γ-agop will be an object of the class of generalized agops introduced in Section 4.1.

Theorem 1. *If $\tilde{A}$ is a pointwise extension of a γ-agop A, and $\gamma > \alpha$, then it is generalized agop w.r.t. order relation $\subseteq^{\alpha}_{F1}$.*

Proof. We need to show $\tilde{A}1$–$\tilde{A}3$ from the Definition 14. Consider $\tilde{A}1$. According to definitions of pointwise extension and $\tilde{0}$ for an arbitrary $x \in \mathbb{R}$ we can write:

$$\tilde{A}(\tilde{0}, \ldots, \tilde{0})(x) = A(\tilde{0}(x), \ldots, \tilde{0}(x)) = A(\alpha_x, \ldots, \alpha_x), \tag{4.5}$$

where $\tilde{0}$ is general representative of the class of minimal elements. $\alpha_x \leq \alpha < \gamma$ then according to the Proposition 2 we can continue (4.5):

$$A(\alpha_x, \ldots, \alpha_x) = A(0, \ldots, 0) = 0.$$

So, $\forall x \in \mathbb{R}$ $\tilde{A}(\tilde{0}, \ldots, \tilde{0})(x) = 0 \leq \alpha$ and this means that

$$\tilde{A}(\tilde{0}, \ldots, \tilde{0})(x) = \tilde{0}(x). \tag{4.6}$$

Consider $\tilde{A}2$. According to definitions of pointwise extension and $\tilde{1}$ for an arbitrary $x \in \mathbb{R}$ we can write:

$$\tilde{A}(\tilde{1}, \ldots, \tilde{1})(x) = A(\tilde{1}(x), \ldots, \tilde{1}(x)) = A(1, \ldots, 1).$$

The equality $A(1, \ldots, 1) = 1$ follows from the boundary condition for γ-agop. Thus,

$$\tilde{A}(\tilde{1}, \ldots, \tilde{1})(x) = 1 = \tilde{1}(x), \ \forall x \in \mathbb{R}.$$

Consider $\tilde{A}3$. It is given that $(\forall i = \overline{1, n})$ $(P_i \subseteq^{\alpha}_{F1} Q_i)$ and we need to show that

$$\tilde{A}(P_1, \ldots, P_n) \subseteq^{\alpha}_{F1} \tilde{A}(Q_1, \ldots, Q_n) \tag{4.7}$$

According to definition of pointwise extension $\forall x \in \mathbb{R}$ we can write:

$$\begin{aligned} \tilde{A}(P_1, \ldots, P_n)(x) &= A(P_1(x), \ldots, P_n(x)), \\ \tilde{A}(Q_1, \ldots, Q_n)(x) &= A(Q_1(x), \ldots, Q_n(x)). \end{aligned} \tag{4.8}$$

If $P_i(x) \geq \alpha$ then according to the inclusion $P_i(x) \subseteq^{\alpha}_{F_1} Q_i(x)$ $P_i(x) \leq Q_i(x)$. If $P_i(x) < \alpha$ and thus $P_i(x) < \gamma$ then $\varphi_\gamma(P_i(x)) = 0$ and according to Proposition 2 formula (4.8) can be continued in the following way:

$$\tilde{A}(P_1, \ldots, P_n)(x) = A(P_1(x), \ldots, 0, \ldots, P_n(x)),$$

where 0 stands on the positions, which belong to the index set

$$I_1 = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} : P_i(x) < \alpha, \forall i \in I_1.$$

Anyhow for $i \in I_1$ $Q_i(x) \geq 0$ and therefore $P_i(x) \leq Q_i(x)$. Thus $P_i(x) \leq Q_i(x)$ $\forall i$ and thus monotonicity of A provides the following inequality:

$$A(P_1(x), \ldots, P_n(x)) \leq A(Q_1(x), \ldots, Q_n(x)).$$

So, we have shown (4.7). $\square$

Remark. The correct way to write formula (4.6) would be $\tilde{A}(\tilde{0}, \ldots, \tilde{0})(x) = \tilde{0}^*(x)$, where $\tilde{0}^*(x) = 0$, $\forall x \in \mathbb{R}$ is just one particular representative of the class of minimal elements. This means that any finite subset of elements from the class of minimal elements will be aggregated into one particular element from the same class. This obstacle modifies boundary condition $(\tilde{A}1)$, but we assume that all elements from the class of minimal elements are equivalent (have the same properties), therefore we do not distinguish between them and agree with such deviation of aggregated result from the input.

4.2.3 Special class of horizontal orders

We study a special class of horizontal orders and pointwise extension of γ-agops in the sequel. Further we take $[a,b] \subseteq \mathbb{R}$ in the role of X and consider

$$F([a,b]) = \{P | P : [a,b] \to [0,1]\}.$$

Let's observe an order relation $\subseteq^{\alpha}_{F_2}$ defined on $F([a,b])$ in the following way:

$$P, Q \in F([a,b]), \alpha \in (0,1], P \subseteq^{\alpha}_{F_2} Q \Leftrightarrow \overline{P}^{\alpha} \leq \underline{Q}^{\alpha},$$

where

$$\begin{aligned} P^{\alpha} &= \{x : P(x) \geq \alpha\}, & \min P^{\alpha} &= \underline{P}^{\alpha}, & \max P^{\alpha} &= \overline{P}^{\alpha}, \\ Q^{\alpha} &= \{x : Q(x) \geq \alpha\}, & \min Q^{\alpha} &= \underline{Q}^{\alpha}, & \max Q^{\alpha} &= \overline{Q}^{\alpha}. \end{aligned}$$

It immediately follows from the properties of α-cuts that

$$P \subseteq^{\alpha_1}_{F_2} Q \Rightarrow P \subseteq^{\alpha_2}_{F_2} Q, \quad \forall \alpha_2 > \alpha_1.$$

$\tilde{0}$ and $\tilde{1}$ are defined in the following way:

$$\tilde{0}(x) = \begin{cases} 1, & \text{if } x = a, \\ \alpha_x < \alpha, & \text{otherwise.} \end{cases}, \quad \tilde{1}(x) = \begin{cases} 1, & \text{if } x = b, \\ \alpha_x < \alpha, & \text{otherwise.} \end{cases}$$

Again definitions of $\tilde{0}$ and $\tilde{1}$ are not unique. But we assume that all representatives from the class of the minimal (accordingly maximal) elements are equivalent.

4.2.4 Pointwise extension of γ-agop w.r.t. $\subseteq^{\alpha}_{F_2}$

It is easy to construct examples showing that even if the input information by pairs is ordered w.r.t. $\subseteq^{\alpha}_{F_2}$ $(\forall i = \overline{1,n}, P_i \subseteq^{\alpha}_{F_2} Q_i)$ the aggregation result need not be ordered in the same manner. As a result $\tilde{A}$ is not a generalized agop w.r.t. $\subseteq^{\alpha}_{F_2}$ for an arbitrary γ-agop.

Example 3. Let's consider triangular numbers

$$P_1 = (1,2,3),\ P_2 = (5,6,7),\ Q_1 = (3,4,5),\ Q_2 = (7,8,9)$$

and pointwise extension of γ-agop $A_{\gamma} = \max(\omega_1 x_1, \omega_2 x_2)$, where

$$\omega_i = \begin{cases} 0, & \text{if } x < \gamma, \\ 1, & \text{if } x \geq \gamma, \end{cases} \quad \forall i = 1,2, \quad \forall \alpha \in (0,1] \quad P_i \subseteq^{\alpha}_{F_2} Q_i.$$

$$\tilde{A}(P_1,P_2)(x) = \max(\omega_1 P_1(x), \omega_2 P_2(x)) = \begin{cases} x-1, & \text{if } x \in [1+\gamma; 2], \\ 3-x, & \text{if } x \in (2; 3-\gamma], \\ x-5, & \text{if } x \in [5+\gamma; 6], \\ 7-x, & \text{if } x \in (6; 7-\gamma], \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

$$\tilde{A}(Q_1,Q_2)(x)=\max(\omega_1 Q_1(x),\omega_2 Q_2(x))=\begin{cases} x-3, & \text{if } x\in[3+\gamma;4],\\ 5-x, & \text{if } x\in(4;5-\gamma],\\ x-7, & \text{if } x\in[7+\gamma;8],\\ 9-x, & \text{if } x\in(8;9-\gamma],\\ 0, & \text{otherwise.}\end{cases} \tag{4.10}$$

According to formulas (4.9) and (4.10)

$$\tilde{A}(P_1,P_2)(6)=1, \quad \tilde{A}(Q_1,Q_2)(4)=1$$

thus for an arbitrary $\alpha^*\in(0,1]$

$$\max\left\{x:\tilde{A}(P_1,P_2)\geq\alpha^*\right\}\geq 6, \quad \min\left\{x:\tilde{A}(Q_1,Q_2)\geq\alpha^*\right\}\leq 4$$

and therefore $\tilde{A}(P_1,P_2)\neg\subseteq^{\alpha}_{F_2}\tilde{A}(Q_1,Q_2),\forall\alpha\in(0,1]$.

Inconsistency between input information order and aggregated information order comes from different approaches in definitions: pointwise extension is defined pointwise for $\forall x\in[a,b]$, but $\subseteq^{\alpha}_{F_2}$ is defined on a fixed level α.

5 Some Remarks on Practical Application

Agops include roughly all nondecreasing mappings with finite (sometimes infinite) number of arguments, preserving the boundaries. Such a large class of mappings obviously has found a broad practical application in different areas.

Many real-world problems can be considered within the information aggregation framework where separate information sources are combined to produce more accurate and simpler evaluation. This final evaluation is a base for conclusion or decision, therefore agops are widely used in multi-criteria decision making and multi-attributes classification. An example of application of information aggregation in financial decision making can be found in [5]. More sophisticated decision and classification problems based on interacting criteria or attributes can be solved by means of fuzzy integrals, which are special class of agops ([2, 3]).

Aggregation of information represented by fuzzy sets plays the central role in intelligent systems where fuzzy rule base and reasoning mechanism are applied. Pointwise extension (similarly to other approaches considered e.g. in [6]) observed in this paper can be successfully applied in this area.

Extension of the γ-agop can be used in this way as well. Additionally it allows some pieces of information to be ignored and researcher can leverage the amount of ignored information by choosing an appropriate γ. On the one hand this obviously extends a range of problems solved by means of γ-agop extension versus extension of usual agops, on the other hand the class of γ-agops is smaller and therefore sometimes a more appropriate agop can be chosen from the bigger class of all agops.

6 Conclusion

The notion of γ-agop and its pointwise extension is central in this paper. We have studied it in details and proved that γ-agop can be generalized w.r.t. the special class of order relations. Practical value of information aggregation and by means of γ-agops and generalized γ-agops in a particular case is a good motivation to continue the study in this area.

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