Mathematical Modelling and Analysis
Volume 14 Number 1, 2009, pages 1–9
Doi:10.3846/1392-6292.2009.14.1-9

ISSN 1392-6292 print
ISSN 1648-3510 online
www.vgtu.lt/mma/

Mathematical Modelling of Alternating Electromagnetic and Hydrodynamic Fields, Induced by Bar Type Conductors in a Cylinder

A. Buikis, L. Buligins and H. Kalis

Institute of Mathematics and Informatics and
Faculty of physics and mathematics University of Latvia
Raina bulvāris 29, Rīga LV-1459, Latvija
E-mail: `buikis@latnet.lv; leonids.buligins @lu.lv`
E-mail(*corresp.*): `kalis@lanet.lv`

Received October 16, 2008; revised December 03, 2008; published online February 25, 2009

Abstract. The heating of buildings by ecologically clean and compact local devices is an interesting and actual problem. One of the modern areas of applications developed during last ten years is an effective usage of electrical energy by alternating current to produce heat energy. This work presents the mathematical model of one of such devices. It is a finite cylinder with viscous incompressible liquid and with metal electrodes of the form of bars placed parallel to the cylinder axis in the liquid. These conductors are connected to the alternating current.

Key words: alternating current, electrically conducting liquid, Navier-Stokes equations.

1 Introduction

In many technological applications it is important to mix an electrically conducting liquid by using various magnetic fields. In papers [1, 2, 3, 4] we had modelled cylinder form electrical heat generators with six or nine circular conductors–electrodes. In this work we analyze different types of conductors, with the forms of bars and they are placed parallel to the cylinder axis in the electrically conducting liquid. It means that in distinction with the case of circular electrodes here we can't assume the axis symmetry and we must consider full 3D mathematical model based on the system of Navier-Stokes equations.

Let the cylindrical domain

$$\Omega = \{(r, \phi, z): \quad 0 < r < R, \;\; 0 \le \phi \le 2\pi, \;\; 0 < z < Z\}$$

contains conducting liquid – electrolyte, where R, Z are the radius and length of the cylinder. The alternating current is fed to N discrete conductors of forms of bars, which are placed parallel to the cylinder axis in the liquid. The current creates in the weakly conductive liquid-electrolyte the radial B_r and the azimuthal B_ϕ components of the magnetic field as well the axial component of the induced electric field E_z, which, in its turn, creates the radial F_r and azimuthal F_ϕ components of the Lorentz force.

For calculation of the electromagnetic fields outside the electrodes, the averaging method over the time interval $2\pi/\omega = 1/f$ is used. The averaged values of force $< F_r >$, $< F_\phi >$ give rise to a liquid motion, which can be described by the stationary Navier-Stokes equation.

2 Calculation of the Electromagnetic Field and Force

Applying the Biot-Savar law we obtain the azimuthal component of the magnetic field B_ϕ and the axial component of vector potential A_z created by the current of density j from one infinite long circular conductor $L = \{r \leq a, 0 \leq \phi \leq 2\pi, -\infty \leq z \leq +\infty\}$ with radius a in following form [5]:

$$B_\phi(r,\phi) = \frac{\mu j a^2}{2\rho}, \quad A_z(r,\phi) = -\frac{\mu j a^2}{2}\ln(\rho),$$

where $\rho = r > a$, $\mu = 4\pi 10^{-7}\frac{mkg}{s^2A^2}$ is the magnetic permeability in vacuum.

For the limit case when the radius of the conductor tends to zero the magnitude $a^2/2$ must be replaced by $1/2\pi$. If the bar type electrode has finite length $z \in [C, D]$, then the azimuthal component of the magnetic field in point P outside of electrodes is given in form $B_\phi(P) = \frac{\mu j}{4\pi\rho}(\cos(\alpha_1) + \cos(\alpha_2))$, where $\alpha_1 = \angle PCD$, $\alpha_2 = \angle PDC$. If α_1 and α_2 tends to zero, then we obtain the previous expression. The magnetic field inside the electrode is not considered here.

For the circular conductor defined by using polar coordinates (r, ϕ)

$$L_i = \{r - r_i \leq a, \ \ \phi_i - \alpha_i \leq \phi \leq \phi_i + \alpha_i, \ \ -\infty \leq z \leq +\infty\}$$

it follows that

$$B_i(r,\phi) = \frac{\mu j_i a^2}{2\rho_i}, \quad A_i(r,\phi) = -\frac{\mu j_i a^2}{2}\ln(\rho_i),$$

where $\rho_i = \sqrt{(r_i^2 + r^2 - 2rr_i\cos(\phi - \phi_i))}$, $\alpha_i = \arcsin(a/r_i)$ and (r_i, ϕ_i) are the polar coordinates of the centers of circular wires L_i. In the cases of alternating current

$$j_i = j_0\cos(\omega t + (i-1)\theta), i = 1, \ldots, N.$$

Here $j_0 = \frac{I}{\pi a^2}$ is the amplitude of density, $\omega = 2\pi f$ is the angular frequency, f is the frequency of the alternating current, $\theta = const$ is the phase (usually $\theta = 120^0$, $f = 50Hz$), t is the time and I is the effective current intensity.

We assume that the azimuthal vector $B_i(r,\phi)$ with respect to the planes at point (r_i, ϕ_i) can be divided in the sum of two vector components $B_{r,i}$, $B_{\phi,i}$, where $B_{r,i} = B_i \sin(\alpha_i)$, $B_{\phi,i} = B_i \cos(\alpha_i)$ and α_i is the angle between vectors B_i and $B_{\phi,i}$. Then

$$\cos(\alpha_i) = \frac{r_i \cos(\phi - \phi_i) - r}{\rho_i}, \quad \sin(\alpha_i) = \frac{r_i \sin(\phi - \phi_i)}{\rho_i}.$$

Therefore we obtain two components of the magnetic field induced by each current wire L_i in the following form

$$\begin{cases} B_{r,i}(r,\phi,t) = \dfrac{\mu j_i a^2}{2\rho_i^2} r_i \sin(\phi - \phi_i), \\ B_{\phi,i}(r,\phi,t) = \dfrac{\mu j_i a^2}{2\rho_i^2} (r_i \cos(\phi - \phi_i) - r), \quad i = 1, \dots, N. \end{cases}$$

We can see that

$$div \mathbf{B}_i = \frac{1}{r}\frac{\partial}{\partial r}(r B_{r,i}) + \frac{1}{r}\frac{\partial B_{\phi,i}}{\partial \phi} = 0.$$

Since

$$B_{r,i} = \frac{1}{r}\frac{\partial A_{z,i}}{\partial \phi}, \quad B_{\phi,i} = -\frac{\partial A_{z,i}}{\partial r},$$

then the axial component of vector-potential $\mathbf{A}$ (i.e., $\mathbf{B} = \mathrm{rot}\mathbf{A}$) is given by

$$A_{z,i}(r,\phi,t) = -\frac{\mu j_i a^2}{2} \ln(\rho_i).$$

It follows from Ohm's law that the axial components j_z of the induced current density are defined as $j_{z,i} = -\sigma \partial A_{z,i}/\partial t$, where σ is the electric conductivity.

From the vector of electromagnetic (the Lorenz) force $\mathbf{F} = \mathbf{j} \times \mathbf{B}$ we can obtain the radial and azimuthal components $F_r = -B_\phi j_z$, $F_\phi = B_r j_z$ as the sum of all induced fields

$$B_\phi = \sum_{i=1}^{N} B_{\phi,i}, \quad B_r = \sum_{i=1}^{N} B_{r,i}, j_z = \sum_{i=1}^{N} j_{z,i}.$$

Therefore, we obtain a system

$$\begin{cases} F_r(r,\phi,t) = K_0 \sum_{i,j=1}^{N} \alpha_{i,j} cs(t), \\ F_\phi(r,\phi,t) = K_0 \sum_{i,j=1}^{N} \beta_{i,j} cs(t), \end{cases}$$

where

$$K_0 = \Big(\frac{a^2 \mu j_0}{2}\Big)^2 \sigma\omega, \quad cs(t) = 0.5 \sin\big(2\omega t + (i+j-2)\theta\big) + 0.5 \sin\big(\theta(j-i)\big),$$

$$\alpha_{i,j} = -\frac{\ln(\rho_i)(r_j \cos(\phi - \phi_j) - r)}{\rho_j^2}, \quad \beta_{i,j} = \frac{\ln(\rho_i) r_j \sin(\phi - \phi_j)}{\rho_j^2}.$$

Similarly, the source term for heat transport equation has the form

$$j_z^2(r,\phi,t) = K_0 \sigma\omega \sum_{i,j=1}^{N} \gamma_{i,j} ss(t),$$

where $\gamma_{i,j} = \ln(\rho_i)\ln(\rho_j)$,

$$ss(t) = -0.5\cos\big(2\omega t + (i+j-2)\theta\big) + 0.5\cos\big(\theta(j-i)\big).$$

Denoting $A_i = \ln(\rho_i)$, we obtain

$$\alpha_{i,j} = A_i\frac{\partial A_j}{\partial r}, \quad \beta_{i,j} = \frac{A_i}{r}\frac{\partial A_j}{\partial \phi}, \quad \gamma_{i,j} = A_i A_j,$$
$$\frac{\partial A_j}{\partial \phi} = \frac{r_j\sin(\phi-\phi_j)}{\rho_j^2}, \quad \frac{\partial A_j}{\partial r} = -\frac{r_j\cos(\phi-\phi_j)-r}{\rho_j^2}.$$

By averaging quantities in the time interval of length $\dfrac{2\pi}{\omega}$ we get

$$\begin{cases} < F_r(r,\phi) >= 0.5K_0 S_N^{\alpha}, \\ < F_\phi(r,\phi) >= 0.5K_0 S_N^{\beta}, \\ < j_z^2(r,\phi) >= 0.5K_0\sigma\omega S_N^{\gamma}, \end{cases}$$

where

$$S_N^{\alpha} = \sum_{i,j=1}^{N}\sin((j-i)\theta)\alpha_{i,j},\ S_N^{\beta} = \sum_{i,j=1}^{N}\sin((j-i)\theta)\beta_{i,j},\ S_N^{\gamma} = \sum_{i,j=1}^{N}\cos((j-i)\theta)\gamma_{i,j}.$$

We can see that

$$S_N^{\alpha} = 2\sum_{k=1}^{N-1}\sin(k\theta)\sum_{i=1}^{N-k}\bar{\alpha}_{i,k+i}, \quad S_N^{\beta} = 2\sum_{k=1}^{N-1}\sin(k\theta)\sum_{i=1}^{N-k}\bar{\beta}_{i,k+i},$$
$$S_N^{\gamma} = 2\sum_{k=1}^{N-1}\cos(k\theta)\sum_{i=1}^{N-k}\gamma_{i,k+i} + \sum_{i=1}^{N}\gamma_{i,i},$$

where

$$\bar{\alpha}_{i,j} = -0.5\big(A_i\frac{\partial A_j}{\partial r} - A_j\frac{\partial A_i}{\partial r}\big), \quad \bar{\beta}_{i,j} = -0.5\frac{1}{r}\big(A_i\frac{\partial A_j}{\partial \phi} - A_j\frac{\partial A_i}{\partial \phi}\big).$$

Using the following formula for axial component of the curl of force vector

$$f = rot_z\mathbf{F} = \frac{1}{r}(\frac{\partial(rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}) = B_r\frac{\partial j_z}{\partial r} + \frac{B_\phi}{r}\frac{\partial j_z}{\partial \phi},$$

we analogously obtain its average value

$$< f(r,\phi) >= 0.5K_0 S_N^{\delta},$$

where

$$S_N^{\delta} = \sum_{i,j}^{N}\sin((j-i)\theta)\delta_{i,j}, \quad \delta_{i,j} = \frac{1}{r}\Big[\frac{\partial}{\partial r}(r\beta_{i,j}) - \frac{\partial}{\partial \phi}(\alpha_{i,j})\Big] = g_{i,j} - g_{j,i},$$
$$g_{i,j} = \frac{r_i\sin(\phi-\phi_i)(r_j\cos(\phi-\phi_j)-r)}{\rho_i^2\rho_j^2}.$$

3 The Mathematical Model

The stationary flow of incompressible viscous liquid in a cylinder is described by the system of the Navier–Stokes equations, which are given in the cylindrical coordinates (r, ϕ, z) [6]:

$$\begin{cases} M(V_z) = -\tilde{\rho}^{-1}\dfrac{\partial p}{\partial z} + \nu \Delta V_z, \\ M(V_r) - r^{-1}V_\phi^2 = -\tilde{\rho}^{-1}\dfrac{\partial p}{\partial r} + \nu(\Delta V_r - r^{-2}V_r - 2r^{-2}\dfrac{\partial V_\phi}{\partial \phi}) + \tilde{\rho}^{-1} < F_r >, \\ M(V_\phi) + r^{-1}V_r V\phi = -(\tilde{\rho} r)^{-1}\dfrac{\partial p}{\partial \phi} + \nu(\Delta V_z - r^{-2}V_\phi + 2r^{-2}\dfrac{\partial V_r}{\partial \phi}) + \tilde{\rho}^{-1} < F_\phi >, \\ \dfrac{\partial (rV_r)}{\partial r} + \dfrac{\partial (V_\phi)}{\partial \phi} + \dfrac{\partial (rV_z)}{\partial z} = 0. \end{cases} \tag{3.1}$$

Here V_r, V_z, V_ϕ are the radial, axial and azimuthal components of velocity vector $\mathbf{V}$, depending on the coordinates r, ϕ, z; and Δ is the Laplace operator,

$$\Delta g = r^{-1}\frac{\partial}{\partial r}\Big(r\frac{\partial g}{\partial r}\Big) + r^{-2}\frac{\partial^2 g}{\partial \phi^2} + \frac{\partial^2 g}{\partial z^2},$$

$< F_r >, < F_\phi >$ are the components of the external averaged force $< \mathbf{F} >$,

$$M(g) = V_r\frac{\partial g}{\partial r} + r^{-1}V_\phi\frac{\partial g}{\partial \phi} + V_z\frac{\partial g}{\partial z}$$

are the convective parts of the equations , $\tilde{\rho}$, ν are the density and kinematic viscosity, p is the pressure, $g = V_r; V_\phi; V_z$. On the walls (the surfaces of the cylinder and electrodes) we have the non-slipping conditions $\mathbf{V} = 0$.

In the cross-section $z = const$ we assume that $V_z = 0$, $\dfrac{\partial^j g}{\partial z^j} = 0$ for all $j \geq 1$ and therefore we can consider the 2D problem. In this case by the elimination of pressure from the second equation of the system of PDEs (3.1) we obtain

$$M(\tilde{\omega}) = \nu \Delta \tilde{\omega} + \tilde{\rho}^{-1} < f >, \tag{3.2}$$

where $\tilde{\omega} = r^{-1}\partial(rV_\phi)/\partial r - \partial V_r/\partial \phi$ is the axial component of vector's $curl\mathbf{V}$ or the function of the vorticity, f is the axial component of the vector's $curl\mathbf{F}$. The stream function ψ can be determined with formulas

$$V_r = r^{-1}\frac{\partial \psi}{\partial \phi}, V_\phi = -\frac{\partial \psi}{\partial r}.$$

From the equation of continuity and from vorticity function it follows, that

$$\tilde{\omega} = -\Delta \psi. \tag{3.3}$$

From (3.2), (3.3) we obtain the system of two PDEs for solving the vorticity function $\tilde{\omega}$ and stream function ψ:

$$\begin{cases} \Delta \psi = -\tilde{\omega} \\ r^{-1}J(\tilde{\omega}, \psi) = \nu \Delta \tilde{\omega} + \tilde{\rho}^{-1} < f >, \end{cases}$$

where $J(\tilde{\omega},\psi) = (\partial\tilde{\omega}/\partial r)(\partial\psi/\partial\phi) - (\partial\tilde{\omega}/\partial\phi)(\partial\psi/\partial r)$ is the Jacobian of the functions ψ and $\tilde{\omega}$.

In the 2D case we have the following boundary conditions:

1. The conditions of periodicity $g(r,0) = g(r,2\pi)$, $\frac{\partial g(r,0)}{\partial\phi} = \frac{\partial g(r,2\pi)}{\partial\phi}$, where $g = \psi, \tilde{\omega}$;

2. The non-slipping conditions on walls $\psi = \partial\psi/\partial\mathbf{n} = 0$, and special conditions for vorticity function $\tilde{\omega} = \omega_w$, where ω_w is the value of the vorticity on the walls (the modificated Wood's conditions [3]) which characterizes the non-slip of the liquid on the wall, $\mathbf{n}$ is the external normal on the walls surfaces.

4 Some Numerical Experiments

The liquid has the following parameters: kinematic viscosity $\nu \approx 10^{-5}\frac{m^2}{s}$, density of liquid $\tilde{\rho} \approx 1000\frac{kg}{m^3}$ and the electric conductivity $\sigma \approx 1000\Omega^{-1}m^{-1}$. The parameter $K_0 = 1$, radius R of the cylinder is $0.035m$, the length Z of the cylinder is $0.35m$, the density of the current amplitude $j_0 \approx 10^8\frac{A}{m^2}$ and the radius a of the electrodes is $0.005m$. At the inlet of the cylinder we assume a uniform velocity $U_0 = 0.1\frac{m}{s}$.

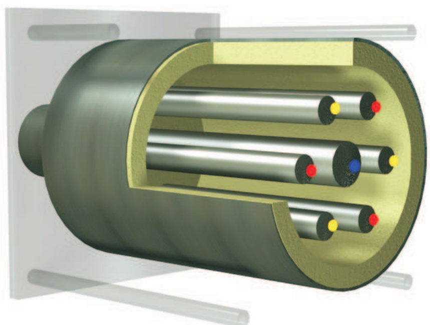

Figure 1. The heat generator.

Calculations and graphic visualization were done with the help of the computer tools MATLAB and FLUENT. In Fig. 1 we can see the electrical heat generator with 7 bar type electrodes. We consider 3 conductors ($N = 3$) placed parallel to the cylinder axis creating the regular triangle with following coordinates of their center $(r_1,\phi_1) = (r_0,0), (r_2,\phi_2) = (r_0,120^0), (r_3,\phi_3) = (r_0,240^0), r_0 = 0.02m$. In the case $N = 2$, $\theta = \pi$, $(r_1,\phi_1) = (r_0,0)$, $(r_2,\phi_2) = (r_0,\pi)$ we have $< F_r >=< F_\phi >= 0$, $< j_z^2 >= 0.5K_0\sigma\omega S_2^\gamma$, where $S_2^\gamma = \ln^2\dfrac{r_0^2 + r^2 - 2rr_0\cos(\phi)}{r_0^2 + r^2 + 2rr_0\cos(\phi)}$. In this case the Lorenz force is zero, but the heat source can depend on the temperature distribution in the liquid. The phase is $2\pi/3$ and the frequency is $50Hz$.

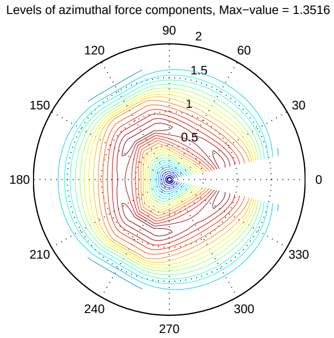

Figure 2. Azimuthal Lorenz force.

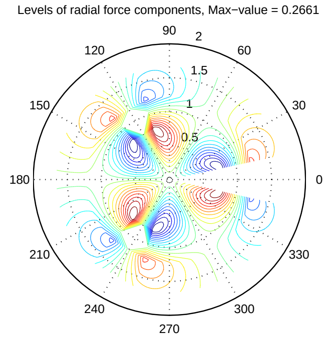

Figure 3. Radial Lorenz force.

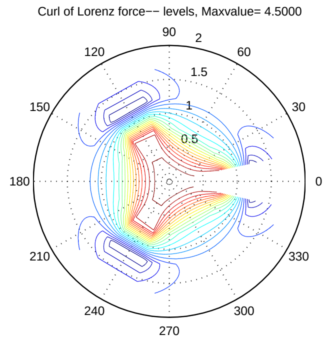

Figure 4. Curl of Lorenz force.

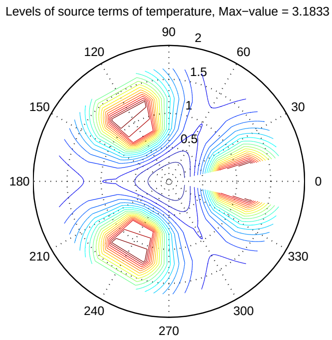

Figure 5. Source terms of temperature.

In this case $\theta = \frac{2\pi}{3}$, $\sin(\theta) = \frac{\sqrt{3}}{2}$, $\sin(2\theta) = -\frac{\sqrt{3}}{2}$, $\cos(\theta) = \cos(2\theta) = -\frac{1}{2}$, and we correspondingly obtain

$$\begin{cases} S_3^\alpha = \frac{\sqrt{3}}{2}(\alpha_{1,2} + \alpha_{2,3} + \alpha_{3,1} - \alpha_{1,3} - \alpha_{2,1} - \alpha_{3,2}), \\ S_3^\beta = \frac{\sqrt{3}}{2}(\beta_{1,2} + \beta_{2,3} + \beta_{3,1} - \beta_{1,3} - \beta_{2,1} - \beta_{3,2}), \\ S_3^\gamma = \gamma_{1,1} + \gamma_{2,2} + \gamma_{3,3} - \gamma_{1,2} - \gamma_{2,3} - \gamma_{1,3}, \\ S_3^\delta = \sqrt{3}(\delta_{1,2} + \delta_{2,3} + \delta_{3,1}). \end{cases} \tag{4.1}$$

In the Figures 2–5 we presents the results of calculations obtained by computer program MATLAB in cross-section $z = const$:

1. Distribution of the averaged azimuthal Lorenz force $< F_\phi >$ (see, Fig. 2);

2. Distribution of the averaged radial Lorenz force $< F_r >$ (see, Fig. 3);

3. Distribution of the averaged axial curl of Lorenz force $< f >$ (see, Fig. 4);

4. Distribution of the heat sours term $< j_z^2 >$ (see, Fig. 5).

These results are nondimensionalized by scaling all the lengths to r_0. Then we have the nondimensional radius of cylinder and electrodes $R/r_0 = 1.75$, $a/r_0 = 0.25$. It seems that the electromagnetic forces are concentrated in the interior of electrodes. In Figures 6–9 the results for 3D problem are obtained

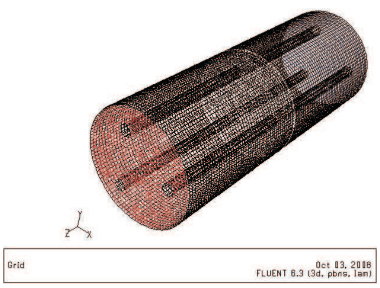

Figure 6. 3D grid.

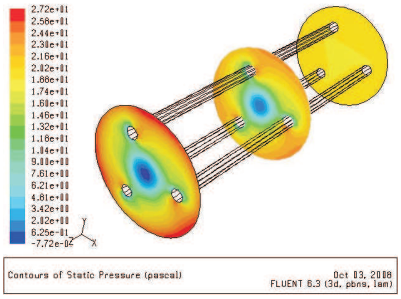

Figure 7. Pressures in 3 cross-section.

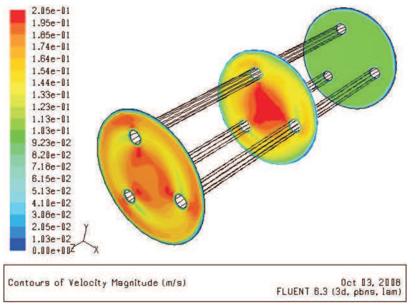

Figure 8. Magnitude of velocity.

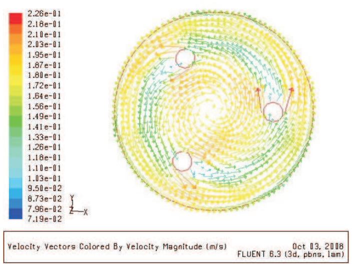

Figure 9. Vectors of velocity in output.

by computer program FLUENT:

1. 3D grid for finite elements method (see, Fig. 6);

2. Distribution of the pressure in three cross-section of the cylinder $z = 0$, $z = 0.16$, $z = 35$ (see, Fig. 7);

3. Distribution of the magnitude for the velocity in three cross-section of the cylinder (see, Fig. 8);

4. Velocity vectors $\mathbf{V}$ in the cross-section $z = 35$ (see, Fig. 9).

We see the vortex formation in the cylinder and how the uniform flow in the output becomes the vortex flow.

5 Conclusions

The distribution of electromagnetic fields and forces induced by a three – phase system of the alternating electric current in the conducting liquid in the cylinder of finite length has been calculated. An original method was used to calculate the mean values of magnetic field and electromagnetic forces. The 2D averaged magnetic field, source terms for the temperature and the Lorenz forces, induced by alternating current with three bar type electrodes are calculated in cross-section of cylinder by computer program MATLAB. 3D magnetohydrodynamics flow of the liquid is calculated with the help of the computer programs FLUENT.

In future it is interesting to apply the finite difference method and to calculate the distributions of magnetohydrodynamical and termodinamical fields for 2D problem in the fixed cross-section of cylinder depending on the electromagnetic and thermodinamical forces.

References

[1] A. Buikis and H. Kalis. Numerical modelling of heat and magnetohydrodinamics flows in a finite cylinder. *Computational methods in applied mathematics*, **2**(3):243–259, 2002.

[2] A. Buikis and H. Kalis. Creation of temperature field in a finite cylinder by alternated electromagnetic force. In A.Buikis, R. Čiegis and A. D. Fitt(Eds.), *Progress in industrial mathematics at ECMI 2002*, pp. 247–251. Springer, 2004.

[3] A. Buikis and H. Kalis. Flow and temperature calculation of electrolyte for a finite cylinder in the alternating field of finite number circular wires. *Magnetohydrodynamics*, **40**(1):77–90, 2004.

[4] A. Buikis and H. Kalis. The vortex formation in horizontal finite cylinder by alternating electric current. *Math. Model. Anal.*, **10**(1):9–18, 2005.

[5] N.E. Kochin, I.A. Kibelj and N.V. Roze. *Theoretical hydrodynamics, part 1.* Moscow, Nauka, 1963. (in Russian)

[6] J.M. Milne-Thomson. *Theoretical hydrodynamics.* St.Martin's Press, London, New York, 1960.