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ON SOLUTIONS OF ONE 6-TH ORDER NONLINEAR BOUNDARY VALUE PROBLEM *

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Abstract. A special technique based on the analysis of oscillatory behaviour of linear equations is applied to investigation of a nonlinear boundary value problem of sixth order. We get the estimation of the number of solutions to boundary value problems of the type $x^{(6)} = f(t, x)$, $x(a) = A$, $x'(a) = A_1$, $x''(a) = A_2$, $x'''(a) = A_3$, $x(b) = B$, $x'(b) = B_1$, where f is continuous together with the partial derivative f'_x which is supposed to be positive. We assume also that at least one solution of the problem under consideration exists.

Key words: nonlinear boundary value problem, multiplicity of solutions, oscillation, differential equations of 6-th order.

1 Introduction

We employ the idea by A. Perov (see [4], ch. 15) who studied the multiplicity of solutions to two-point the second order nonlinear boundary value problems. His approach is based on comparison of the behaviour of solutions of the equation with some given solution of the BVP and at infinity. The first type of behaviour is described in terms of the linear equation of variations and the second type of the behaviour is a consequence of requirements on a function f on the right side of the equation. Using this idea and the technique of the angular function the multiplicity results were obtained. To apply this idea to the study of higher order equations one have to choose other methods since the angular function technique hardly can be applied in this case. As an alternative the theory of oscillation of linear equations of higher order can be used. Multiplicity results for the third order BVPs were obtained in [1] by combining the idea of A. Perov and some facts from the linear theory of conjugate points. The notion

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of a conjugate point is useful in our considerations. We use the definition of conjugate point by Kiguradze [3].

Definition 1. The minimal value of $t = \eta_{m+n-1}$, where $t = \eta_{m+n-1}$ is a $(m+n-1)st$ zero of solution of n-th order equation which has a zero at $t = a$, counting multiplicities, is called m-th conjugate point of a.

As a by-product, a technique was elaborated for treatment of the higher dimensional case. In [6], n-th order BVPs were considered by straight-forward generalizations of the results of [1]. However, only the case of $n-1$ (of total number n) boundary conditions prescribed at one end of the interval was treated. In this paper we study the boundary value problem

$$x^{(6)} = f(t,x), \tag{1.1}$$

$$x(a) = A, \quad x'(a) = A_1, \quad x''(a) = A_2, \quad x'''(a) = A_3, \tag{1.2}$$

$$x(b) = B, \quad x'(b) = B_1, \tag{1.3}$$

under the assumption that functions f, $f_x = \frac{\partial f}{\partial x}$ are continuous and $f_x > 0$. We suppose also that there exists a particular solution $\xi(t)$ of the above problem. A solution (1.1) can be described in terms of oscillatory behaviour of the corresponding linear equation of variations

$$y^{(6)} = f_x(t,\xi(t))y, \tag{1.4}$$

which will play a significant role in our considerations. Our results are based on the theory of 6-th order linear boundary value problem for 6-th order linear differential equations of the form

$$y^{(6)} = p(t)y, \tag{1.5}$$

with boundary conditions (1.2)–(1.3). We assume that p is continuous and $p > 0$. We refer to the book by Kiguradze and Chanturia [3] with respect to two termed equations $y^{(n)} = p(t)y$, and to the article by Hunt [2] for the 6-th order equation.

Our principal result consists of estimation from below of the number of solutions to the boundary value problem (1.1)–(1.3) provided that it has at least one solution $\xi(t)$. This estimate depends on the oscillatory behaviour of the equation (1.4).

The paper is organized as follows. In the subsequent Section 2 we investigate linear equation of the form (1.5). The main theorem of this section describes the two-parametric set of solutions to the equation (1.5), subject to initial data

$$y(a) = y'(a) = y''(a) = y'''(a) = 0. \tag{1.6}$$

In Section 3 the multiplicity result is proved. The nonlinear problem (1.1)–(1.3) is considered, provided that f is bounded. Our method of the proof differs from that used by Perov [4] and it is based on representation of the nonlinear equation (1.1) as a family of linear equations, the coefficients of which depend on solutions of (1.1) satisfying the initial value conditions (1.2).

2 Linear Equation

In this section we consider equation (1.5) with continuous coefficient $p > 0$. We show that lemmas similar to Lemma 2.1 and Lemma 2.2 in [5] are valid for equation (1.5).

Lemma 1. *If $y(t)$ is a solution of (1.5) and the values of $y^{(i)}$, $i = 0, \dots, 5$ are non-negative (but not all zero) at $t = a$, then functions $y^{(i)}(t)$, $i = 0, \dots, 5$ are positive for $t > a$.*

Proof. Consider the case of $y'(a) > 0$. First we show that $y'(t) > 0$, $t > a$. We assume that there exists $t_0 > a$ such that $y'(t_0) = 0$ and $y'(t) > 0$, if $t \in [a; t_0)$. Then it follows that there exists $t_1 \in [a; t_0)$ such that $y''(t_1) < 0$, therefore there exists $t_2 \in [a; t_1)$ for which $y^{(3)}(t_2) < 0$. Similarly there exist

$$t_k \in [a; t_{k-1}), \quad y^{(k+1)}(t_k) < 0, \quad k = 3, 4, 5.$$

Since $y^{(6)} = p(t)y$, where $p(t)$ is positive valued function and $y^{(6)}(t_5) < 0$, therefore $y(t_5) < 0$, but $y(a) \geq 0$, therefore $\exists t_6 \in [a; t_5) \subset [a; t_0)$, $y'(t_6) < 0$. We got a contradiction.

Now we show that $y(t) > 0$, $t > a$. Assume that there exists $t_7 > a$, such that $y(t_7) < 0$, then it follows that there exists $t_8 \in [a; t_7)$ such that $y'(t_8) < 0$. Again we got a contradiction with the fact that $y'(t) > 0$, $t > a$.

Since $y^{(6)} = p(t)y$, where $p(t)$ is a positive valued function and $y(t) > 0$, $t > a$, therefore $y^{(6)}(t) > 0$, $t > a$. If $y^{(6)}(t) > 0$, $t > a$, and $y^{(i)}(a) \geq 0$, $i = 5, 4, 3, 2$, then we get that $y^{(5)}(t) > 0$, and it follows that $y^{(4)}(t) > 0$, $y^{(3)}(t) > 0$, $y^{(2)}(t) > 0$, $t > a$.

In the other cases, if it is given that $y^{(i)}(a) > 0$, $i = 0, 2, 3, 4, 5$, the proofs are similar. □

Lemma 2. *If $y(t)$ is a solution of (1.5) and $y^{(i)}(a) \geq 0$, $i = 0, 2, 4$, $y^{(i)}(a) \leq 0$, $i = 1, 3, 5$, then the functions $y^{(i)}(t)$, $i = 0, 2, 4$ are positive and the functions $y^{(i)}(t)$, $i = 1, 3, 5$ are negative for $t < a$.*

Proof. The proof follows from Lemma 1 by making the variable change $\tau = -t$. □

Lemma 3. *If $u(t)$ and $v(t)$ are two different solutions of (1.5) and $u^{(i)}(a) \geq v^{(i)}(a)$, $i = 0, \dots, 5$ (and at least one nonequality is strong), then $u^{(i)}(t) > v^{(i)}(t)$, $i = 0, \dots, 5$ for $t > a$.*

Proof. The result follows from Lemma 1, if we use function $w(t) = u(t) - v(t)$. This function is a solution of (1.5) and $w^{(i)} \geq 0$, $i = 0, \dots, 5$ (but not all equal to zero). □

Let us state the result by Kiguradze adapted to the case of the 6-th order equation.

Lemma 4. *The first conjugate point η_1 to the point $t = a$ is given by a solution, which has quadruple zero at the point $t = a$ and double zero at the point η_1.*

Now we can prove the following result.

Lemma 5. *The first conjugate point* η_1 *continuously depends on the coefficient* $p(t) > 0$ *of equation* (1.5).

Proof. Consider equation (1.5) together with a boundary conditions

$$y(a) = y'(a) = y''(a) = y'''(a) = 0, \quad y^{(4)}(a) = \alpha, \quad y^{(5)}(a) = \beta.$$

Due to linearity of (1.5) we can consider the initial angle $\Theta = \arctan\left(\frac{\beta}{\alpha}\right)$. Let Θ_1 be the angle corresponding to a solution with the conjugate point η_1 which is a double zero, unstable under perturbations of the coefficient $p(t)$. However, by Lemma 1 a solution with $\Theta < \Theta_1$ satisfies $y(t) < y_1(t)$ for $t > 0$. If Θ is close enough to Θ_1 the respective $y(t)$ has two simple zeros on opposite sides of η_1 and close to it. These simple zeros change continuously together with $p(t)$. Since η_1 lies between these zeros, it changes continuously too. □

3 Nonlinear Equation

In this section, results for the nonlinear boundary value problem (1.1)–(1.3) are stated and proved. Suppose $\xi(t)$ is a solution of the boundary value problem (1.1)–(1.3). Let $x(t, \alpha, \beta)$ be a solution of the equation (1.1), subject to the initial value conditions (1.6) and

$$x^{(4)}(a) - \xi^{(4)}(a) = r\cos\Theta, \quad x^{(5)}(a) - \xi^{(5)}(a) = r\sin\Theta. \tag{3.1}$$

We denote $z(t) = x(t) - \xi(t)$. Consider auxiliary linear equations

$$z^{(6)} = \varphi(t, r, \Theta)z, \tag{3.2}$$

where coefficient φ depends on solutions of the problem (1.1), (1.6), (3.1):

$$\varphi = \frac{f(t, x(t, r, \Theta)) - f(t, \xi(t))}{x(t, r, \Theta) - \xi(t)}.$$

We assume that at the points where denominators vanish the right hand sides are substituted by appropriate value of f_x.

Our further considerations are based on the following observation.

Lemma 6. *A solution* $x(t)$ *of the initial value problem* (1.1), (1.6), (3.1) *satisfies also conditions* (1.3) *(i.e. it is a solutions of the problem* (1.1)–(1.3)*) if and only if there exists an extremal function* $Z_1(t)$ *of the linear equation* (3.2), *realizing conjugate point to* $t = a$ *at* $t = b$, *and such that*

$$\arctan\frac{z^{(5)}(a)}{z^{(4)}(a)} = \Theta.$$

Proof. Let $x(t)$ be a solution $(x(t) \neq \xi(t))$ of the problem (1.1)–(1.3). Necessity then follows from the observation that $x - \xi$ is an extremal function, since it satisfies the equation (3.2) and has a quadruple zero at $t = a$ and a double zero at $t = b$.

Consider now linear equation (3.2) corresponding to some solution $x(t)$ of (1.1), (1.6), (3.1). Suppose that $Z(t)$ is an extremal function for (3.2), realizing a conjugate point at $t = b$. Without a loss of generality we may assume that

$$Z^{(4)}(a) = x^{(4)}(a) - \xi^{(4)}(a), Z^{(5)}(a) = x^{(5)}(a) - \xi^{(5)}(a).$$

Otherwise $Z(t)$ should be multiplied by an appropriate constant. Both functions $Z(t)$ and $x(t) - \xi(t)$ are solutions of the same Cauchy problem for linear equation (3.2). Thus they are identical, and $x(t) = Z(t) + \xi(t)$ satisfies also the boundary condition (1.3) at the right end of interval (a, b). □

Lemma 7. *Let f in* (1.1) *be bounded. Let $\xi(t)$ be a solution of the problem* (1.1)–(1.3) *and $x(t)$ be a solution of the initial value problem* (1.1)–(1.2). *Then the difference $x(t) - \xi(t)$ cannot have more than five zeros* (*counting multiplicities*) *in (a, b) for large enough $r^2 = (x^{(4)}(a) - \xi^{(4)}(a))^2 + (x^{(5)}(a) - \xi^{(5)}(a))^2$.*

Proof. The proof follows from the fact that

$$(x - \xi)^{(6)}(t)/r^2 = (f(t, x(t)) - f(t, \xi(t)))/r^2$$

and the right hand side tends to zero uniformly in $t \in [a, b]$, as $r \to \infty$. Then functions $(x(t) - \xi(t))/r^2$ tend to solutions of the equation $y^{(6)} = 0$, none of which, having fourth order zero at $t = a$, can have more than one zero in (a, b). □

The linear sixth order differential equation is said to be disconjugate on the interval $[a, b]$ if each of its nontrivial solutions has at most five zeros, counting multiplicities, on $[a, b]$.

Lemma 8. *Equations* (3.2) *are disconjugate on $[a, b]$ for large enough values of r.*

Proof. By Lemma 7, functions $(x(t) - \xi(t))/r^2$ tend to $z(t)$ as $r \to \infty$, where $z(t) = t^4(k - t)$ with appropriate $k > 0$ (the solution of equation $z^{(6)} = 0$ having quadruple zero at $t = a$). Thus, both functions have at most two zeros in $[a, b]$ (the first zero at $t = a$). We may include these zeros in a subset U of $[a, b]$ of arbitrarily small measure. On the complement of U, function $x(t) - \xi(t)$ is large enough (for $r \to \infty$) to make denominators in coefficients of (3.2) arbitrarily small. Thus φ in (3.2) can be made small enough in the integral sense. The lemma now follows from (1.6). □

Theorem 1. *Let $\xi(t)$ be a solution of the problem* (1.1)–(1.3). *Suppose that function f in* (1.1) *is bounded and there exists a solution of the problem* (1.1)–(1.3) *such that the interval (a, b) contains exactly one conjugate point η_1* (*to $t = a$*) *with respect to the equation of variation* (1.4). *Than the boundary value problem* (1.1)–(1.3) *has at least $2 + 1$ solutions* (ξ *counted*).

Proof. Fix $\Theta \in (-\pi/2, 0)$ and consider one parametric family of linear equations (3.2). For small r, solutions of (3.2) behave like solutions of the equation of variations. Hence, there exists one conjugate point η_1 in the interval (a, b) for small r. On the other hand for large r, this interval does not contain point η_1. The only way for conjugate point to leave the interval (a, b), as r varies from zero to infinity, is to pass over $t = b$. Let $S_1(\Theta) = \max[r : \eta_1 = b]$. For any $\Theta \in [-\pi/2, 0)$ such S_1 exists and forms a continuous one-parametric (dependent on Θ) curve. Denote by $w_1(\Theta)$ the angle defining the first extremal function of the equation

$$z^{(6)} = \varphi(t, S_1(\Theta), \Theta)z,$$

and consider the difference $w_1(\Theta) - (\Theta)$. Note that $w_1(0)$ is positive and $w_1(-\frac{\pi}{2}) - (-\frac{\pi}{2})$ is negative. Hence, there exists $\Theta_1 \in [-\pi/2, 0)$ such that $w_1(\Theta_1) = \Theta_1$. By Lemma 6, the 1-th extremal function of the equation (3.2) where $r = S_1(\Theta_1)$, $\Theta = \Theta_1$, generates a solution to the boundary value problem (1.1)–(1.3). $\square$

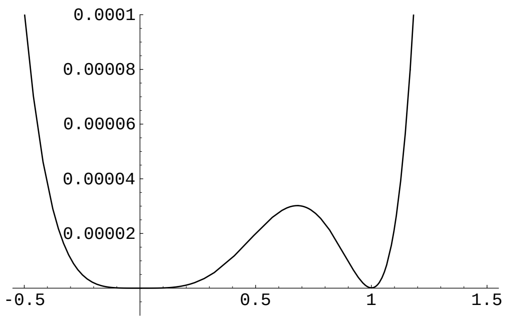

Figure 1. The solution of nonlinear boundary value problem (3.3)–(3.4), which satisfies $x^{(4)}(0) = 0.02279$, $x^{(5)}(0) = -0.155$.

Example 1. Here we will apply the results of Theorem 1 to the following nonlinear boundary value problem

$$x^{(6)} = \arctan(8^6 x), \tag{3.3}$$

$$x(0) = x'(0) = x''(0) = x'''(0) = x(1) = x'(1) = 0. \tag{3.4}$$

$\xi \equiv 0$ is a solution of problem (3.3)–(3.4). We have such equation of variation

$$z^{(6)} = 8^6 z$$

which has conjugate point $\eta_1 \approx 0.838454$ in $(0, 1)$. Therefore, by Theorem 1, the nonlinear problem (3.3)–(3.4) has at least three solutions. We have computed the first solution of nonlinear equation (see, Figure 1). The second solution is obtained by multiplying the first solution by -1. The third one is the trivial solution $\xi \equiv 0$.

4 Conclusions

The classical object of oscillation theory is a second order linear equation. There are less but also many papers devoted to the fourth order linear equation. Articles on the sixth order linear equation are lacking. We prove some additional results for the sixth order two-termed linear equation concerning conjugate points and oscillatory behavior of solutions. These results are specific for the sixth order equation. Doe to simple form of the linear equation the structure of a set of solutions is quite clear. The results for linear equations are used then to establish the multiplicity results for two-termed nonlinear equation which is represented as a family of linear equations dependent on parameters. The same scheme can be used to prove the multiplicity results for higher order equations, and not only for the case of the first conjugate point.

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