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MODELLING OF ROTATIONS BY USING MATRIX SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Abstract. A family of matrix solutions of nonlinear wave equations is extended and its application to modelling is given. It is shown that a similarity transformation, induced by the matrix solution, is equivalent to the rotation. Matrix solutions are used for modelling helical motions and vortex rings, simultaneous rotations and particles collision, mapping contraction and pulsating spheres. Geometrical interpretation of the doubling of rotation angle in each step of sequential mapping contraction is given.

Key words: anti-commuting matrices, mapping contraction, matrix solution, nonlinear wave equation, particles collision, rotation, vortex ring

1. Introduction

In the present paper we develop some results of papers [4, 5, 6], where matrix solutions u_n of the order $n = 2^k$ were constructed for nonlinear wave equations. Now we show how matrix solutions of arbitrary even or odd order can be constructed. However, the main aim of the paper is to propose some applications of matrix solutions. In particular, by using matrix solutions we describe a toroidal motion in vortex rings. We also describe a helical motion of pulsating spheres. On the base of the theorem of composition and decomposition of simultaneous rotations we propose a model of particles collision. We also propose a geometrical interpretation of the doubling of rotation angle on each step of mapping contraction and we give a relation with Euler angles.

Let us remind the basic results from papers [4, 5, 6] starting from the Klein-Gordon eguation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{dQ}{du} = 0, \quad Q(u) = \frac{\mu^2}{4}(u^2 - 1)^2. \tag{1.1}$$

The corresponding sequence of matrix solutions u_n $(n = 2^k, k = 1, 2, \ldots)$ is constructed as

$$u_n(\phi, \mathbf{a}) = \cos(\phi)E_n + \mathbf{a}\sin(\phi) = \exp(\phi\mathbf{a}), \quad \mathbf{a} = \sum_{j=1}^{2k+1} a_j M_j, \quad |\mathbf{a}| = 1. \quad (1.2)$$

Here we use the following notation: E_n is the unit $n \times n$-matrix, M_j are unitary $M_j\, M_j^* = E_n$, anti-Hermitian $M_j^* = -M_j$, anti-commuting $n \times n$-matrices, $*$ means transposition and conjugation. The angular parameter ϕ is defined as follows

$$\phi \equiv \phi(\alpha z) = \operatorname{arccot}(-\sinh(\alpha z)), \quad \alpha = \mu\sqrt{2/(1 - v^2)}, \qquad (1.3)$$

where $z = x - vt$ is the moving frame of reference, x is an arbitrary point along chosen direction in 3D-space of coordinates x_1, x_2, x_3. Note that ϕ varies from 0 to π to within period π of function cot when z varies from $-\infty$ to ∞. For definiteness we assume that $0 \le \phi \le \pi$, where boundary points 0 and π correspond to elementary solutions $u = 1$ and $u = -1$, respectively.We also use functions $u_n(m\phi, \mathbf{a})$, of the same structure as (1.2), which satisfy a nonlinear wave equation like (1.1) with more complicated potentials Q_m (see [4]).

The complete the system $2k + 1$ anti-commuting, unitary, anti-Hermitian, linearly independent matrices $M_j(n) \equiv M_j$ of the order of $n = 2^k$ $(k = 1, 2, \ldots)$ are constructed for $k = 2, 3, \ldots$ as follows:

$$M_j(n) = \begin{pmatrix} M_j(n/2) & 0 \\ 0 & -M_j(n/2) \end{pmatrix}, \qquad j = 1, 2, \ldots, 2k - 1,$$

$$M_{2k}(n) = \begin{pmatrix} 0 & -E_{n/2} \\ E_{n/2} & 0 \end{pmatrix}, \quad M_{2k+1}(n) = \begin{pmatrix} 0 & iE_{n/2} \\ iE_{n/2} & 0 \end{pmatrix}.$$

For $k = 1$ we take $M_j(2) = H_j$ $(j = 1, 2, 3)$, where H_j are unit quaternions.

2. Additional Construction

In this section we propose a simple procedure for construction of matrices $M_j(n)$ of an arbitrary even or odd order.

Even order $n = 2N$, where $2^k < 2N < 2^{k+1}$, can be always represented in the form

$$2N = 2^k + 2^{k-i_1} + \ldots + 2^{k-i_m}, \quad 1 \le i_1, \ldots < i_m \le (k - 1).$$

Hence, matrix $M_j(2N)$ can be constructed as a diagonal block matrix with the blocks of the orders $2^k, 2^{k-i_1}, \ldots, 2^{k-i_m}$. It is clear that matrices $M_j(2N)$ are anti-commuting, unitary, anti-Hermitian, and linearly independent.

Matrices of the odd order $n = 2N + 1$ are constructed as follows

$$M_j(2N + 1) = \operatorname{diag}(M_j(2N), -i), \quad j = 1, 2, \ldots, 2(k - i_m) + 1.$$

These matrices are unitary and anti-Hermitian but not anti-commuting, however, as is proved below, matrix function $u_{2N+1}(\phi, \mathbf{a})$ also satisfies (1.1) provided an additional condition on unit vector $\mathbf{a}$.

Theorem 1. *Function*

$$u_{2N+1}(\phi, \mathbf{a}) = E_{2N+1} \cos\phi + \mathbf{a} \sin\phi, \quad \mathbf{a} = \sum_{j=1}^{2(k-i_m)+1} a_j M_j(2N+1), \quad |\mathbf{a}| = 1$$

satisfies equation (1.1) and it has the exponential representation

$$u_{2N+1} = \exp(\phi, \mathbf{a}),$$

provided additional condition $\sum a_j = \pm 1$ *is valid.*

Proof. Let us reduce equation (1.1) to the ordinary differential equation

$$(v^2 - 1)\frac{d^2 u}{dz^2} + \mu^2 u(u^2 - 1) = 0. \tag{2.1}$$

The first derivative of u_n is $du/dz = (-E_n \sin\phi + \mathbf{a}\cos\phi) d\phi/dz$. It follows both from the definition (1.3), i.e. from $\cot\phi = -\sinh(\alpha z)$, and from equality $\sin^2\phi = (1 + \cot^2\phi)^{-1}$ that $d\phi/dz = \alpha \sin\phi$. Thus we find

$$\frac{du_n}{dz} = \frac{\alpha}{2}\big(- 2E_n \sin^2\phi + 2\mathbf{a} \sin\phi \cos\phi\big) = \frac{\alpha}{2}\big(u_n(2\phi, \mathbf{a}) - 1\big).$$

We identify E_n with 1 both in the last term and in the term $u^2 - 1$ of equation (2.1). It follows from the definition of $M_j(2N+1)$, the anti-commuting property of matrices $M_j(2N)$, and from condition $\sum a_j = \pm 1$ that equality $\mathbf{a}^2 = -E_{2N+1}$ is valid. Thus for $n = 2N+1$ the equality $(u_n(\phi, \mathbf{a}))^2 = u_n(2\phi, \mathbf{a})$ is also valid. Therefore

$$\frac{du_n}{dz} = \frac{\alpha}{2}(u_n^2 - 1), \qquad \frac{d^2 u_n}{dz^2} = \frac{\alpha^2}{2} u_n (u_n^2 - 1)$$

and, hence, equation (2.1) is reduced to the form

$$((v^2 - 1)\frac{\alpha^2}{2} + \mu^2) u_n (u_n^2 - 1) = 0. \tag{2.2}$$

It is evident that function u_{2N+1} satisfies this equation, provided $\alpha^2 = \dfrac{2\mu^2}{1 - v^2}$, and, hence, it satisfies equation (1.1). Moreover, equation (2.2) has also the trivial solutions $u_n = 0$, $u_n = \pm 1$. Finally, due to the basic equality $\mathbf{a}^2 = -E_{2N+1}$, we find the following matrix expansion

$$\exp(\phi\mathbf{a}) = E_n + \phi\mathbf{a} - \frac{\phi^2}{2}E_n - \frac{\phi^3}{6}\mathbf{a} + \ldots = E_n \cos\phi + \mathbf{a}\sin\phi = u_n(\phi, \mathbf{a}).$$

The theorem is proved. ■

3. Mapping and Rotation

In this section we consider a more wide set of matrix functions $u_n(\phi, \mathbf{a})$ for $(j = 1, \ldots, 2k+1)$, constructed as (1.2), provided that angular parameter ϕ can be an arbitrary real number or differentiable increasing function on time.

It is easy to verify that similarity transformation

$$M_j \to M_j' = u_n(\phi, \mathbf{a}) M_j u_n^*(\phi, \mathbf{a}), \quad j = 1, 2, \ldots, 2k+1 \tag{3.1}$$

maps system $\{M_j\}$ to a similar system $\{M_j'\}$ which satisfies the same properties as the initial system. For example, if $\mathbf{a} = M_{2k+1}$ and $\phi = \pi/4$ then mapping (3.1) transfers system $\{M_j(n)\}$ to the following system:

$$M_j'(n) = M_{2k+1} M_j = \begin{pmatrix} 0 & -iM_j(n/2) \\ iM_j(n/2) & 0 \end{pmatrix}, \quad j = 1, 2, \ldots, 2k-1,$$

$$M_{2k}'(n) = \begin{pmatrix} iE_{n/2} & 0 \\ 0 & -iE_{n/2} \end{pmatrix}, \quad M_{2k+1}'(n) = \begin{pmatrix} 0 & iE_{n/2} \\ iE_{n/2} & 0 \end{pmatrix}.$$

Let V_{2k+1} be a vector space spanned over basis vectors $M_1, M_2, \ldots, M_{2k+1}$ and let vector $\mathbf{b}$ and unit vector $\mathbf{a}$ belong to V_{2k+1}.

Theorem 2. *Mapping $\mathbf{b} \to u_n \mathbf{b} u_n^*$ induced by function $u_n(\phi, \mathbf{a})$ is equivalent to rotation $u_n(2\phi, \mathbf{a})\mathbf{b}$ of vector $\mathbf{b}$ about unit vector $\mathbf{a}$ by angle 2ϕ, provided orthogonality condition $\sum a_j b_j = 0$ is valid.*

Proof. It follows from the anti-commuting property of matrices M_j and from the orthogonality condition that

$$\mathbf{ba} = \sum_{i,j} b_i a_j M_i M_j = \sum_{i \neq j} a_j b_i (-M_j M_i) + \sum_j a_j b_j (-E_n) = -\mathbf{ab}.$$

Thus, taking into account $\mathbf{a}^2 = -E_n$ and $(u_n(\phi, \mathbf{a}))^2 = u_n(2\phi, \mathbf{a})$ we have

$$\mathbf{b} \to u_n \mathbf{b} u_n^* = u_n(\phi, \mathbf{a})(\mathbf{b} \cos\phi - \mathbf{ba} \sin\phi) = u_n(2\phi, \mathbf{a})\mathbf{b}.$$

The last term is equal to $(\mathbf{b} \cos 2\phi + \mathbf{ab} \sin 2\phi)$. It means that vector $\mathbf{b}$ is turned by angle 2ϕ about vector $\mathbf{a}$ in the plane of vectors $\mathbf{b}, \mathbf{ab}$. The theorem is proved. ■

It should be noted that when we say that $u_n(\phi, \mathbf{a})$ accomplishes a rotation, we mean an action $u_n(2\phi, \mathbf{a})\mathbf{b}$. The angular parameter can be replaced by any divisible by ϕ angle. Moreover, in the case of unit vector $\mathbf{b}$ we simply define a rotation u_n.

Now we consider a mapping contraction. We consider a condition when rotation u_n in V_{2k+1} induces a rotation $u_{n/2}$ in V_{2k-1}, i.e. in the space spanned over basis vectors $M_1(n/2), \ldots, M_{2(k-1)}(n/2)$. It follows from the construction of matrices M_j that they have the block-diagonal form

$$M_j(n) = \operatorname{diag}(M_j(n/2), M_j^*(n/2)), \quad j = 1, 2, \ldots, 2k-1.$$

Therefore function u_n also has a diagonal form $u_n = \mathrm{diag}(u_{n/2}, u^*_{n/2})$ provided $\mathbf{a} = (a_1, \ldots, a_{2k-1}, 0, 0)$. Remind that $u^*_n = u_n^{-1}$ for arbitrary index n. On this base we define the next formula for mapping contraction.

DEFINITION 1. Let assume that orthogonality condition $\sum a_j b_j = 0$ is valid. Then mapping contraction from V_{2k+1} to V_{2k-1} induced by diagonal matrix $u_n = \mathrm{diag}(u_{n/2}, u^*_{n/2})$ is defined in the form of linear fractional transformation (it is denoted by symbol $\Rightarrow$):

$$u_n(2\phi, \mathbf{a})\mathbf{b} \;\Rightarrow\; \frac{u_{n/2}(2\phi, \mathbf{a})}{u^*_{n/2}(2\phi, \mathbf{a})}\mathbf{b} = u_{n/2}(4\phi, \mathbf{a})\mathbf{b}.$$

Note that in the right-hand side of this formula vectors $\mathbf{a}$ and $\mathbf{b}$ belong to V_{2k-1}, i.e. they are matrices of the order of $n/2$. The last term in the defined formula means a rotation of vector $\mathbf{b}$ about vector $\mathbf{a}$ by angle 4ϕ, i.e. this formula leads to the doubling of rotation angle. The defined linear fractional transformation is equivalent to the similarity transformation in V_{2k-1}

$$u_{n/2}(2\phi, \mathbf{a})\mathbf{b}u^*_{n/2}(2\phi, \mathbf{a}) = u_{n/2}(4\phi, \mathbf{a})\mathbf{b}.$$

The process of mapping contraction can be continued to lowering vector spaces. Namely, for orthogonal vectors $\mathbf{a}, \mathbf{b} \in V_{2k+1}$, of the form $\mathbf{a} = (a_1, a_2, a_3, 0, \ldots, 0)$ and $\mathbf{b} = (b_1, b_2, b_3, 0, \ldots, 0)$, a chain of contractions from V_{2k+1} to V_{2k-1} and so on until V_3 is valid in the form

$$u_n(2\phi, \mathbf{a})\mathbf{b} \Rightarrow u_{n/2}(4\phi, \mathbf{a})\mathbf{b} \Rightarrow \ldots \Rightarrow u_2(n\phi, \mathbf{a})\mathbf{b}. \tag{3.2}$$

Now we propose a geometrical interpretation of the doubling of rotation angle in the chain (3.2). It is known that block-diagonal matrix can be represented as direct sum. Thus for a special case of vector $\mathbf{b}$ we have

$$\mathbf{b} = \sum_{j=1}^{2k-1} b_j M_j(n) = \sum_{j=1}^{2k-1} b_j M_j(n/2) \oplus \sum_{j=1}^{2k-1} b_j M^*_j(n/2) \equiv \mathbf{b}(n/2) \oplus \mathbf{b}^*(n/2).$$

Moreover, if we take the linear norm of matrices, then we find

$$|\mathbf{b}(n)| = |\mathbf{b}(n/2)| + |\mathbf{b}^*(n/2)| = 2|\mathbf{b}(n/2)|.$$

Analogously on each step in chain (3.2) we find that vector $\mathbf{b}$ is two times shorter than in the previous step. Let us take a unit circle C_1 with center O_1 and take a unit vector $\mathbf{b}(n) = O_1A_1$ directed along horizontal line up to point A_1 on C_1. Let O_2 be a middle point of O_1A_1 and let $\mathbf{b}(n/2) = O_2A_1$, $\mathbf{b}^*(n/2) = O_2O_1$. Let C_2 be a circle with center O_2. Now we turn vector $\mathbf{b}(n)$ by angle $\pi/2$ until vertical position O_1B_1. Simultaneously circle C_2 is rolled along C_1 in such a way that diameter O_1A_1 of C_2 takes a vertical position, i.e. point A_1 in the system C_2 takes place of point O_1 of system C_1. It means that vector $\mathbf{b}(n/2)$ is turned from position O_2A_1 to O_2O_1 in the system C_2, i.e. it is turned by angle π. Analogously in a circle C_3 with center O_3 (middle point of O_2A_1) we find that vector $\mathbf{b}(n/4) = O_3A_1$ is turned by angle 2π. Thus, when circle C_k is rolled along C_1 the vector $\mathbf{b}(2)$ is turned by angle $(\pi/2)2^{k-1}$. The geometrical interpretation is completed.

4. Some Applications

The first application deals with modelling of vortex rings. In [5] we use matrix solutions of a non-linear wave equation for modelling a vortex ring arising behind acute edge of a cylindrical bar when a fluid flow is striving along the bar in acute edge direction. In [6] matrix solutions of linear wave equation are used for modelling well-known vortical rings of smoke when its portions are flying out of an aperture. Let us consider how pair of solutions u_1 and u_2 of linear wave equation can be used for visual demonstration of toroidal motion in vortex rings arising in different plastic mediums. Let us represent solution $u_1 = \exp(i\phi)$, where $\phi = \alpha z$ for linear case, by unit circle S^1 and take local frame $\mathbf{e_j}(j = 1,2,3)$ with the current point $\exp(i\phi)$ as its origin. Let unit vector $\mathbf{a} = \mathbf{e_3}$ lies in the plane of the circle and it is tangent to S^1. When ϕ is increasing, matrix u_2 rotates arbitrary vector $\mathbf{b}$ (orthogonal to $\mathbf{a}$) about vector $\mathbf{a}$ and, hence, about circle S^1 which forms circular axis of torus. As a result, an arbitrary vector $\mathbf{b}$ describes a helical motion around circle S^1. This is the desired toroidal motion.

Atsukovsky [1], by use of equations of fluid dynamics, describes the formation of basic elementary particles (proton, neutron, electron, etc.) in the form of stable vortex rings in gas-like ether.

The second application is a modelling of particles collision. A scheme of collision of two particles [6] can be applied to a model of collision of many particles. For this purpose we generalize results on composition and decomposition [6] from two simultaneous rotations onto arbitrary number of simultaneous rotations.

Theorem 3. *Arbitrary N of simultaneous rotations $u_2(n_j\phi, \mathbf{a_j})$ $(j = 1,\ldots,N)$ on unit sphere S^2 can be composed into one whole rotation $u_2(n_0\phi, \mathbf{a_0})$, where $n_0\mathbf{a_0} = \sum n_j\mathbf{a_j}$ and $\mathbf{a_0}$ is a unit vector in S^2; inversely, rotation $u_2(n_0\phi, \mathbf{a_0})$ can be decomposed on arbitrary number L of simultaneous rotations $u_2(m_j\phi, \mathbf{c_j})$ $(j = 1,\ldots,L)$, where $\sum m_j\mathbf{c_j} = n_0\mathbf{a_0}$ and $\mathbf{c_j}$ are radial unit vectors in S^2.*

Proof. It follows from Theorem 2 in [4] that rotation $u_2(n_j\phi, \mathbf{a_j})$ can be decomposed into three simultaneous rotations $u_2(n_{ji}\phi, \mathbf{e_i})$ about coordinate vectors $\mathbf{e_i}(i = 1,2,3)$, where $n_j\mathbf{a_j} = \sum_{i=1}^{3} n_{ji}\mathbf{e_i}$. The N of simultaneous rotations about fixed $\mathbf{e_i}$-axis are simply composed into one rotation $u_2(n_{0i}\phi, \mathbf{e_i})$, where $n_{0i} = \sum_{j=1}^{N} n_{ji}$. The three obtained simultaneous rotations about coordinate axes are composed into one whole rotation $u_2(n_0\phi, \mathbf{a_0})$ about unit vector $\mathbf{a_0}$, where $n_0\mathbf{a_0} = \sum_{i=1}^{3} n_{0i}\mathbf{e_i} = \sum_{i=1}^{3}\sum_{j=1}^{N} n_{ji}\mathbf{e_i} = \sum_{j=1}^{N} n_j\mathbf{a_j}$. By use of an inverse procedure it is easy to show that rotation $u_2(n_0\phi, \mathbf{a_0})$ can be decomposed on arbitrary number L of simultaneous rotations $u_2(m_j\phi, \mathbf{c_j})$, where $(j = 1,\ldots,L)$, $\sum m_j\mathbf{c_j} = n_0\mathbf{a_0}$ and $\mathbf{c_j}$ are radial unit vectors in S^2. Theorem is proved. ■

Now by use of this result we construct a model of particles collision. The scheme of the model is close to the scheme of Green-Schwarz-Witten [3] model of collision of superstrings. According to the chain (3.2) we pass from many dimensional space, where matrix solution u_n acts, to 3D-space with mapped function $u_2(n\phi, \mathbf{a})$. Consider a model of collision of three incoming and three outgoing particles. In spite of the opinion that three-particles collision is improbable there are some reactions in rotational water, described by Fominsky [2], where such three-particles collision are natural phenomena.

Let functions $u_2(n_j\phi, \mathbf{a_j})$ $(j = 1, 2, 3)$ correspond to three incoming particles, where each unit vector $\mathbf{a_j}$ moves with its own local frame along a direction of corresponding particle motion. It follows from the rotations $u_2(n_j\phi, \mathbf{a_j})$ that for each $\mathbf{a_j}$-direction there exists a tube (of unit radius) as a carrier of rotation traces. Let O be an intersection point of three $\mathbf{a_j}$-directions. A region of intersection of the tubes is isomorphic to unit sphere S^2 with center O. Composition of simultaneous rotations $u_2(n_j\phi, \mathbf{a_j})$ gives a whole rotation $u_2(n_0\phi, \mathbf{a_0})$, where $n_0\mathbf{a_0} = \sum n_j\mathbf{a_j}$ and $\mathbf{a_0}$ is a unit vector. Furthermore, the whole rotation can be decomposed on arbitrary number L of simultaneous rotations. Let be $L = 3$ and let unit vectors $\mathbf{c_j}(j = 1, 2, 3)$ correspond to three directions of outgoing particles motion. Then we denote corresponding simultaneous rotations by $u_2(m_j\phi, \mathbf{c_j})$, where $\sum m_j\mathbf{c_j} = n_0\mathbf{a_0}$. Now it is sufficient to take on each particle its own local frame with corresponding vector $\mathbf{c_j}$ as one of the basis vectors. This completes the mathematical model of particles collision.

The third application is a modelling of pulsating spheres. As is shown in [4] the relations $x_0 = \cos\phi$ and $x_j = a_j \sin\phi$ $(j = 1, 2, 3)$ establish the one-to-one correspondence $u_2 \leftrightarrow S^3$ for all unit radial vectors $\mathbf{a} \in S^2$ and for all points $(x_0, x_1, x_2, x_3) \in S^3$ in 4D-space. Coordinates $x_j(j = 1, 2, 3)$ form a sphere S_ϕ of radius $\rho = |\sin\phi|$. Note that for linear wave equation, when Q in (1.1) is replaced by $Q_0 = -\mu^2u^2 + Const$, formula (1.3) is replaced by $\phi = \alpha z$ with the same α as in (1.3). Let us show that in the linear case the sphere S_ϕ pulsates. Indeed, S_ϕ expands to S^2 when ϕ increases from 0 to $\pi/2$, then it contracts to the point at $\phi = \pi$. After that coordinates x_j change signs, $x_j = -a_j\rho$, for $\pi < \phi < 2\pi$. It means that S_ϕ transfers to the left oriented sphere S_ϕ^- with local frame $(-\mathbf{e_1}, -\mathbf{e_2}, -\mathbf{e_3})$, i.e. S_ϕ turns inside out to S_ϕ^-. Further, passing through point at $\phi = 2\pi$, sphere S_ϕ^- turns inside out to initial right oriented sphere S_ϕ. This process can be continued up to infinity. We note that matrix solution $u_n(\phi, \mathbf{a})$ can be geometrically represented as helix in 3D-space of coordinates $E_n, \mathbf{a}, z$, where $\mathbf{a}$ plays a role of imaginary axis. If we combine these two representations – pulsating sphere and helical motion – then we obtain a motion of pulsating sphere along the helix. This is close to the phenomena what is observed in the Potapov's heatgenerator [2]. Fominsky [2] shows that an acceleration of water rotation in Potapov's heatgenerator leads to the arising a sequence of bubbles along helical streamlines. During the many cycles of extension and contraction of bubbles a pulsating luminescence with stable periodicity appears in each bubble. The vortical rotation of the water leads to extra connections between the particles, in particular, to three-particle collision, for example, deuteron-proton-electron. This reaction leads to the increasing of output energy in the Potapov's heatgenerator.

5. Relation with Euler Angles

From the solid state theory it is known that one turn about some **a**-direction in 3D-space can be replaced by three sequential turns by the Euler angles ψ_1, θ, ψ_2 about basis vectors $\mathbf{e_3}, \mathbf{e_1}, \mathbf{e_3}$, respectively. The same turn can be derived by transformation $\mathbf{b} \to U\mathbf{b}U^*$ of arbitrary vector $\mathbf{b} = \sum_{j=1}^{3} b_j H_j$, where unitary 2×2-matrix $U \in SU(2)$. Since matrix solution $u_2(\phi, \mathbf{a})$ is in the one-to-one correspondence with $SU(2)$ [4], the comparison of matrices

$$U = \begin{pmatrix} f & g \\ -\bar{g} & \bar{f} \end{pmatrix}, \quad u_2(\phi, \mathbf{a}) = \begin{pmatrix} \cos\phi + ia_3 \sin\phi & (-a_2 + ia_1)\sin\phi \\ (a_2 + ia_1)\sin\phi & \cos\phi - ia_3 \sin\phi \end{pmatrix}$$

gives the relation $f = \cos\phi + ia_3 \sin\phi, \quad g = (-a_2 + ia_1)\sin\phi$.

It follows from [7], p.107 that

$$|f| = \cos(\theta/2),\ |g| = \sin(\theta/2),\ \arg(f) = (\psi_1+\psi_2)/2,\ \arg(g) = (\psi_1-\psi_2+\pi)/2.$$

Now we can easily obtain the relation between the Euler angles and the parameters of matrix solution $u_2(\phi, \mathbf{a})$ in the form

$$\tan\frac{\psi_1 + \psi_2}{2} = a_3 \tan\phi, \quad \tan\frac{\psi_1 - \psi_2}{2} = \frac{a_2}{a_1}, \quad \tan\frac{\theta}{2} = \frac{\sqrt{1 - a_3^2}}{\sqrt{a_3^2 + \cot^2\phi}}.$$

Solution $u_2(\phi, \mathbf{a})$ describes the rotation about **a**-direction by angle 2ϕ, the same turn by 2ϕ is found after performing sequential turns by the mentioned Euler angles ψ_1, θ, ψ_2. Solution u_2 has a remarkable property, namely, $u_2(\phi, \mathbf{a})$, being decomposed into the three solutions $u_2(\phi a_j, \mathbf{e_j})$, gives an equivalent interchangeability between the whole rotation about vector **a** by angle 2ϕ and three simultaneous rotations about coordinate vectors $\mathbf{e_j}$ by angles $2\phi a_j$. However, this interchangeability take place in that medium, where each fraction can have independent motion, i.e. in the plastic matter.

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