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ON POSITIVE EIGENFUNCTIONS OF STURM-LIOUVILLE PROBLEM WITH NONLOCAL TWO-POINT BOUNDARY CONDITION

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Abstract. Positive eigenvalues and corresponding eigenfunctions of the linear Sturm-Liouville problem with one classical boundary condition and another nonlocal two-point boundary condition are considered in this paper. Four cases of nonlocal two-point boundary conditions are analysed. We get positive eigenfunctions existence domain for each case of these problems. This domain depends on the parameters of the nonlocal boundary problem and it gives necessary and sufficient conditions for existing positive eigenvalues with positive eigenfunctions.

Key words: Sturm-Liouville problem, nonlocal two-point condition, positive eigenfunctions, positive eigenvalues

1. Introduction

Investigation of the spectrums of differential equations with nonlocal conditions is quite a new area in scientific literature. Eigenvalue problems for differential operators with the nonlocal boundary conditions are considerably less investigated than cases of classical boundary conditions. Eigenvalue problems with nonlocal conditions are closely linked with boundary problems for differential equations with nonlocal conditions [5, 6, 7, 8]. In the papers [1, 2, 4, 12, 13] the similar problems are investigated for the operators with nonlocal integral condition. In papers [6, 7, 8, 11, 14, 15] problems with nonlocal boundary condition of Samarskii-Bitsadze type are analysed. Existence of positive solutions of stationary problem was investigated in [3, 9, 10]. Eigenvalues and corresponding eigenfunctions were analysed in [8].

In various papers eigenvalues and eigenfunctions of selfajoint and nonselfajoint Sturm-Liouville problem are investigated in complex and real cases. Differential problems of this type are among investigations analysed by G. Infante in paper [6]. In this paper he studied the existence of eigenvalues of a Hammerstein integral equation

$$\tilde{\lambda} u(t) = \int_G k(t,s) f(s,u(s))\,ds.$$

Here G is a compact set in $\mathbb{R}^n$, k and $f \geqslant 0$ are allowed to be discontinuous, and k may change the sign. The obtained results are applied to the second order differential equation

$$\tilde{\lambda} u''(t) + f(t,u(t)) = 0, \quad a.e. \quad t \in [0,1], \tag{1.1}$$

with classical boundary condition (the first or the second type) on the left side of interval ($t=0$) and various nonlocal two-point boundary conditions in the right side point:

$$u(0) = 0, \quad u(1) = \gamma u'(\xi); \qquad \text{(Case 1)} \qquad (1.2_1)$$
$$u(0) = 0, \quad u(1) = \gamma u(\xi); \qquad \text{(Case 2)} \qquad (1.2_2)$$
$$u'(0) = 0, \quad u(1) = \gamma u'(\xi); \qquad \text{(Case 3)} \qquad (1.2_3)$$
$$u'(0) = 0, \quad u(1) = \gamma u(\xi). \qquad \text{(Case 4)} \qquad (1.2_4)$$

Note that the index in references denotes the case. If the positive function $f(s,u)$ satisfies some additional conditions (for more details, see [6]) then:

- the problem (1.1), (1.2_1) has a positive eigenvalue when $\gamma < 0$ and $0 < \gamma < 1-\xi$ and a corresponding eigenfunction that is positive on $(0,\xi]$ when $\gamma < 0$, and on $(0, 1-\gamma]$ when $0 < \gamma < 1-\xi$;
- the problem (1.1), (1.2_2) has a positive eigenvalue when $\gamma\xi < 0$ and $0 < \gamma\xi < 1$ and a corresponding eigenfunction that is positive on $(0,\xi]$ when $\gamma\xi < 0$, and $(0,1]$ when $0 < \gamma\xi < 1$, as well as this problem has a negative eigenvalue when $\gamma\xi > 1$ and a corresponding eigenfunction that is negative on $[\xi, 1]$;
- the problem (1.1), (1.2_3) has a positive eigenvalue when $\gamma < 0$ and $0 < \gamma < 1-\xi$ and a corresponding eigenfunction that is positive on $[0,1)$ when $\gamma < 0$, and on $[0,\xi]$ when $0 < \gamma < 1-\xi$;
- the problem (1.1), (1.2_4) has a positive eigenvalue when $\gamma < 0$ and $0 < \gamma < 1$ and a corresponding eigenfunction that is positive on $[0,\xi]$ when $\gamma < 0$ and $[0,1]$ when $0 < \gamma < 1$. This problem has a negative eigenvalue when $\gamma > 1$ and a corresponding eigenfunction that is negative on $[a,b]$, where $a = \xi, b \in (\xi, 1]$.

In this paper we present analogous results for the linear Sturm-Liouville problem with four cases of nonlocal two-point boundary conditions. Here the existence of positive eigenvalues and eigenfunctions is analysed.

2. Positive Eigenvalues and Positive Eigenfunctions of the Linear Sturm-Liouville Problem

We will analyze the Sturm-Liouville problem

$$-u''(t) = \lambda u(t), \qquad t \in (0,1), \tag{2.1}$$

with one classical boundary condition and another nonlocal two-point boundary condition (1.2) with the parameters $\gamma \in \mathbb{R}$ and $\xi \in (0,1)$.

Remark 1. The linear problem is not a separate case of equation (1.1), because in [6] the function $f(t,u)$ must satisfy some additional conditions and must be positive.

When $\gamma = 0$ in problem (2.1), (1.2), we get a problem with classical boundary conditions. Then eigenvalues and eigenfunctions do not depend on the parameter ξ:

$$\lambda_k = (\pi k)^2, \qquad u_k(t) = \sin(\pi k t), \qquad k \in \mathbb{N} := \{1,2,\dots\}, \qquad (2.2_{1,2})$$

$$\lambda_k = \pi^2\left(k - \tfrac{1}{2}\right)^2, \quad u_k(t) = \cos\left(\pi(k - \tfrac{1}{2})t\right), \quad k \in \mathbb{N}. \qquad (2.2_{3,4})$$

If $u(t)$ is eigenfunction of the linear problem (2.1) then functions $cu(t)$, $0 \neq c \in \mathbb{R}$ will be eigenfunction too. We say that this problem has positive eigenfunction $u(t)$ in the interval $(a,b) \subset (0,1)$ if eigenfunction $u(t) > 0$ or $u(t) < 0$ (in this case $-u(t) > 0$) exists for all $t \in (a,b)$.

2.1. Eigenvalues, constant eigenvalues and characteristic function

In previous papers [11, 15] it is proved that the eigenvalue $\lambda = 0$ exists if and only if $\gamma = \frac{1}{\xi}$ (Case 1,2) and $\gamma = 1$ (Case 4).

Lemma 1. *The eigenvalue $\lambda = 0$ does not exist in problem* (2.1), (1.2$_3$).

Proof. We will search for a general solution of equation (2.1) in the form $u(t) = c_1 t + c_2$. Computing the derivative of the solution and using (1.2$_3$) we get that $c_1 = 0$, i. e. $u(t) = c_2$. It follows from (1.2$_3$) that $c_2 = \gamma 0$, i.e. $u(t) \equiv 0$ and a zero eigenvalue does not exist. ∎

For $\lambda = 0$ we always have positive eigenfunction $u(t) = t$ in $(0,1)$ in Case 1,2 and $u(t) \equiv 1$ in Case 4.

In the general case, for $\lambda \neq 0$, eigenfunctions are $u(t) = \sin(qt)$ (Case 1,2) and $u(t) = \cos(qt)$ (Case 3,4) and eigenvalues are $\lambda = q^2$, where $q \in \mathbb{C}_q$,

$$\mathbb{C}_q := \{q \in \mathbb{C} \mid \operatorname{Re} q > 0 \text{ or } \operatorname{Re} q = 0, \operatorname{Im} q > 0 \text{ or } \ q = 0\}.$$

We can find q from *characteristic equations*[11, 15] with $q \in \mathbb{C}_q$

$$\sin q = \gamma q \cos(\xi q); \qquad \sin q = \gamma \sin(\xi q); \qquad (2.3_{1;2})$$

$$\cos q = -\gamma q \sin(\xi q); \qquad \cos q = \gamma \cos(\xi q). \qquad (2.3_{3;4})$$

Let q_c be a solution of a system

$$\begin{cases} \sin q &= 0, \\ \cos(\xi q) &= 0; \end{cases} \qquad \begin{cases} \sin q &= 0, \\ \sin(\xi q) &= 0; \end{cases} \tag{2.4$_{1;2}$}$$

$$\begin{cases} \cos q &= 0, \\ \sin(\xi q) &= 0; \end{cases} \qquad \begin{cases} \cos q &= 0, \\ \cos(\xi q) &= 0. \end{cases} \tag{2.4$_{3;4}$}$$

In this case we say that q_c is a *constant eigenvalues point*, and *constant eigenvalues* $\lambda = q_c^2$ exist for all γ. In all cases constant eigenvalues are positive real numbers and exist only for some rational ξ.

We can get all *nonconstant eigenvalues* (which depend on the parameter γ) as square of the γ-points of *complex-real characteristic function* $\gamma : \mathbb{C}_q \to \mathbb{R}$, $D(\gamma) = \{q \in \mathbb{C}_q \,|\, \mathrm{Im}\gamma(q) = 0\}$, (see, [11], Cases 1 and 2):

$$\gamma(q) = \frac{\sin q}{q\cos(\xi q)}; \qquad \gamma(q) = \frac{\sin q}{\sin(\xi q)}; \tag{2.5$_{1;2}$}$$

$$\gamma(q) = -\frac{\cos q}{q\sin(\xi q)}; \qquad \gamma(q) = \frac{\cos q}{\cos(\xi q)}. \tag{2.5$_{3;4}$}$$

We name these γ-points of the function γ as *eigenvalues points.* Negative and nonconstant positive eigenvalues points we find as γ-points of *real (first type) characteristic function* $\gamma_1 : \mathbb{R} \to \mathbb{R}$:

$$\gamma_1(x) := \begin{cases} \gamma_-(x) := \gamma(\mathrm{i}x) & \text{for } x < 0, \\ \lim_{q\to 0}\gamma(q) & \text{for } x = 0, \text{ if the limit exists,} \\ \gamma_+(x) := \gamma(x) & \text{for } x > 0. \end{cases} \tag{2.6}$$

If x_- is such a negative eigenvalue point then corresponding eigenvalue $\lambda_- = -x_-^2$. If x_+ is such a positive eigenvalue point then corresponding eigenvalue $\lambda = x_+^2$. For the problem (2.1)-(1.2) we have

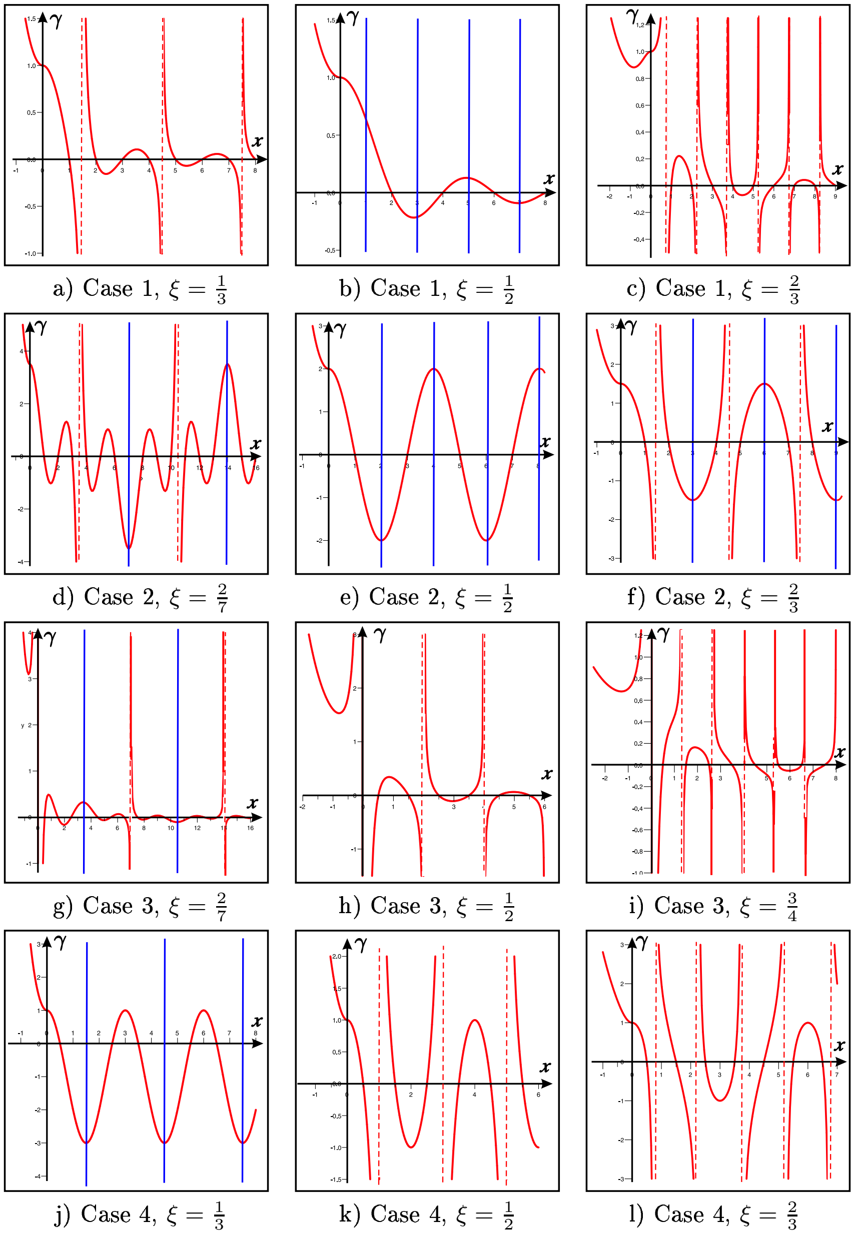
$$\gamma_-(x) = \frac{\sinh x}{x\cosh(\xi x)}, \qquad \gamma_+(x) = \frac{\sin x}{x\cos(\xi x)}, \tag{2.7$_1$}$$

$$\gamma_-(x) = \frac{\sinh x}{\sinh(\xi x)}, \qquad \gamma_+(x) = \frac{\sin x}{\sin(\xi x)}, \tag{2.7$_2$}$$

$$\gamma_-(x) = \frac{\cosh x}{x\sinh(\xi x)}, \qquad \gamma_+(x) = -\frac{\cos x}{x\sin(\xi x)}, \tag{2.7$_3$}$$

$$\gamma_-(x) = \frac{\cosh x}{\cosh(\xi x)}, \qquad \gamma_+(x) = \frac{\cos x}{\cos(\xi x)}. \tag{2.7$_4$}$$

Properties of these functions were investigated in papers [11] (Case 1,4) and [15](Case 2). Graphs of $\gamma_1(x)$ for various values of parameter ξ are presented in Fig. 1. In our paper [11] we investigated Sturm-Liuville problem (2.1) with boundary conditions $u(0) = 0$ and $u'(1) = \gamma u(\xi)$ and investigated characteristic function $\gamma_+(x) = \dfrac{x\cos x}{\sin(\xi x)}$. If we compare this function and the characteristic function in Case 3 we get that many properties of the characteristic



a) Case 1, $\xi = \frac{1}{3}$ b) Case 1, $\xi = \frac{1}{2}$ c) Case 1, $\xi = \frac{2}{3}$

d) Case 2, $\xi = \frac{2}{7}$ e) Case 2, $\xi = \frac{1}{2}$ f) Case 2, $\xi = \frac{2}{3}$

g) Case 3, $\xi = \frac{2}{7}$ h) Case 3, $\xi = \frac{1}{2}$ i) Case 3, $\xi = \frac{3}{4}$

j) Case 4, $\xi = \frac{1}{3}$ k) Case 4, $\xi = \frac{1}{2}$ l) Case 4, $\xi = \frac{2}{3}$

Figure 1. Function $\gamma_1(x\pi)$. Vertical lines describe constant eigenvalues.

function related with constant eigenvalues, zeroes and poles are investigated in [11]. We note that all zeroes and poles of the characteristic function are nonnegative real numbers.

Lemma 2. *Constant eigenvalues do not exist for irrational ξ, while for rational $\xi = r = \frac{m}{n} \in [0,1]$ they exist for even m and odd n and constant eigenvalues are equal to $\lambda_k = c_k^2$, $c_k = \pi(k - \frac{1}{2})n$, $k \in \mathbb{N}$.*

When γ is real, multiple and complex eigenvalues can exist for all $\gamma \neq 0$ and for all $\xi \in (0,1)$. In the case $\xi = 1$ (the classical third type condition) we have one negative eigenvalue for $\gamma > 0$, because function $\dfrac{\cosh x}{x \sinh x}$ is monotone increasing function when $x \in (-\infty, 0)$ and the other eigenvalues are positive and simple, because function $\Big(-\dfrac{\cos x}{x \sin x}\Big)$ is monotone increasing function when $x \in (\pi(k-1), \pi k)$, $k \in \mathbb{N}$. We formulate a lemma on negative eigenvalues in Case 3.

Lemma 3. *If $\xi \in (0,1)$, then there exists $x_* = x_*(\xi) < 0$ and $\gamma_* = \gamma_-(x_*) \in (0, +\infty)$ such that there exists one double negative eigenvalue for $\gamma = \gamma_*$ and two negative eigenvalues exist for $\gamma \in (\gamma_*, +\infty)$, and negative eigenvalues do not exist for $\gamma < \gamma_*$.*

Proof. Function $y_1(x) := \tanh x - 1/x$ is a monotone increasing function when $x \in (0, +\infty)$, because $y_1'(x) := 1/\cosh^2(x) + 1/x^2 > 0$ and $y_1(0) = -\infty$, $y_1(+\infty) = 1$. So, there exist $x_0 \approx 1.199678$ such that $y_1(x_0) = 0$ and $y_1(x) < 0$ for $x \in (0, x_0)$, $y_1(x) > 0$ for $x > x_0$. Function $y_2(x) := \tanh x$ is a positive monotone increasing function when $x \in (0, +\infty)$. In paper [11] we proved that function $\tanh(\xi x)/\tanh x$ (and function $y_3(x;\xi) := \tanh(\xi x)/\tanh x/\xi$) is a positive increasing function when $x \in (0,1)$ for all $\xi \in (0,1)$. Then function

$$y(x;\xi) := \frac{x \tanh x - 1}{\xi x \coth(\xi x)} = y_1(x) y_2(x) y_3(x;\xi)$$

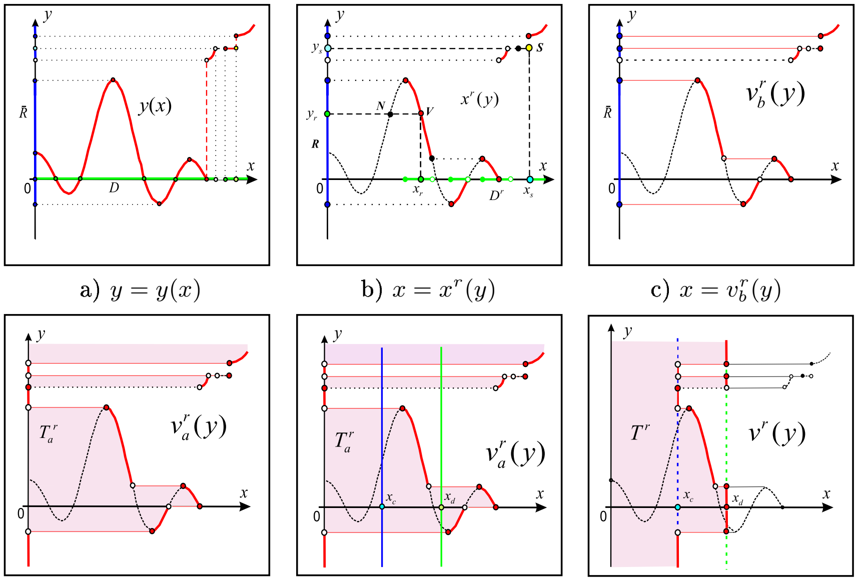
is a positive monotone increasing function when $x > x_0$ and it is negative when $x \in (0, x_0)$ and $y(0;\xi) = -1$, $y(x_0;\xi) = 0$, $y(+\infty;\xi) = 1/\xi$. So, there exists only one point $x_* = x_*(\xi) \in (x_0, +\infty)$ such that $y(x_*) = 1$.

Function $y_4(t) := \tanh t/t$ is a monotone decreasing function when $t \in (0, +\infty)$, because $y_4'(t) := \big(2t - \sinh(2t)\big)/(2t^2 \cosh^2 t) < 0$. Since $x\xi \in (0, +\infty)$, we have that function $\tanh(\xi x)/(\xi x)$ (and functions $y_3(x,\xi)$, $y(x;\xi)$ too) are monotone decreasing function when $\xi \in (0,1)$ for all fixed x. So, $x_* \geqslant x_1 \approx 2.065338$ where x_1 is the root of equation $y(x_1;0) = 1$, i.e. $x_1 \tanh x_1 = 2$.

Since $\xi x \coth(\xi x) > 0$ function $f(x;\xi) := x \tanh -1 - \xi x \coth(\xi x) < 0$ for $x \in (0, x_*(\xi))$, $f(x;\xi) > 0$ for $x \in (x_*(\xi), +\infty)$ and $f(x_*(\xi);\xi) = 0$. Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\cosh x}{x \sinh(\xi x)} = f(x;\xi) \frac{\cosh x}{x^2 \sinh(\xi x)}.$$

The function $\gamma_-(x;\xi)$, $x \in \mathbb{R}$ is an even function. Therefore, monotonicity properties of the function $\gamma_-(x;\xi)$, $x < 0$ follow from the properties of the function $f(x;\xi)$: if $\xi \in (0,1)$, then there exists $x_{\min} = -x_*(\xi) < x_1$ such that $\gamma_-(x;\xi)$ is a decreasing function for $x \leqslant x_{\min}$ and $\gamma_-(x;\xi)$ is an increasing function for $x_{\min} \leqslant x < 0$ for all $\xi \in (0,1)$. ■



a) $y = y(x)$ b) $x = x^r(y)$ c) $x = v_b^r(y)$

d) $x = v_a^r(y)$, T_a^r e) $x \equiv x_c$, $x \equiv x_d$ f) $x = v^r(y)$, T^r

Figure 2. Functions $y(x)$, $x^r(y)$, $v_b^r(y)$, $v_a^r(y)$, $v^r(y)$, domains T_a^r, T^r, $I = (0, +\infty)$.

For the negative eigenvalue $\lambda = -x^2$ eigenfunction is $u(t) = \sin(xt)$ (Case 1,2) and $u(t) = \cos(xt)$ (Case 3,4). So, all eigenfunctions are positive or negative in the interval $(0, 1)$.

Remark 2. For all $\gamma \in \mathbb{R}$ the least positive eigenvalue exists.

Proof. Let characteristic function is not an entire function. In Case 1,2,3,4 we have the first order poles and between two poles (or between pole and constant eigenvalues point) finite number (> 0) zeroes exist. Let z_- and z_+ are two next zeroes for the pole p and $z_- < p < z_+$, then $\gamma_+([z_-, z_+]) = \mathbb{R}$. So, the least positive eigenvalue exists. If characteristic function is an entire function then a positive constant eigenvalue exists and the least positive eigenvalue exists, too. ■

2.2. Visible from the right functions

We introduce a few definitions which we use to describe positive eigenfunctions intervals for positive eigenvalues.

Suppose that function $y : I \to \mathbb{R}$, where $I = [a, b]$, (a, b), $[a, b)$, (a, b), $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, a domain of this function is $D \subset I$ and a range of this function is a set $\bar{R} \subset \mathbb{R}$ (see, Fig. 2a). We call the point (x_r, y_r) of the functions $y(x)$ graph ***visible from the right point***, if $y_r = y(x_r) \neq y(x)$ for all $x \in (x^r, b] \cap D$ (see, Fig. 2b). In this figure V is visible from the right point, and N, S are not such points. If such point exists it is the rightmost point of the function $y(x)$ graph for fixed y_r. Let R be a set of the points

$y_r \in \bar{R}$ such that exists visible from the right point (x_r, y_r). In Fig. 2b $y_r \in R$, but $y_s \notin R$. For each $y \in R^r$ we can assign only one visible from the right point $x_r(y)$, i. e. we can define single-valued function $x^r : y_r \mapsto x_r$ and this function is bijection between R and $D^r := x^r(R) \subset D$. If $y_s \in \bar{R}^r \setminus R^r$ then y_s-value of the function $y(x)$ is obtained in the interval $(a, b]$ infinitely many times. In this case we assign value $v_b^r(y_s) := \sup_{x \in I \cap D}\{x \,|\, y(x) = y_s\}$. If $y \in R^r$, then $v_b^r(y) := x^r(y)$ (see, Fig. 2,c). Finally, we define $v_a^r(y) := v_b^r(y)$ for $y \in \bar{R}$ and $v_a^r(y) := a$ for $y \notin R$. So, we have the function $v_a^r : \mathbb{R} \to [a, b]$ and a set $T_a^r := \{(x, y) \in \mathbb{R}^2_{x,y} \,|\, a < x < v_a^r(y)\}$ (see, Fig. 2d). We call this function as *visible from the right function* generated by function $y = y(x)$. Let $[x_c, x_d] \in [a, b]$. Then we consider *truncated visible from the right function* (see, Fig. 2e,f) $v^r(y) := \max\{x_c, \min\{x_d, v_a^r(y)\}\}$. If

$$T_c^r := \{(x, y) \in \mathbb{R}^2_{x,y} \,|\, a < x < x_c\}, \quad T_d^r := \{(x, y) \in \mathbb{R}^2_{x,y} \,|\, a < x < x_d\}$$

then domain

$$T^r := \{(x, y) \in \mathbb{R}^2_{x,y} \,|\, a < x < v^r(y)\} = (T_a^r \cap T_d^r) \cup T_c^r.$$

2.3. Positive eigenfunctions for positive eigenvalues

Eigenfunctions for positive eigenvalues of the linear problem (2.1)-(1.2) are $u(t) = \sin(xt)$ (Case 1, 2) and $u(t) = \cos(xt)$ (Case 3, 4). Function $\sin(xt)$ is positive iff $xt \in (0, \pi)$ and function $\cos(xt)$ is positive iff $xt \in (0, \pi/2)$. The interval is biggest when $x = x_+$ where x_+ is the least positive eigenvalue. This eigenvalue can be constant or nonconstant. If the right side of the interval (where eigenfunction is positive) is $t = y$ then $x_+ = \frac{\pi}{y}$ (Case 1, 2) and $x_+ = \frac{\pi}{2y}$ (Case 3, 4).

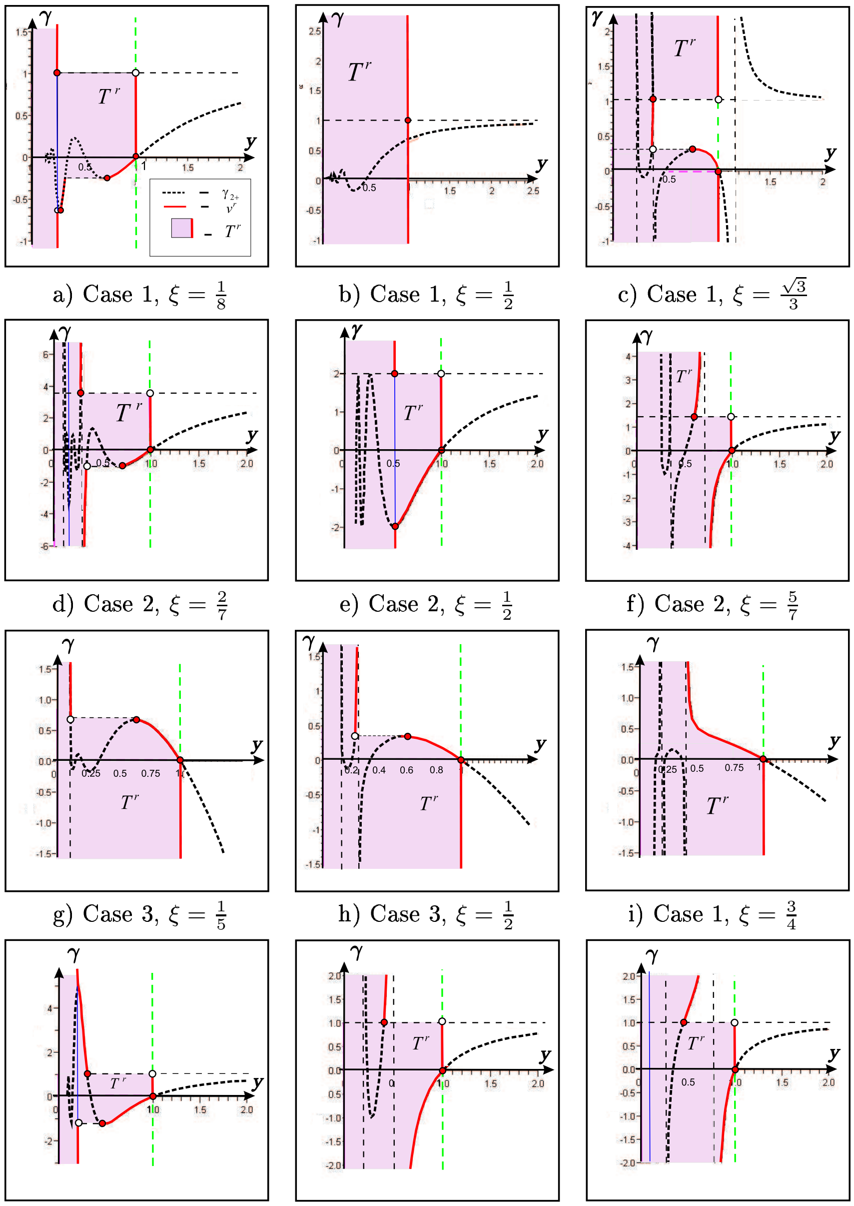
Let define *real (second type) characteristic function* $\gamma_{2+} : \mathbb{R}_+ \to \mathbb{R}$:

$$\gamma_{2+}(y) := \gamma_{1+}(\tfrac{\pi}{y}) = \frac{\sin\frac{\pi}{y}}{\frac{\pi}{y}\cos(\xi\frac{\pi}{y})}; \qquad \gamma_{2+}(y) := \gamma_{1+}(\tfrac{\pi}{y}) = \frac{\sin\frac{\pi}{y}}{\sin(\xi\frac{\pi}{y})}; \qquad (2.8_{1;2})$$

$$\gamma_{2+}(y) := \gamma_{1+}(\tfrac{\pi}{2y}) = \frac{-\cos\frac{\pi}{2y}}{\frac{\pi}{2y}\sin(\xi\frac{\pi}{2y})}; \quad \gamma_{2+}(y) := \gamma_{1+}(\tfrac{\pi}{2y}) = \frac{\cos\frac{\pi}{2y}}{\cos(\xi\frac{\pi}{2y})}. \quad (2.8_{3;4})$$

The function $\gamma = \gamma_{2+}(y)$ generates function $y = v_a^r(\gamma)$ which is visible from the right and domain T_a^r. If constant eigenvalues exist then truncated visible from the right function $y = v^r(\gamma)$ we construct with $y_c = \frac{\pi}{x_1}$ (Case 1,2) or $y_c = \frac{\pi}{2x_1}$ (Case 3,4) and $y_d = 1$, else we take $y_c = 0$ and $y_d = 1$. The graphs of functions $\gamma_{2+}(y)$, $v^r(\gamma)$ and domain T^r are presented in Fig. 3 for various ξ (see legend in Fig. 3a). In these figures a part of a graph of functions $\gamma_{2+}(y)$ is hidden in the vicinity of a zero. If the point $(y, \gamma) \in T^r$ then there exists positive eigenvalue $\lambda = \pi^2/(v^r(\gamma))^2$ (Case 1,2) or $\lambda = \pi^2/(2v^r(\gamma))^2$ (Case 3,4) for this γ and corresponding eigenfunctions are positive when $t \in (0, y)$ (Case 1,2) or $t \in [0, y)$ (Case 3,4).

Further, we describe intervals of eigenfunctions, where positive eigenvalues exist and eigenfunctions for these eigenvalues are positive. Graphs of the



a) Case 1, $\xi = \frac{1}{8}$ b) Case 1, $\xi = \frac{1}{2}$ c) Case 1, $\xi = \frac{\sqrt{3}}{3}$

d) Case 2, $\xi = \frac{2}{7}$ e) Case 2, $\xi = \frac{1}{2}$ f) Case 2, $\xi = \frac{5}{7}$

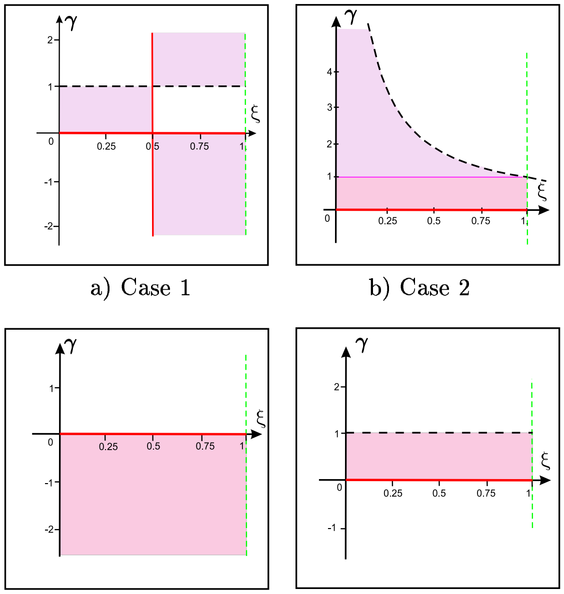
g) Case 3, $\xi = \frac{1}{5}$ h) Case 3, $\xi = \frac{1}{2}$ i) Case 1, $\xi = \frac{3}{4}$

j) Case 4, $\xi = \frac{1}{5}$ k) Case 1, $\xi = \frac{1}{2}$ l) Case 1, $\xi = \frac{7}{9}$

Figure 3. Functions $\gamma_{2+}(y)$, $v^r(\gamma)$ and domain T^r.

functions γ_{2+} (black dashed), v^r (solid line) and domains T^r are shown in Fig. 3.

Case 1. For the problem (2.1)–(1.2$_1$) domain T^r is equal (see, Fig. 3a-c):



a) Case 1 b) Case 2

c) Case 3 d) Case 4

Figure 4. The positive eigenfunctions existence domain in the interval $(0,1)$.

$$T^r = \begin{cases} 0 < y < 1 & \text{when } 0 \leqslant \gamma < 1, \\ 0 < y < v^r(\gamma;\xi) & \text{otherwise}, \end{cases} \qquad \text{for } \xi \in (0, \tfrac{1}{2});$$

$$T^r = (0,1) \times \mathbb{R} \qquad \text{for } \xi = \tfrac{1}{2};$$

$$T^r = \begin{cases} 0 < y < v^r(\gamma;\xi) & \text{when } 0 < \gamma \leqslant 1, \\ 0 < y < 1 & \text{otherwise}, \end{cases} \qquad \text{for } \xi \in (\tfrac{1}{2}, 1).$$

Positive eigenfunctions exist in the interval $(0,1)$ when $0 \leqslant \gamma < 1$, $\xi \leqslant \frac{1}{2}$ or $\gamma \leqslant 0$, $\xi \geqslant \frac{1}{2}$ or $\gamma > 1$, $\xi \geqslant \frac{1}{2}$ (see, Fig. 4a).

Corollary 1. Positive in the interval $(0,1)$ for all $\xi \in (0,1)$ eigenfunctions exist only when $\gamma = 0$ (see, Fig. 4a).

Case 2. For the problem (2.1)–(1.2$_2$) domain T^r is equal (see, Fig. 3d-f):

$$T^r = \begin{cases} 0 < y < 1 & \text{when } 0 \leqslant \gamma < 1/\xi, \\ 0 < y < v^r(\gamma;\xi) & \text{otherwise}, \end{cases} \qquad \text{for } \xi \in (0,1).$$

Positive eigenfunctions in the interval $(0,1)$ exist when $\gamma\xi \leqslant 1$ (see, Fig. 4b).

Corollary 2. Positive in the interval $(0,1)$ for all $\xi \in (0,1)$ eigenfunctions exist only when $0 \leqslant \gamma \leqslant 1$ (see, Fig. 4b).

Case 3. For the problem (2.1)–(1.2$_3$) domain T^r is equal (see, Fig. 3g-i):

$$T^r = \begin{cases} 0 < y < 1 & \text{when } -\infty \leqslant \gamma \leqslant 0, \\ 0 < y < v^r(\gamma;\xi) & \text{otherwise,} \end{cases} \qquad \text{for } \xi \in (0,1).$$

Corollary 3. Positive in the interval $(0,1)$ for all $\xi \in (0,1)$ eigenfunctions exist when $\gamma \leqslant 0$ (see, Fig. 4c).

Case 4. For the problem (2.1)–(1.2_4) domain T^r is equal (see, Fig. 3j-l):

$$T^r = \begin{cases} 0 < y < 1 & \text{when } 0 \leqslant \gamma < 1, \\ 0 < y < v^r(\gamma;\xi) & \text{otherwise,} \end{cases} \qquad \text{for } \xi \in (0,1).$$

Corollary 4. Positive in the interval $(0,1)$ for all $\xi \in (0,1)$ eigenfunctions exist when $0 \leqslant \gamma < 1$ (see, Fig. 4d).

Theorem 1. *For the problems* (2.1), (1.2) *positive eigenvalue exists and corresponding eigenfunction is positive on* $(0, y)$ *iff* $(y,\gamma) \in T^r$.

Remark 3. In the case of the linear Sturm-Liouville problem, domain T^r describes necessary and sufficient conditions for the existence of positive eigenfunctions for positive eigenvalues.

3. Conclusions

- We obtain positive eigenfunctions existence intervals of Sturm-Liouville problem with nonlocal two-point boundary condition.
- We find the values of parameter γ when positive eigenfunctions exist in interval $(0,1)$ for each $\xi \in (0,1)$.
- We get necessary and sufficient conditions when at least one positive eigenvalue and corresponding eigenfunction exists.
- Some new results on negative eigenvalues in Case 3 of nonlocal boundary conditions are proved.

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