ON SOLVABILITY OF THE BVPS FOR THE FOURTH-ORDER EMDEN-FOWLER TYPE EQUATIONS[1]

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Abstract. Solvability of the boundary value problems (BVPs) for the fourth-order Emden-Fowler type equations $x^{(4)} = q(t)|x|^p \operatorname{sgn} x$ is investigated by using the quasilinearization process. We modify the equation to a quasi-linear form $x^{(4)} - k^4 x = F_k(t, x)$ for various values of k. Our considerations are based on a fact that the modified quasi-linear problem has a solution of the same oscillatory type as the linear part $x^{(4)} - k^4 x$ has. We show that original problem in some cases also has a solution of definite type and establish sufficient conditions for multiple solutions of the given BVP.

Key words: quasi-linear equation, quasilinearization, i-nonresonant linear part, i-type solution

1. Introduction

This paper is devoted to the boundary value problem (BVP) for the fourth-order Emden-Fowler type equations

$$\begin{cases} x^{(4)} = q(t)\,|x|^p \operatorname{sgn} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1), \end{cases} \tag{1.1}$$

where $p > 1$, $t \in I := [0, 1]$, $q(t) \in C(I, (0, +\infty))$. We investigate the solvability of problem (1.1) with respect to the values of p and prove estimates of the function $q(t)$ by using the so called quasilinearization process. Our aim is to obtain sufficient conditions for existence of multiple solutions

of different types to the given nonlinear problem (1.1). Our considerations are based on the oscillation theory by Leighton and Nehari [1] for linear fourth-order differential equations. We generalize previously obtained multiplicity results (see [2, 3, 4]) for non-autonomous Emden-Fowler type equations.

2. Quasilinearization Process

Our intent is to reduce the original nonlinear problem (1.1) to a quasi-linear one and to prove that both equations are equivalent in some domain Ω (see Table 1).

Table 1. Quasilinearization process.

$$
\boxed{\begin{array}{c} \text{nonlinear problem} \\ x^{(4)} = q(t)\,|x|^p\,\mathrm{sgn}\,x, \\ x(0) = x'(0) = 0 = x(1) = x'(1) \end{array}}
$$

$$\Updownarrow$$

$$
\boxed{\begin{array}{l} \text{quasi-linear problem} \\ (L_4x)(t) := x^{(4)} - k^4x = F(t,\,x), \\ x(0) = x'(0) = 0 = x(1) = x'(1), \\ \Omega = \{(t,x):\ 0 \le t \le 1,\ |x| \le N\} \end{array}}
\qquad
\begin{array}{c} (L_4x)(t) - \text{non-resonant} \\ F, F_x \in C([0,1] \times R,\ R), \\ \max_{\Omega} \big|F(t,\,x)\big| = M \end{array}
$$

$$\Downarrow$$

$$
\boxed{\begin{array}{c} x_{\text{quasi-lin.}}(t) = \displaystyle\int_0^1 G(t,s)F(s,x(s))\,ds, \\ |x_{\text{quasi-lin.}}(t)| \le \Gamma M \end{array}}
\qquad
\max_{0 \le t,s \le 1} |G(t,s)| = \Gamma
$$

If F is continuous along with F_x, F is bounded in Ω and a linear part $(L_4x)(t) := x^{(4)} - k^4x$ is non-resonant (that is the respective homogeneous problem $(L_4x)(t) = 0$, with boundary conditions of (1.1) has only the trivial solution), then a modified quasi-linear problem is solvable. Its solution $x_{\text{quasi-lin.}}(t)$ can be written in the integral form and can be estimated as shown in Table 1. $G(t,s)$ is the Green's function for the respective homogeneous problem $(L_4x)(t) = 0$ with the boundary condition of problem (1.1).
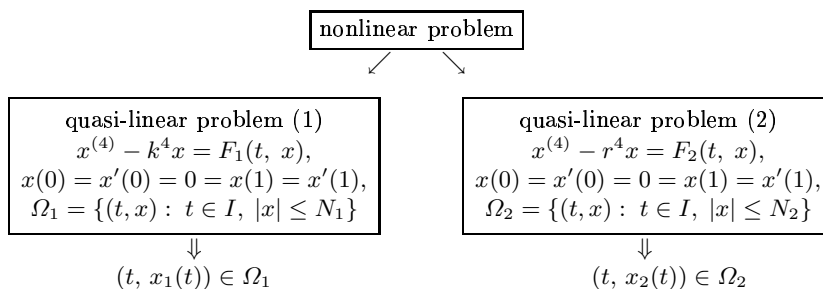
If an inequality

$$\Gamma M \le N \tag{2.1}$$

holds (that is, $\big|x_{\text{quasi-lin.}}(t)\big| \le N$) then the solution of the quasi-linear problem is located in the domain of equivalence Ω, therefore it also solves the nonlinear problem. Thus we can prove the solvability of the original problem (1.1).

If inequality (2.1) is fulfilled we will say for brevity that nonlinear problem (1.1) allows for quasilinearization with respect to the linear part $(L_4x)(t) := x^{(4)} - k^4x$ and the domain Ω.

Suppose that the original problem (1.1) allows for quasilinearization with respect to a different linear part $(l_4x)(t) := x^{(4)} - r^4x$ and different domain

Table 2. Quasilinearization and different solutions.

$$\boxed{\text{nonlinear problem}}$$
$$\swarrow \qquad \searrow$$

quasi-linear problem (1)	quasi-linear problem (2)
$x^{(4)} - k^4 x = F_1(t, \, x),$	$x^{(4)} - r^4 x = F_2(t, \, x),$
$x(0) = x'(0) = 0 = x(1) = x'(1),$	$x(0) = x'(0) = 0 = x(1) = x'(1),$
$\Omega_1 = \{(t,x): \ t \in I, \ |x| \le N_1\}$	$\Omega_2 = \{(t,x): \ t \in I, \ |x| \le N_2\}$

$$\Downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$
$$(t, \, x_1(t)) \in \Omega_1 \qquad\qquad\qquad\qquad\qquad (t, \, x_2(t)) \in \Omega_2$$

(see Table 2). Does that mean that the original problem has another solution, revealed by this quasilinearization? In what follows we try to answer this question.

3. Fourth-Order Quasi-Linear Problems

Consider a quasi-linear problem

$$\begin{cases} x^{(4)} - k^4 x = F(t, x), \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(A1) F and F_x are continuous functions;   **(A2)** $F(t, 0) \equiv 0$;

(A3) $k > 0$ and $\cos k \cosh k \ne 1$;   **(A4)** $k^4 + \dfrac{\partial F(t, \, x)}{\partial x} > 0$.

Condition **(A3)** implies that a linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is non-resonant with respect to the given boundary conditions in (3.1). All proper values of k form the intervals of non-resonance $(0, k_1)$, (k_1, k_2), $\ldots$, $(k_n, k_{n+1}), \ldots$, where $\cos k_n \cosh k_n = 1$, $n = 0, 1, 2, \ldots$, $k_0 = 0$.

Our considerations are based on the oscillation theory by Leighton and Nehari [1] for the fourth-order linear differential equations of the form

$$x^{(4)} - p(t)x = 0, \quad p(t) > 0. \tag{3.2}$$

If the coefficients k_i and k_j $(i \ne j)$ belong to different intervals of non-resonance then the solutions of the respective problems have different oscillatory properties. We can illustrate this fact considering a Cauchy problem

$$\begin{cases} x^{(4)} - k^4 x = 0, \\ x(0) = 0, \ x'(0) = 0, \ x''(0) = A, \ x'''(0) = -B, \end{cases} \tag{3.3}$$

where A, B are some positive numbers. Figure 1 presents the solutions of problem (3.3) for different values of k from the first and the second intervals of non-resonance.
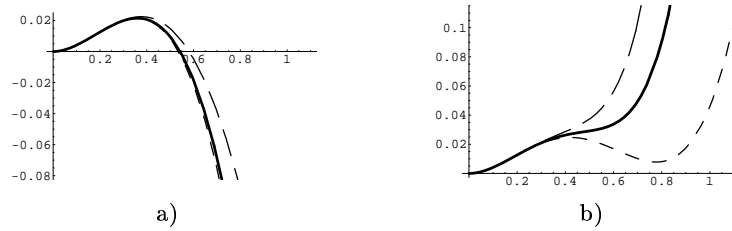
Figure 1. Solutions of problem (3.3) ($\alpha = 1$, $\beta = 5.6$) for different values k: a) $k \in (0, k_1)$,   b) $k \in (k_1, k_2)$.

If the numbers k_i and k_j belong to different intervals then the linear parts $x^{(4)} - k_i^4 x$ and $x^{(4)} - k_j^4 x$ have different types of non-resonance. We define an i-nonresonance of the linear part (i.e. non-resonance of i-type) using a notion of the conjugate point [1].

DEFINITION 1. A point $\eta > 0$ is called a *conjugate point* for the point $t = 0$, if there exists a nontrivial solution $x(t)$ of the equation (3.2) such that

$$x(0) = x'(0) = 0 = x(\eta) = x'(\eta).$$

DEFINITION 2. The linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is called *i–nonresonant* with respect to the boundary conditions in (3.1), if there are exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point.

If the linear parts $x^{(4)} - k_i^4 x$ and $x^{(4)} - k_j^4 x$ have different types of non-resonance we say for brevity that they are essentially different.

Now we give a definition of an i-type solution of the quasi-linear problem in a slightly different form than it was done in [2, 3].

DEFINITION 3. $\xi(t)$ is *an i-type solution* of the quasi-linear problem (3.1) if for small enough $\alpha, \beta > 0$ the difference $u(t; \alpha, \beta) = x(t; \alpha, \beta) - \xi(t)$ has at most $(i + 1)$ zeros in the interval $(0, 1]$ (counting multiplicities), where $x(t; \alpha, \beta)$ is a solution of the same quasi-linear equation in (3.1), which satisfies the initial conditions

$$\begin{cases} x(0; \alpha, \beta) = \xi(0), & x'(0; \alpha, \beta) = \xi'(0), \\ x''(0; \alpha, \beta) = \xi''(0) + \alpha, & x'''(0; \alpha, \beta) = \xi'''(0) - \beta. \end{cases} \qquad (3.4)$$

Such a solution $x(t; \alpha, \beta)$ is called a *neighbouring* solution.

Remark 1. It follows from the theory of Leighton and Nehari [1] that if conditions of Definition 3 are satisfied, then there exist exactly i solutions $x(t; \alpha_i, \beta_i)$ of the initial value problem $x^{(4)} - k^4 x = F(t, x)$, (3.4) such that the difference $u(t; \alpha_i, \beta_i)$ has a double zero $\eta_i \in (0, 1)$.

The following theorem was proved in [2, 4].

Theorem 1. *Suppose that conditions* **(A1)**–**(A4)** *are satisfied. Quasi-linear problem* (3.1) *has an i-type solution, if the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ is i-nonresonant.*

We can now answer the question posed at the end of the previous section. If a nonlinear problem allows for quasilinearization with respect to the linear parts of different types of non-resonance (that is, with respect to essentially different linear parts), then the solutions, revealed by these quasilinearizations, are different.

4. BVPs for Non-Autonomous Emden-Fowler Type Equations

Consider BVPs for the fourth-order non-autonomous Emden-Fowler type equations (1.1).

Theorem 2. *Suppose that $0 < q_1 \leq q(t) \leq q_2 \quad \forall t \in [0,\ 1]$. If there exists some k in the form $k = \pi i,\ (i = 1,\ 2,\ldots)$, which satisfies the inequality*

$$k\frac{e^k(4\sqrt{2}+3)-1}{4(e^k+1)} < \beta\,\frac{p^{\frac{p}{p-1}}}{(p-1)}\Big(\frac{q_1}{q_2}\Big)^{\frac{1}{(p-1)}} \quad for \quad k = (2n-1)\pi \qquad (4.1)$$

or

$$k\frac{e^k(4\sqrt{2}+3)+1}{4(e^k-1)} < \beta\,\frac{p^{\frac{p}{p-1}}}{(p-1)}\Big(\frac{q_1}{q_2}\Big)^{\frac{1}{(p-1)}} \quad for \quad k = 2n\pi, \qquad (4.2)$$

where β is a positive root of the equation

$$\beta^p = \beta + (p-1)p^{\frac{p}{1-p}}, \qquad (4.3)$$

then there exists an $(i-1)$-type solution of problem (1.1).

Proof. The given nonlinear equation (1.1) is equivalent to the equation

$$x^{(4)} - k^4 x = q(t)\,|x|^p \operatorname{sgn} x - k^4 x. \qquad (4.4)$$

Suppose that k satisfies $\cos k \cosh k \neq 1$ in order the linear part $\big(L_4 x\big)(t) := x^{(4)} - k^4 x$ to be non-resonant with respect to the given boundary conditions of problem (1.1). We wish to make the right side in (4.4) bounded. The function $f_k(t,x) := q(t)\,|x|^p \operatorname{sgn} x - k^4 x$ is odd in x for a fixed t. Let us consider it for nonnegative values of x. There exists a positive point of local extremum x_0. For a fixed $t = t^*$ we calculate the value of the function $f_k(t,x)$ at the point of extremum x_0

$$m_k(t^*) = |f_k(t^*,\ x_0)| = \Big(\frac{k^4}{p}\Big)^{\frac{p}{p-1}} |p-1|\,q(t^*)^{\frac{1}{1-p}}, \qquad (4.5)$$

and choose $n_k(t^*)$ such that $|x| \leq n_k(t^*) \Rightarrow |f_k(t^*,\ x)| \leq m_k(t^*)$. Computation gives that $n_k(t^*) = \big(\frac{k^4}{q(t^*)}\big)^{\frac{1}{p-1}}\beta$, where a constant β is described in (4.3). Set

$$M_k = \max\{m_k(t^*) : t^* \in [0,\,1]\}, \quad N_k = \min\{n_k(t^*) : t^* \in [0,\,1]\}. \quad (4.6)$$

Then carry out the appropriate smooth truncation ("cutoff") of the function $f_k(t, x)$ (a similar approach was applied in [5]) and get multiple quasi-linear equations for different values of k

$$x^{(4)} - k^4 x = F_k(t, x), \quad (4.7)$$

where the right sides in (4.4) and (4.7) coincide for $\left|x(t)\right| \leq N_k$ and $F_k(t, x)$ is bounded in modulus by a constant M_k. So the original problem (1.1) and each quasi-linear one (4.7) are equivalent in a domain

$$\Omega_k = \{(t, x) : \ 0 \leq t \leq 1, \ |x| \leq N_k\}.$$

Notice that a linear part $\left(L_4 x\right)(t) := x^{(4)} - k^4 x$ for the numbers k in the form $k = \pi i$, $(i = 1,\, 2, \ldots)$ is $(i-1)$-nonresonant. Besides, in the domain of equivalence Ω_k the conditions **(A1)** $-$ **(A4)** are fulfilled. Therefore in accordance with Theorem 1 the quasi-linear problems (4.7) for the numbers k in the form $k = \pi i$, $(i = 1,\, 2, \ldots)$ have $(i-1)$-type solutions.

Next we need to verify, whether the original problem (1.1) allows for quasi-linearization with respect to the linear parts $\left(L_4 x\right)(t) := x^{(4)} - k^4 x$ and the domains Ω_k. In this case inequality (2.1) (its fulfillment is necessary in order the quasilinearization in the above sense be possible) has the form

$$\Gamma_k M_k \leq N_k, \quad (4.8)$$

where the numbers M_k and N_k are described in (4.6) and Γ_k is an estimate of the respective Green's function $G_k(t, s)$ for the homogeneous problem $\left(L_4 x\right)(t) = 0$ with boundary conditions of (1.1). Since $p > 1$ and $0 < q_1 \leq q(t) \leq q_2, \ \forall t \in [0,\,1]$, then

$$M_k = \max_{t^* \in [0,\,1]} m_k(t^*) = \left(\frac{k^4}{p}\right)^{\frac{p}{p-1}} (p-1)\, q_1^{\frac{1}{1-p}},$$

$$N_k = \min_{t^* \in [0,\,1]} n_k(t^*) = \left(\frac{k^4}{q_2}\right)^{\frac{1}{p-1}} \beta. \quad (4.9)$$

The Green function $G_k(t, s)$ was constructed explicitly and estimated in [3, 4]. These estimates were improved in [6]. For values of k in the form $k = \pi i$, $(i = 1,\, 2, \ldots)$ the Green function satisfies the estimates

$$\left|G_k(t, s)\right| < \frac{e^k(4\sqrt{2} + 3) - 1}{4k^3(e^k + 1)} =: \Gamma_1(k), \quad \text{if} \quad k = (2n - 1)\pi, \quad (4.10)$$

$$\left|G_k(t, s)\right| < \frac{e^k(4\sqrt{2} + 3) + 1}{4k^3(e^k - 1)} =: \Gamma_2(k), \quad \text{if} \quad k = 2n\pi. \quad (4.11)$$

It follows from (4.9), (4.10), (4.11) that the inequality (4.8) reduces respectively either to (4.1) or (4.2). Therefore if there exists some k in the form $k = \pi i$, $(i = 1,\, 2, \ldots)$, which satisfies an inequality (4.1) or (4.2), then the original problem (1.1) allows for the quasilinearization with respect to the corresponding linear part $\left(L_4 x\right)(t) := x^{(4)} - k^4 x$ and the domain Ω_k and therefore this problem has an $(i-1)$- type solution. The proof is complete. ∎

Corollary 1. If there exist the numbers $k = \pi i$, $i = 1, 2, \ldots, m$, which satisfy the inequalities (4.1), (4.2), then there exist at least m solutions of different types to the problem (1.1).

Table 3. Results of calculations.

p	β	$\frac{q_1}{q_2}$	k_i
$\dfrac{5}{4}$	1.2813	$\frac{q_1}{q_2} \geq \frac{29}{30}$	$k_1 = \pi$; $k_2 = 2\pi$
$\dfrac{6}{5}$	1.2884	$\frac{q_1}{q_2} \geq \frac{15}{16}$	$k_1 = \pi$; $k_2 = 2\pi$
$\dfrac{7}{6}$	1.2933	$\frac{q_1}{q_2} \geq \frac{12}{13}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{53}{54}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
$\dfrac{8}{7}$	1.2969	$\frac{q_1}{q_2} \geq \frac{11}{12}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{27}{28}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
$\dfrac{9}{8}$	1.2998	$\frac{q_1}{q_2} \geq \frac{10}{11}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{21}{22}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
		$\frac{q_1}{q_2} \geq \frac{81}{82}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$; $k_4 = 4\pi$
$\dfrac{10}{9}$	1.3019	$\frac{q_1}{q_2} \geq \frac{10}{11}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{18}{19}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
		$\frac{q_1}{q_2} \geq \frac{43}{44}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$; $k_4 = 4\pi$
$\dfrac{11}{10}$	1.3038	$\frac{q_1}{q_2} \geq \frac{10}{11}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{17}{18}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
		$\frac{q_1}{q_2} \geq \frac{32}{33}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$; $k_4 = 4\pi$
		$\frac{q_1}{q_2} \geq \frac{111}{112}$	k_1; k_2; $k_3 = 3\pi$; $k_4 = 4\pi$; $k_5 = 5\pi$
$\dfrac{12}{11}$	1.3053	$\frac{q_1}{q_2} \geq \frac{10}{11}$	$k_1 = \pi$; $k_2 = 2\pi$
		$\frac{q_1}{q_2} \geq \frac{16}{17}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$
		$\frac{q_1}{q_2} \geq \frac{27}{28}$	$k_1 = \pi$; $k_2 = 2\pi$; $k_3 = 3\pi$; $k_4 = 4\pi$
		$\frac{q_1}{q_2} \geq \frac{60}{61}$	k_1; k_2; $k_3 = 3\pi$; $k_4 = 4\pi$; $k_5 = 5\pi$

In Table 3 the results of calculations are provided. For certain values of p and $\dfrac{q_1}{q_2}$ the numbers k in the form $k = \pi i$, $i = 1, 2, \ldots$ are given, which

satisfy the inequalities (4.1), (4.2). This table may be interpreted as a set of multiplicity results for the BVP (1.1).

Let us try to describe for fixed values of k given in the form $k = \pi i$, $i = 1, 2, \ldots$ the domains $D\left(p, \frac{q_1}{q_2}\right)$, in which the inequalities (4.1), (4.2) hold. Denote

$$\overline{q} := \frac{q_1}{q_2}, \quad \gamma(k) := \begin{cases} k\dfrac{e^k(4\sqrt{2} + 3) - 1}{4(e^k + 1)}, & \text{if } k = (2n - 1)\pi, \\[2mm] k\dfrac{e^k(4\sqrt{2} + 3) + 1}{4(e^k - 1)}, & \text{if } k = 2n\pi. \end{cases}$$

Then both inequalities (4.1) and (4.2) can be rewritten in the form

$$\overline{q} > \left(\frac{\gamma(k)}{\beta}\right)^{p-1} \frac{(p-1)^{(p-1)}}{p^p}. \tag{4.12}$$

We have computed the domains $D_k\left(p, \overline{q}\right)$, in which the inequality (4.12) holds. If in problem (1.1) coefficient p and function $q(t)$ are such that the point $(p, \overline{q})$ is located in the domain, which corresponds to $k = \pi m$, then the considered problem (1.1) has at least m solutions of different types. Figure 2 shows four
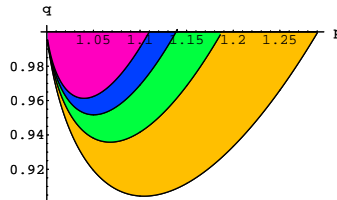

Figure 2. Domains, in which the inequality (4.12) holds.

embedded domains, which illustrate possibilities of application of the quasi-linearization process to problem (1.1). It is characterized by the parameters p and $\overline{q} = \frac{q_1}{q_2}$. If a point $(p, \overline{q})$ is located in the largest domain, then at least 2 essentially different quasilinearizations are possible with $k_1 = \pi$, $k_2 = 2\pi$. If a point $(p, \overline{q})$ belongs to the smallest domain, it means that at least 5 essentially different quasilinearizations are possible with $k_i = i\pi$, $i = 1, \ldots, 5$. These considerations agree well with the data in Table 3.

5. Example

Consider the fourth-order nonlinear boundary value problem

$$\begin{cases} x^{(4)} = 50(81 + \sin \frac{\pi}{2} t)|x|^{\frac{9}{8}} \operatorname{sgn} x, \\ x(0) = x'(0) = 0 = x(1) = x'(1). \end{cases} \tag{5.1}$$

It is a special case of the problem (1.1), when $p = \frac{9}{8}$ and $q(t) = 50(81+\sin\frac{\pi}{2}t)$. Since $\min_{[0, 1]} q(t) = 4050$ and $\max_{[0, 1]} q(t) = 4100$ than $\frac{q_1}{q_2} = \frac{81}{82}$. So in accordance with calculations (see Table 3 and Figure 2) and Corollary 1 there exist at least four solutions of different types to the given problem (5.1). We have computed them (see Fig. 3, Fig. 4, Fig. 5, Fig. 6).
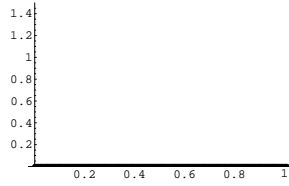

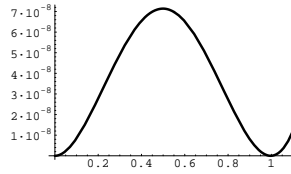**Figure 3.** 0-type solution of problem (5.1).

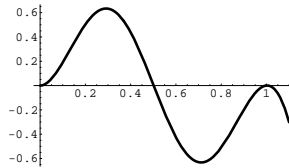**Figure 4.** 1-type solution of problem (5.1).

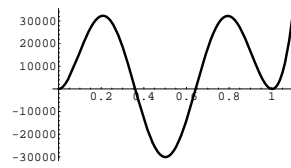**Figure 5.** 2-type solution of problem (5.1).

Figure 6. 3-type solution of problem (5.1).

The differences between a particular solution of the problem (5.1) and the respective neighboring solutions for different (α, β) were computed and a type of each solution was determined in accordance with Definition 3. A trivial solution of the problem (5.1) is depicted in Fig. 3. All neighbouring solutions are such that the differences between neighbouring solution and the trivial one have at most one zeros in the interval $(0, 1]$, therefore the trivial solution is a 0-type solution. Fig. 4 shows another solution of the problem (5.1). This solution is an 1-type solution because the differences between neighbouring solutions (for different pairs (α, β)) and it have either one simple zero or one double zero in $(0, 1]$. The initial data of the 1-type solution are $x''(0) = 0,000002$, $x'''(0) = -0,000009223$. Fig. 5 illustrates a 2-type solution of the problem (5.1), its initial data are $x''(0) = 51$, $x'''(0) = -395,08258$. The differences between certain neighbouring solutions and this solution are depicted in Fig. 7. This solution, actually, is a 2-type solution, because respective differences for different values of α, β have at most 3 zeros in the interval $(0, 1]$, counting multiplicities. Fig. 6 shows a 3-type solution of the problem (5.1). The initial data of this solution are $x''(0) = 5100000$, $x'''(0) = -55374924,809$. Comparing with a case of the 2-type solution, for the 3-type solution there exist such

neighbouring solutions that respective differences have two simple zeros and one double zero in the interval $(0, 1]$.
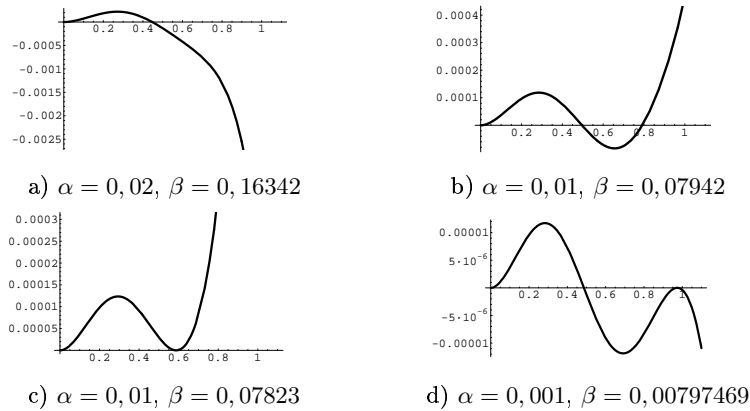

a) $\alpha = 0,02,\ \beta = 0,16342$ b) $\alpha = 0,01,\ \beta = 0,07942$

c) $\alpha = 0,01,\ \beta = 0,07823$ d) $\alpha = 0,001,\ \beta = 0,00797469$

Figure 7. Differences between the neighbouring solutions and 2-type solution of problem (5.1): a) one simple zero; b) two simple zeros; c) one double zero; d) one simple and one double zero.

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