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STEADY HEAT CONDUCTION OF MATERIAL WITH COATED INCLUSION IN THE CASE OF IMPERFECT CONTACT

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Abstract. We discuss a potential steady heat conduction for composites with the coated cylindrical inclusion under an imperfect contact condition. We rewrite equivalently the considered problem to a conjugation problem for analytic functions which is reduced then to functional–differential equations. Solution of the obtained system of functional–differential equations is given in the closed form.

Key words: Functional equations in complex domains, iteration and composition of analytic functions, boundary value problems

1. Statement of Boundary Value Problem

Let $D_1 = \{z \in \mathbb{C} : |z| < r\}$ be the disc of radius $r < 1$ and $D = \{z \in \mathbb{C} : r < |z| < 1\}$ be the annulus on the complex plane $\mathbb{C}$. Let $D_0 = \{z \in \mathbb{C} : |z| > 1\}$ be an exterior of the unit disc on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Denote by $\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}$, $\Gamma_2 = \{z \in \mathbb{C} : |z| = r\}$ the corresponding boundary curves. Let the curve Γ_2 be orientated in counter clockwise sense, and Γ_1 be orientated in clockwise sense. Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal vector to Γ_k, $(k = 1, 2)$. The normal derivative is introduced as follows

$$\frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}.$$

We consider the steady heat conduction in the long cylindrical composites with coated cylindrical inclusion in the direction orthogonal to the axis of the cylinders. The problem on the description of the heat field can be understood as purely two-dimensional one. In this paper we consider only potential case.

We suppose that all parts of the complex plane D, D_0, D_1 are occupied by materials with the positive thermal conductivities 1, λ_0, λ_1, respectively (here λ_0 and λ_1 are positive). Let the contact between materials D, D_0 is perfect, and between D, D_1 is imperfect. We are interested in the temperature distribution on $\widehat{\mathbb{C}}$ which is described by a function $u(z)$. It is assumed that the external field is described by a singularity at infinity.

The problem is to find a function $u(z)$ harmonic in D, D_0, D_1, continuously differentiable in the closures of the considered domains except $z = \infty$, and satisfying the following conditions

$$\frac{\partial u^-}{\partial \mathbf{n}}(t) = \lambda_1 \frac{\partial u^+}{\partial \mathbf{n}}(t), \qquad |t| = r, \tag{1.1}$$

$$\lambda_1 \frac{\partial u^+}{\partial \mathbf{n}}(t) + \gamma_1 (u^+(t) - u^-(t)) = 0, \qquad |t| = r, \tag{1.2}$$

$$u^+ = u^-, \quad \frac{\partial u^-}{\partial \mathbf{n}}(t) = \lambda_0 \frac{\partial u^+}{\partial \mathbf{n}}(t), \qquad |t| = 1, \tag{1.3}$$

It is supposed also that $u(z) - \mathrm{Re} p_N(z)$ is bounded at infinity, where

$$p_N(z) = A_1 z + A_2 z^2 + \ldots + A_N z^N \tag{1.4}$$

is a given polynomial of degree N corresponding to the external field applied at infinity.

The conjugation conditions (1.1)–(1.2) describe the imperfect contact between materials on Γ_2 (see [2]), and conditions (1.3) correspond to the perfect contact on Γ_1 (see, e.g., [4]).

In the following we identify the normal vector $\mathbf{n} = (n_1, n_2)$ with the complex number $n = n_1 + in_2$. Let us introduce the complex potential

$$\varphi(z) = u(z) + iv(z), \tag{1.5}$$

which is analytic in D, D_0, D_1, continuously differentiable in the closures of the considered domains except $z = \infty$. $\varphi(z)$ has the principal part $p_N(z)$ at infinity, i.e., $\varphi(z) - p_N(z)$ is bounded at infinity. The problem (1.1)–(1.3) is equivalent to the following $\mathbb{R}$-linear problem (see [1, 4])

$$\varphi^-(t) = \phi_1(t) - \rho_1 \overline{\phi_1(t)} + \mu_1 t \phi_1'(t) + \mu_1 \frac{r^2}{t} \overline{\phi_1'(t)}, \qquad |t| = r, \tag{1.6}$$

$$\varphi^-(t) = \phi_0(t) - \rho_0 \overline{\phi_0(t)}, \qquad |t| = 1, \tag{1.7}$$

where

$$\phi_k(z) = \frac{\lambda_k + 1}{2} \varphi^+(z), \quad z \in D_k, \tag{1.8}$$

$\rho_k = \frac{\lambda_k - 1}{\lambda_k + 1}$ $(k = 0, 1)$, $\mu_1 = \frac{1 + \rho_1}{2r\gamma_1}$. Moreover, the functions $\phi_1(z)$ and $\varphi(z)$ are analytic in D_1 and D respectively. The function $\phi_0(z)$ is analytic in $D_0 - \infty$, continuously differentiable in its closure and has the representation

$$\phi_0(z) = \varphi_0(z) + p_N(z), \quad |z| \geq 1. \tag{1.9}$$

Despite the significant interest in applications [2] the conjugation problem in the case of imperfect contact is not deeply studied. To the best of author's knowledge there are only a few results devoted to the constructive solution of the problem. Benveniste and Miloh [3] expanded solution of such problem in series of special functions in order to estimate the effective conductivity. Gonsalves and Kolodziej [2] solved this problem numerically in a class of doubly periodic function by the collocation method.

2. Functional-Differential Equation

In this section the problem (1.6)–(1.7) is reduced to functional–differential equations. Let us introduce the function

$$\Phi(z)=\begin{cases}\phi_1(z)+\mu_1 z\phi_1'(z)+\rho_0\overline{\phi_0(1/\overline{z})}, & |z|\le r,\\ \varphi(z)+\rho_1\overline{\phi_1(r^2/\overline{z})}-\mu_1\frac{r^2}{z}\overline{\phi_1'(r^2/\overline{z})}+\rho_0\overline{\phi_0(1/\overline{z})}, & r<z<1,\\ \phi_0(z)+\rho_1\overline{\phi_1(r^2/\overline{z})}-\mu_1\frac{r^2}{z}\overline{\phi_1'(r^2/\overline{z})}, & z\ge 1.\end{cases} \tag{2.1}$$

First, consider $\Phi(z)$ in $|z|<r$. The functions $\phi_1(z)$ and $z\phi_1'(z)$ are analytic in $|z|<r$. Since $\phi_0(z)$ is analytic in $1<|z|<\infty$, hence the function $\overline{\phi_0\left(1/\overline{z}\right)}$ is analytic in $0<|z|<1$. Therefore, $\Phi(z)$ is analytic in $0<|z|<r$. Analogously, $\Phi(z)$ is analytic in D and $D_0\setminus\{\infty\}$.

Let us prove that $\Phi^+(t)=\Phi^-(t)$ on the circle $|t|=r$, where

$$\Phi^+(t)=\lim_{\substack{z\to t\\|z|<r}}\Phi(z)=\phi_1(t)+\mu_1 z\phi_1'(t)+\rho_0\overline{\phi_0\left(1/\overline{t}\right)},$$

$$\Phi^-(t)=\lim_{\substack{z\to t\\|z|>r}}\Phi(z)=\varphi^-(t)+\rho_1\overline{\phi_1\left(r^2/\overline{t}\right)}-\mu_1\frac{r^2}{t}\overline{\phi_1'\left(r^2/\overline{t}\right)}+\rho_0\overline{\phi_0\left(1/\overline{t}\right)}.$$

For this we calculate the jump $\Delta:=\Phi^+(t)-\Phi^-(t)$ across Γ_2. Using the relation $t=r^2/\overline{t}$ on the circle $|t|=r$ we have

$$\Delta=\phi_1(t)+\mu_1 t\phi_1'(t)-\varphi^-(t)-\rho_1\overline{\phi_1(t)}+\mu_1\frac{r^2}{t}\overline{\phi_1'(t)}.$$

It follows from (1.6) that $\Delta=0$. The same consideration yields $\Phi^+(t)=\Phi^-(t)$ on the circle $|t|=1$. Then Analytic Continuation Principle implies that the function $\Phi(z)$ is analytic in $\mathbb{C}\setminus\{0\}$. In the force of (1.9) and (2.1) we conclude that the principal part of $\Phi(z)$ is equal to $p_N(z)$ at $z=\infty$ and is equal to $\rho_0\overline{p_N\left(1/\overline{z}\right)}$ at $z=0$. Then the Liouville theorem yields

$$\Phi(z)-\rho_0\overline{p_N\left(1/\overline{z}\right)}-p_N(z)=c,\quad z\in\mathbb{C}, \tag{2.2}$$

where c is constant. Let us introduce the function

$$g(z)=\rho_0\overline{p_N\left(1/\overline{z}\right)}+p_N(z)+c. \tag{2.3}$$

Consider (2.2) in $|z|\le r$ and in $|z|\ge 1$. Using (2.1) we obtain

$$\phi_1(z)+\mu_1 z\phi_1'(z)-\rho_0\overline{\phi_0\left(1/\overline{z}\right)}=g(z),\quad |z|\le r, \tag{2.4}$$

$$\phi_0(z) + \rho_1 \overline{\phi_1\left(r^2/\overline{z}\right)} - \mu_1 \frac{r^2}{z} \overline{\phi_1'\left(r^2/\overline{z}\right)} = g(z), \quad |z| \geq 1. \tag{2.5}$$

Substituting $w = 1/\overline{z}$ in (2.5) and taking the complex conjugation we have

$$\overline{\phi_0\left(1/\overline{w}\right)} + \rho_1 \phi_1\left(r^2 w\right) - \mu_1 r^2 w \phi_1'\left(r^2 w\right) = \overline{g\left(1/\overline{w}\right)} \quad |w| \leq 1,$$

or in variable z

$$\overline{\phi_0\left(1/\overline{z}\right)} = \overline{g\left(1/\overline{z}\right)} - \rho_1 \phi_1\left(r^2 z\right) + \mu_1 r^2 z \phi_1'\left(r^2 z\right) \quad |z| \leq 1.$$

Substitute now the value of the function $\overline{\phi_0(1/\overline{z})}$ from the latter relation into (2.4). As a result, we obtain the functional–differential equation

$$\phi_1(z) + \mu_1 z \phi_1'(z) + \rho_0 \rho_1 \phi_1\left(r^2 z\right) - \rho_0 \mu_1 r^2 z \phi_1'\left(r^2 z\right) = h(z), \quad |z| \leq r, \tag{2.6}$$

where

$$h(z) = g(z) - \rho_0 \overline{g\left(1/\overline{z}\right)}. \tag{2.7}$$

In (2.6), the unknown function $\phi_1(z)$ is analytic in $|z| < r$ and continuously differentiable in $|z| \leq r$. We note that the known function $h(z)$ contains the undetermined constant c (see (2.3)).

3. Explicit Solution to the Functional–Differential Equations

Theorem 1. *Let $\rho_0\rho_1 \neq -1$. Denote by h_m the Taylor coefficients of the function $h(z)$ at $z = 0$. Then equation (2.6) has the unique solution*

$$\phi_1(z) = \sum_{m=0}^{\infty} \frac{h_m}{B_m} z^m, \tag{3.1}$$

where

$$B_m = 1 + \mu_1 m + \rho_0 \rho_1 r^{2m} - \rho_0 \mu_1 m r^{2m+2} > 0. \tag{3.2}$$

Proof. We are looking for the function $\phi_1(z)$ analytic in the disc $|z| < r$, in the form of series

$$\phi_1(z) = \sum_{m=0}^{\infty} \alpha_m z^m.$$

Calculate

$$z\phi_1'(z) = \sum_{m=0}^{\infty} m \alpha_m z^m, \ \ |z| < r, \quad \phi_1'(r^2 z) = \sum_{m=0}^{\infty} m \alpha_m r^{2m} z^{m-1}. \tag{3.3}$$

Substituting (3.3) into (2.6) and comparing coefficients at the same powers of z we obtain equations on α_m

$$\alpha_m \left(1 + \mu_1 m + \rho_0 \rho_1 r^{2m} - \rho_0 \mu_1 m r^{2m+2}\right) = h_m, \quad m = 0, 1, \ldots. \tag{3.4}$$

It follows from the above assumptions that

$$-1 < \rho_0\rho_1 \leq 1, \quad 0 \leq \mu_1 \leq \infty. \tag{3.5}$$

Using (3.5) one can see that

$$0 < 1 - \rho_0 r^{2m+2} < 2 \ \text{ and } \ -1 < \rho_0\rho_1 r^{2m} \leq 1.$$

Therefore, the constants B_m in (3.2) are positive for all $m = 0, 1, \ldots$. Hence (3.4) yields $\alpha_m = h_m B_m^{-1}$, $m \geq 0$, and we arrive at (3.1). The function $\phi_0(z)$ is found from (2.5). This completes the proof. ■

4. Solution to Boundary Value Problem

We now apply Theorem 1 to the problem (1.6)–(1.7). Using (1.4), (2.3) and (2.7) we rewrite $g(z)$ in the form

$$g(z) = \rho_0 \sum_{m=1}^{N} A_m z^{-m} + \sum_{m=1}^{N} A_m z^m + c. \tag{4.1}$$

Theorem 2. *Let $\rho_0\rho_1 \neq -1$. Then the problem (1.6)–(1.7) has the unique solution*

$$\phi_0(z) = \sum_{m=-N}^{N} \beta_m z^m, \tag{4.2}$$

where

$$\beta_m = \begin{cases} A_m, & m = 1, 2, \ldots, N, \\ -\rho_1(\overline{c} - \rho_0 c)(1 + \rho_0\rho_1)^{-1} + c, & m = 0, \\ -\rho_1 \overline{A}_{-m} \overline{B}_{-m}^{-1} r^{-2m} - \mu_1 m \overline{A}_{-m} \overline{B}_{-m}^{-1} r^{-2m} + \rho_0 \overline{A}_{-m}, & m = -1, \ldots, -N, \end{cases}$$

and

$$\phi_1(z) = a_0 + \sum_{m=1}^{N} \frac{A_m}{B_m} z^m, \tag{4.3}$$

where $a_0 = (c - \rho_0\overline{c})(1 + \rho_0\rho_1)^{-1}$.

Proof. Represent $\phi_0(z)$ in the form of the Laurent series

$$\phi_0(z) = \sum_{m=-\infty}^{\infty} \beta_m z^m.$$

Applying Theorem 1 and using equations (2.5), (4.1) we have

$$\sum_{m=-\infty}^{\infty} \beta_m z^m = \sum_{m=0}^{\infty} (\mu_1 m - \rho_1)\, \overline{h}_m \overline{B}_m^{-1} r^{2m} z^{-m} + \rho_0 \sum_{m=1}^{N} A_m z^{-m} + \sum_{m=1}^{N} A_m z^m + c.$$

Selecting the coefficient at z^m in the latter relation we calculate β_m. It follows from (2.3) and (2.7) that function $h(z)$ is a polynomial of degree N

$$h(z) = (1-\rho_0^2)p_N(z) + c - \rho_0\overline{c}.$$

Then the series (3.1) becomes the polynomial (4.3). ■

5. Solution to the Steady Heat Conduction Problem

The functions $\phi_0(z)$ and $\phi_1(z)$ are given in Theorem 2. Determine now the function $\varphi(z)$ using (2.1) in $r < |z| < 1$. After tedious calculations we obtain

$$\varphi(z) = \sum_{m=-N}^{N} \alpha_m z^{-m}, \tag{5.1}$$

where

$$\alpha_m = \begin{cases} \rho_0 A_m - \rho_0 \overline{A}_m + (m\mu_1 - \rho_1) A_m B_m^{-1} r^{2m}, \quad m = 1, 2, \ldots, N, \\ c - \rho_1 \overline{\eta}(1+\rho_0\rho_1)^{-1} - \rho_0(\overline{c} - \rho_1 \eta), \quad m = 0, \\ A_{-m} - \rho_0 A_{-m}\left(\rho_0 - \rho_1 B_{-m}^{-1} r^{-2m} - \mu_1 m B_{-m}^{-1} r^{-2m}\right), \; m = -1, \ldots, -N. \end{cases}$$

Applying the relations between analytic and harmonic functions (1.5) and (1.8) we arrive at the following theorem.

Theorem 3. *Let $\rho_0\rho_1 \neq -1$. The conjugation problem* (1.1)–(1.3) *has the unique solution determined up to an additive constant via the following relations*

$$u(z) = \begin{cases} \frac{2}{\lambda_0+1} Re\phi_0(z), & |z| \geq 1, \\ \frac{2}{\lambda_1+1} Re\phi_1(z), & |z| \leq r, \\ Re\varphi(z), & r < |z| < 1, \end{cases}$$

where functions ϕ_0, ϕ_1, φ are given of the formulas (4.2), (4.3), (5.1), *where c is an arbitrary complex constant.*

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