NUMERICAL INVESTIGATION OF ANISOTROPICALLY DRIVEN DEVELOPED TURBULENCE

E.A. HAYRYAN[1], E. JURCISINOVA[1,2], M. JURCISIN[2,3], M. STEHLIK[2]

[1] *Laboratory of Information Technologies, Joint Institute for Nuclear Research*

Joliot-Curie 6, 141980, Dubna, Moscow region, Russia

E-mail: `ayrjan@jinr.ru`; `jurcisinova@jinr.ru`

[2] *Institute of Experimental Physics, Slovak Academy of Sciences*

Watsonova 47, 040 01 Košice, Slovakia

E-mail: `stehlik@saske.sk`

[3] *Laboratory of Theoretical Physics, Joint Institute for Nuclear Research*

Joliot-Curie 6, 141980, Dubna, Moscow region, Russia

E-mail: `jurcisin@thsun1.jinr.ru`

Abstract. The fully developed turbulence with axial anisotropy for dimensions $d > 2$ is investigated by means of renormalization group approach. The corresponding system of strongly nonlinear renormalization group equations which contain angle integrals is solved numerically. Possible utilization of the parallel programming methods is discussed. As a result, the influence of anisotropy on the stability of the Kolmogorov scaling regime is analyzed. The borderline dimension between stable scaling regime and unstable one is calculated as a function of the anisotropy parameters. Obtained results are compared with results calculated in [7].

Key words: Numerical investigation, developed turbulence, renormalization group

1. Introduction

One of the most suitable and also powerful tool for investigation of fully developed turbulence based on the stochastic Navier-Stokes equation [21] is the so-called quantum-field renormalization group (RG) [2, 19]. In early papers, the RG approach was applied only to isotropic models of developed turbulence. However, the method can also be used in the theory of anisotropically developed turbulence. The reason for theoretical study of the influence of

anisotropy on the behavior of developed turbulence is given by the fact that a variety of experimental studies as well as computer simulations indicate the existence of deviation from the isotropic statistics of the fully developed turbulence. A crucial question immediately arises here: whether the scaling regime remains stable under transition from the isotropic to the anisotropic case. In other words, do the stable fixed points of the RG equations remain stable under the influence of anisotropy?

During the last decade a few papers have appeared in which the above question was considered in the framework of the RG approach in fully developed turbulence and related problems (magnetohydrodynamic developed turbulence, advection of passive and vector fields by a given turbulent environment, etc.). In some cases, it was found that the stability actually takes place (see, e.g., [13, 18]). On the other hand, the existence of systems without such a stability has also been proven. As was shown in [3] in the anisotropic[1] magnetohydrodynamic developed turbulence a stable regime generally does not exist. In [7, 13] d-dimensional models of fully developed turbulence with $d > 2$ were investigated for two cases: weak anisotropy [13] and strong anisotropy [7], and it was shown that the stability of the isotropic fixed point is lost for dimensions $d < d_c \simeq 2.685$. In [7], where strong anisotropy was investigated, it was also stated that stability of the fixed point, even for dimension $d = 3$, takes place only for sufficiently weak anisotropy. In present paper, we would like to return to the problem of the influence of strong anisotropy on the stability of the scaling regime in fully developed turbulence which was studied in [7]. The reason is the suspicion that their results are, at least, not precise (an evidence for this can be also seen by comparison of the results obtained in [7] with results shown in [6]. Thus, in what follows, we shall return to the description and detailed analysis of the numerical methods used in [7]. Our conclusion is the following: the numerical results and conclusions of [7] are not exact and must be specified, although the conceptual framework of their approach is accurate. We shall try to find and to understand the critical points in numerical calculations which could lead to the non accurate results. Another question which will be studied is the possible exploitation of the parallel programming methods to speed up the calculations. As a result, we shall calculate the dependence of the "critical" dimension d_c between stable and unstable scaling regimes as a function of the anisotropy parameters and compare our results to that ones obtained in [7].

The paper is organized as follows: In Section 2 we discuss the field theoretic functional formulation of the stochastic problem of fully developed turbulence with strong anisotropy. In Section 3 we discuss the stability of the fixed point under influence of strong anisotropy. In Section 4 we present a detail analysis of numerical methods which was used in calculations. In Section 5 the discussion of the results is present. Appendix I contains explicit expressions of the coefficients of β functions. In Appendix II the necessary and sufficient conditions for convergence of needed integrals are proven.

---

[1] In what follows we always have in mind the uniaxial anisotropy, i.e., the anisotropy defined by one specific direction (see the next section).

2. Description of Model. Field Theoretic Formulation

In the statistical theory of anisotropically developed turbulence, the turbulent flow is characterized by the random velocity field $\mathbf{v}(\mathbf{x}, t)$, where $\mathbf{v}$ and $\mathbf{x}$ are supposed to be d-dimensional vectors. Its evolution is governed by the randomly forced Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu_0 \Delta \mathbf{v} - \mathbf{f}^A = \mathbf{f}, \qquad (2.1)$$

where the incompressibility of the fluid is assumed, which is given mathematically by the conditions $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{f} = 0$. The parameter ν_0 is the kinematic viscosity (subscript 0 denotes bare parameters, see, e.g., [19]). The term $\mathbf{f}^{\mathbf{A}}$ is related to uniaxial anisotropy, and it has the following form [7]

$$\mathbf{f}^{\mathbf{A}} = \nu_0 \left[\chi_{10}(\mathbf{n}\nabla)^2 \mathbf{v} + \chi_{20}\mathbf{n}\nabla^2(\mathbf{n}\mathbf{v}) + \chi_{30}\mathbf{n}(\mathbf{n}\nabla)^2(\mathbf{n}\mathbf{v}) \right] . \qquad (2.2)$$

Parameters χ_{10}, χ_{20} and χ_{30} characterize the weight of the individual structures in (2.2), and the unit vector $\mathbf{n}$ specifies the direction of the anisotropy axis. The large-scale random force per unit mass $\mathbf{f}$ is assumed to have the Gaussian statistics defined by the averages

$$\langle f_i \rangle = 0, \ \ \langle f_i(\mathbf{x}_1, t) f_j(\mathbf{x}_2, t) \rangle = D_{ij}(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2). \qquad (2.3)$$

It is convenient to parametrize the two point correlation matrix

$$D_{ij}(\mathbf{x}, t) = \delta(t) \int \frac{d^d \mathbf{k}}{(2\pi)^d} \tilde{D}_{ij}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) \qquad (2.4)$$

in the following way [3, 7, 18]

$$\tilde{D}_{ij}(\mathbf{k}) = g_0 \nu_0^3 k^{4-d-2\epsilon} \left[(1 + \alpha_1 \xi_k^2) P_{ij}(\mathbf{k}) + \alpha_2 R_{ij}(\mathbf{k}) \right] , \qquad (2.5)$$

where vector $\mathbf{k}$ is the wave vector, d is the dimension of the space (in our case: $2 < d$), $\epsilon \geq 0$ is dimensionless parameter of the model. The physical value of this parameter is $\epsilon = 2$ (so-called energy pumping regime). We shall not discussed here more complicated case $d = 2$ (see, e.g., [12] and references cited therein). The value $\epsilon = 0$ corresponds to a logarithmic perturbation theory for a calculation of the Green function when g_0, which plays the role of a bare coupling constant of the model, becomes dimensionless [19]. The problem of the continuation from $\epsilon = 0$ to the physical values was discussed in [1]. The $(d \times d)$-matrices P_{ij} and R_{ij} are the transverse projection operators. Their explicit forms are defined by the relations (in the wave-number space)

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \ \ R_{ij}(\mathbf{k}) = \left(n_i - \xi_k \frac{k_i}{k} \right) \left(n_j - \xi_k \frac{k_j}{k} \right), \qquad (2.6)$$

where ξ_k is given by the equation $\xi_k = \mathbf{k} \cdot \mathbf{n}/k$. The tensor $\tilde{D}_{ij}$, given by (2.5), is the most general form with respect to the condition of incompressibility of the system under consideration and contains two dimensionless free parameters

α_1 and α_2. The positiveness of the correlator tensor D_{ij} leads to restrictions on the above parameters, namely, $\alpha_1 > -1$ and $\alpha_2 > -1$.

Using the well-known Martin–Siggia–Rose formalism [4, 5, 8, 15], the stochastic problem (2.1) with correlator (2.4) can be transformed into the field theoretic model of fields $\mathbf{v}$ and $\mathbf{v}'$ where $\mathbf{v}'$ is independent of the velocity field $\mathbf{v}$ auxiliary incompressible field, which we have to introduce when transforming the stochastic problem into a functional form. After this transformation the action of the fields $\mathbf{v}$ and $\mathbf{v}'$ is given in the form

$$S = \frac{1}{2} \int d^d\mathbf{x}_1 dt_1 d^d\mathbf{x}_2 dt_2 \left[v_i'(\mathbf{x}_1, t_1) D_{ij}(\mathbf{x}_1 - \mathbf{x}_2, t_1 - t_2) v_j'(\mathbf{x}_2, t_2) \right]$$
$$+ \int d^d\mathbf{x} dt \left\{ \mathbf{v}'(\mathbf{x}, t) \left[-\partial_t \mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v} + \nu_0 \nabla^2 \mathbf{v} + \mathbf{f}^A \right] (\mathbf{x}, t) \right\} . \quad (2.7)$$

The functional formulation gives the possibility to use the quantum field theory methods, including the RG technique, to solve the problem. The formulation through action functional (2.7) replaces the statistical averages of random quantities in the stochastic problem (2.1)–(2.5) with equivalent functional averages with weight $\exp[S(\mathbf{v}, \mathbf{v}')]$ (for details see [2, 19]).

3. RG Equations and Stability of the Kolmogorov Scaling Regime

Fully developed turbulence is characterized by the large Reynolds number Re. On the other hand, the large Re corresponds to the existence of a large inertial interval, which is defined by the inequalities $1/\Lambda = l \ll r \ll L = 1/m$, where l corresponds to an inner scale (the scale where dissipation forces are dominated, or the scale of the smallest eddies), and L is an outer scale of the system (the scale of the energy pumping into the system, or the scale of the largest eddies). In fully developed turbulence we are interested in the behavior of the correlation functions of velocity field $\langle v_{i_1}(\mathbf{x}_1, t), \dots, v_{i_N}(\mathbf{x}_N, t) \rangle$ deep inside of the inertial interval, i.e., far away from the dissipation effects as well as far away from energy pumping scale. Within the field theoretic approach they are given by the following functional integral (see, [19])

$$\langle v_{i_1}(\mathbf{x}_1, t), \cdots, v_{i_N}(\mathbf{x}_N, t) \rangle = \int \mathcal{D}\Phi \, v_{i_1}(\mathbf{x}_1, t), \cdots, v_{i_N}(\mathbf{x}_N, t) e^{S(\Phi)}, \quad (3.1)$$

where $\Phi = \{\mathbf{v}, \mathbf{v}'\}$, $1 \le i_j \le d, j = 1, \dots, N$, and $S(\Phi)$ is given by (2.7).

The behavior of the correlation functions inside the inertial interval is the main issue of the famous Kolmogorov-Obukhov phenomenological theory [14, 17] (see also [16]). It was formulated in the form of two hypotheses which lead to the scaling behavior of the correlation functions within the inertial interval. In what follows we shall discuss only the so-called second Kolmogorov hypothesis related to the IR scaling and our aim will be to investigate the influence of the axial anisotropy on this scaling behavior.

As was mentioned in the Introduction the appropriate method to investigate self-similar systems is the RG method. Within the RG technique the correlation functions are obtained directly in the scaling form (with correct critical dimensions) and their large-scale limit (i.e., IR limit) is described by the stable fixed points of the renormalization theory, i.e., the scaling regime is stable if the corresponding fixed point is IR stable. The IR fixed point is obtained by using the system of differential equations (also called the flow equations) which drive the effective variables $\bar{C} = \{\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3\}$ which are the functions of dimensionless scale parameter (wave number) $t = k/\Lambda$. Their explicit form is the following

$$\begin{cases} t\dfrac{d\bar{g}}{dt} = \beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) \,, \\ t\dfrac{d\bar{\chi}_i}{dt} = \beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) \,, \quad i = 1, 2, 3 \,. \end{cases} \tag{3.2}$$

The dimensionless wave number t belongs to the interval $0 \leq t \leq 1$, and the initial conditions for the above differential equations are taken at $t = 1$. The IR stable fixed point corresponds to the values in the limit $t \to 0$, i.e., $(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3)|_{t \to 0} = (g^*, \chi_1^*, \chi_2^*, \chi_3^*)$ (standardly, a quantity with star denotes the fixed point value). The so-called β-functions $\beta_g, \beta_{\chi_i}, i = 1, 2, 3$ are defined by the so-called renormalization constants of the renormalization procedure and their final form is as follows (details see in Ref. [11])

$$\beta_g = g(-2\epsilon + 3Aga_1) \,, \quad \beta_{\chi_i} = -Ag(a_{i+1} - \chi_i a_1) \,, \quad i = 1, 2, 3 \,, \tag{3.3}$$

where parameter A is defined as $A = S_{d-1}/((2\pi)^d(d^2 - 1))$, S_d is the area of the d-dimensional sphere given as $S_d = 2\pi^{(d/2)}/\Gamma(d/2)$, and functions a_i ($i = 1, ..., 4$) are given in Appendix I.

In principle, one has two possible ways how to find the IR fixed point of the model. First of all, one can solve the system of four equations

$$\beta_C(C^*, \alpha_1, \alpha_2, d) = 0 \,, \tag{3.4}$$

where we denote $C^* = \{g^*, \chi_1^*, \chi_2^*, \chi_3^*\}$. In this case, the IR stability of the fixed point is determined by the positive real parts of the eigenvalues of the matrix

$$\omega_{lm} = \left(\frac{\partial \beta_{C_l}}{\partial C_m}\right)_{C=C^*} \,, \quad l, m = 1, \ldots, 4 \,. \tag{3.5}$$

This is a comfortable way for the determination of the fixed point but in our case it cannot be used. The reason is the presence of the integrals in the β functions (see Appendix I) which makes this way rather complicated.

The second possibility is to solve directly the system of the differential equations (3.2). It is the way which will be used in our case. This method was also applied in [7]. More about numerical methods will be said in the next section.

Now we have all necessary tools at hand to investigate the fixed point and its stability. Our aim is to find the so-called borderline dimension d_c between
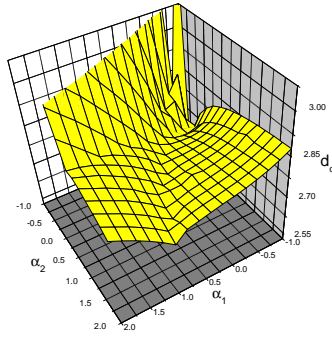
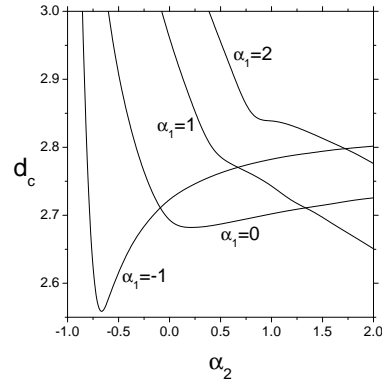**Figure 1.** The three-dimensional view on the dependence of the borderline dimension d_c on the parameters α_1, and α_2.

Figure 2. The dependence of the borderline dimension d_c on the parameter α_2 for concrete values of the parameter α_1.

stable and unstable regimes when moving from the three-dimensional system towards two-dimensional one. In Fig. 1, and Fig. 2 our results for d_c as a function of the anisotropy parameters are present. The difference between our results and those of [7] can be seen immediately, namely, the region of stability of the three-dimensional scaling regime is sufficiently larger in our case than in [7] (see Fig 1 in [7]). Fig. 1, and Fig. 2 show that in the three-dimensional system the Kolmogorov scaling regime is unstable only in the limit $\alpha_{1,2} \to -1$ and for large enough values of parameter α_1 together with negative or relatively small positive values of the parameter α_2. Our conclusion is the following (it differs sufficiently from the conclusion found in [7]): to destroy stability of the Kolmogorov scaling regime in three-dimensional space by the uniaxial anisotropy, which is in our model represented by the parameters α_1, and α_2, it is necessary to apply anisotropy with rather specific values of these parameters.

4. Numerical Methods

One possible way how to solve the problem of the IR fixed point of the four differential RG equations (3.2) with β functions (3.3) and corresponding integrals (5.1) of Appendix I is based on the analytical calculations of the integrals. The integrands of (5.1) have the form of fractions of two polynomials, namely:

$P_i(x^2)/Q(x^2)$ $(i = 1, \ldots, 4)$, with different numerators $P_i(x^2) = b_i(x^2)$ but with the same denominator $Q_i(x^2) = (M_1 M_2 M_3)^3$. It is possible to expand the expression $1/(M_1 M_2 M_3)^3$ into a sum of partial fractions of the type $R(x^2)/(a + x^2)^j$, where $R(x^2)$ is a polynomial, a is, in general, complex function of parameters $\chi_i, i = 1, 2, 3$ and $j = 1, 2, 3$. Now using the following result (see, e.g., [9])

$$\int_{-1}^{1} \frac{(1 - x^2)^{\frac{d-3}{2}} x^{2n}}{(a + x^2)^j} \, dx = \frac{(1 + (-1)^{2n}) \Gamma(\frac{d-1}{2}) \Gamma(\frac{1}{2} + n)}{2 a^j \Gamma(\frac{d}{2} + n)}$$

$$\times \, _2F_1 \left(1, \frac{1}{2} + n; \frac{d}{2} + n; -\frac{1}{a} \right), \qquad (4.1)$$

one can represent the integrals in the form of a combination of hypergeometric functions $_2F_1(a, b; c; z)$ defined as

$$_2F_1(a, b; c; z) = 1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots.$$

Equation (4.1) is held when $Re[d] > 1$ ($Re[x]$ means the real part of x), $Re[n] > -1/2$, and $Arg[a] \neq \pi$. In our case, these conditions are fulfilled because $d \in (2, 3]$, n is zero or positive integer, and it can be shown that the last condition is also valid.

By using this representation of integrals (5.1) it is possible to find the IR fixed point of differential equations (3.2) by solving the system of equations (3.4) together with the matrix of the first derivatives (3.5) to test the IR stability of the fixed point. But this way is rather complicated and we shall not use it here.

The most comfortable way how to find the IR fixed point of the system of four differential RG equations (3.2) with (3.3) is to solve it numerically using some appropriate numerical method. In what follows, we work with the fourth-order Runge-Kutta method with the adaptive choice of the integration step. For this purpose it is convenient to transform the system of differential equations (3.2) into an autonomic system by substitution $t = e^{-s}$. Using this transformation one obtains

$$\begin{cases} \dfrac{d\bar{g}}{ds} = -\beta_g(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) \,, \\[2mm] \dfrac{d\bar{\chi}_i}{ds} = -\beta_{\chi_i}(\bar{g}, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3; \alpha_1, \alpha_2, d) \,, \quad i = 1, 2, 3 \,, \end{cases}$$

where $s \in [0, \infty)$. The initial conditions correspond to $s = 0$ and the IR fixed point is found in the limit $s \to \infty$. The first step for the variable s was taken as $\Delta s = 10^{-3}$. The initial values of the parameters can be chosen arbitrary but the most convenient choice is to take them to be the fixed point of the three-dimensional isotropic model (see [7]).

As was already discussed our differential equations are made of the linear combinations of the following integrals

$$I = \int_{-1}^{1} \frac{(1 - x^2)^{\frac{d-3}{2}} x^{2n}}{(M_1 M_2 M_3)^3} \, dx, \tag{4.2}$$

where explicit forms of the functions $M_i, i = 1, 2, 3$ are given in Appendix I. Therefore, the first step to solve the problem is the necessity to guarantee their convergence within the interval $x \in [-1, 1]$, i.e., to determine the allowed values of the parameters χ_1, χ_2, χ_3. That is why, let us first discuss the conditions under which the integrals will be well defined.

Assume that one or more of the expressions $M_i, (i = 1, 2, 3)$ (defined in Appendix I) are vanished with respect to the variable x within the interval $[-1, 1]$. Let us denote as x_1 the point in which one of the M_i is equal to zero. Then, from the convergence point of view, the integral (4.2) is equivalent to the integral of the type $\int_{-1}^{1} 1 \, (x - x_1)^3 \, dx$. Hence, if one of the expressions $M_i, i = 1, 2, 3$ vanish in the interval $[-1, 1]$ then the integrand will be not integrable. Thus, to guarantee the convergence of the integrals, we are looking for such conditions on variables χ_1, χ_2, χ_3 which give nonzero values for M_i within of the corresponding interval. The necessary and sufficient conditions of the convergence of the integrals (4.2) are as follows

$$\chi_1 > -1, \quad \chi_2 > -1, \quad \chi_3 > -(\sqrt{1 + \chi_1} + \sqrt{1 + \chi_2})^2.$$

The detailed proof of these conditions can be found in Appendix II. They are also important in the numerical solution of our system of differential equations and these conditions must be tested on each step of the Runge-Kutta method.

An important question is related to the choice of a numerical method for calculation of the integrals. It can be shown that the most appropriate method is the using of the Chebyshev quadrature formula. The question of the number of divisions of the integration interval is another important one. In our calculations, we used the division to 1024 subintervals, which was found as the best choice from the point of view of the accuracy and needed time of the calculation. On the other hand, in [7], the division to the 128 subintervals was used. We suppose that this fact could lead to the difference between our and their results because the division to 128 subintervals may not be sufficient in some critical situations.

To find the borderline dimension it is enough to use the bisection method. Our results were calculated with the accuracy of 0.005. The same accuracy was supposed in [7] but, as can be seen from their results, this accuracy was not achieved by them even in the isotropic limit where the exact result exists.

From the numeric calculations point of view, the problem is rather time-consuming, i.e., the calculations take relatively long time. Therefore, the question of using modern computational methods arises. In what follows, we shall analyze a possible speed up of calculations based on the utilization of the parallel programming methods using the Message Passing Interface (MPI) (see, e.g., [10, 20]). Let us discuss this problem in more detail. First of all, we have to solve our system of differential equations at large number of points. For example, Fig. 1 was obtained by using results on the lattice of the size 16×16 in the plane of anisotropy parameters α_1 and α_2, i.e., we had to repeat the numerical procedure 256 times.

Suppose first that we have to calculate the borderline dimension d_c for one concrete value of the parameters α_1 and α_2. How can the parallel algorithms help in this situation? To find the borderline dimension with a precision Δd when the starting interval for d has the length l one has to carry out a pre-scribed number of calculations n. Using one-processor computer the best way how to find the d_c is to use the bisection method. In this case, the result is obtained after $n = \lceil \log_2(l/\Delta d) \rceil$ calculations ($\lceil x \rceil$ means the smallest integer greater or equal to x). On the other hand, in the case of the multi-processor computer with m processors, one can divide the interval into $m + 1$ subinter-vals and carry out the calculations in m points of the division at the same time. Thus, the result is obtained after $n = \lceil \log_{m+1}(l/\Delta d) \rceil$ serial calculations (of course, in this case, the total number of calculations is larger but the total time of the calculations is shorter).

Table 1. The number of needed serial calculations n as a function of the number of processors m for the special case: $l = 1$ and $\Delta d = 0.005$.

m	1	2	$3, 4$	$5 - 13$	$14 - 198$	≥ 199
n	8	5	4	3	2	1

Let us demonstrate this speed-up by an example. Suppose that $l = 1$, and $\Delta d = 0.005$. The results are shown in Tab. 1. The table shows the effective numbers of processors which are $1, 2, 3, 5, 14$, and 199. If we suppose that the calculations take the same time for all values of dimension d then n is directly related to the time of calculation. For example, the calculation with three processors (the same holds also for four processors) is two times shorter then calculation with one processor, see Tab. 1. On the other hand, the calcula-tion takes the same time for the computations with 3 and 4 processors. The same is held for the computers with number of processors from the intervals $m = [5, 13]$, $m = [14, 198]$, and $m = [199, \infty)$. Therefore, our conclusion is the following: if one needs to do only one computational process (in our case, it means to find one borderline dimension d_c for concrete value of the parame-ters of the model) then it is appropriate to use the advantage of the parallel computing.

Now let us analyze the situation when one needs to calculate the borderline dimension d_c as a function of the anisotropy parameters (as it is in our case). Thus, it is necessary to carry out two or more independent calculations for different values of the parameters of the model. The simplest situation occurs when the number of independent calculations are much larger than the number of processors. Because this is our case, we shall analyze it in detail. The situation is shown in Fig. 3, where the total number of computational processes N is shown as a function of the number of processors m and of the desired precision Δd. It is seen immediately that the most effective utilization of the processors is to give to each processor to calculate independent borderline dimension d_c alone.
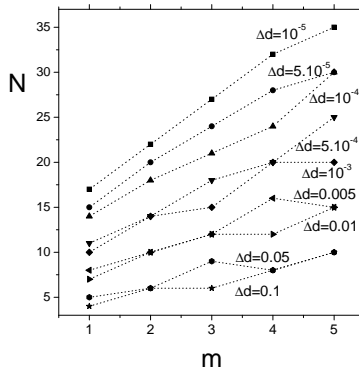
Figure 3. The dependence of the total number of calculational processes as a function of the number of processors m and the precision Δd. The length of the initial interval is $l = 1$.

We have analyzed two special cases, namely, the case with one independent calculation and the case with the large number of independent calculations. The situation becomes more complicated when one needs to carry out the number of calculations which is comparable to the number of processors. But each such situation needs special analysis and we shall not analyze it here.

In concrete calculations we used advantages of the parallel programming. Our situation is the case with a large number of independent calculations, therefore, each processor has calculated borderline dimension for defined values of the parameters of the model.

5. Conclusion

By using the field-theoretic RG method the influence of the uniaxial anisotropy on the stability of the Kolmogorov scaling regime in fully developed turbulence was investigated. The stability of the regime is defined by the very existence of the IR stable fixed point. The fixed point was found numerically by solving the corresponding differential RG equations. It was shown that the earlier results obtained in [7] as well as their conclusions about the dependence of the borderline dimension d_c as a function of the anisotropy parameters $\alpha_{1,2}$ are not precise enough. We have found that the stability of the three-dimensional scaling regime is destroyed only in the case of rather large (in the sense of the absolute value) and special values of the anisotropy parameters. We have also

analyzed the optimal way how to calculate the numerical problem by using the parallel programming methods.

Acknowledgments

Appendix I

The explicit form of the functions a_i, $i = 1, \ldots, 4$ is as follows

$$a_i = \frac{1}{4} \int_{-1}^{1} \frac{(1 - x^2)^{(d-3)/2}}{(M_1 M_2 M_3)^3} \, b_i \, dx, \quad i = 1, \ldots, 4, \tag{5.1}$$

$$\begin{aligned}
b_1 &= c_{1,1} + 4M_2 M_3 x^2 x_1^2 (c_{1,2} + c_{1,3}) \\
&\quad + M_1 x^2 x_1 \Big(c_{1,4} + M_2 M_3^2 r_4 x_1 c_{1,5} + M_2^3 c_{1,6} - 2M_2^2 M_3^2 c_{1,7} \Big), \\
b_2 &= c_{2,1} - M_1^2 M_2^3 r_1 (c_{2,2} + c_{2,3}) - 4M_2 M_3 x_1 (f_3 + x_1) c_{2,4} \\
&\quad - M_1 \Big(-2M_3^3 r_4^2 w_1 x_1^2 (f_3 + x_1) y_4 + M_2^3 (2x_1^2 c_{2,5} - 2f_3 c_{2,6} + d_1 M_3 r_1 r_4^2 x_1 y_7) \\
&\quad + 2M_2^2 M_3^2 (2\alpha_2 M_3 c_{2,7} + c_{2,8}) + M_2 M_3^2 r_4 (c_{2,9} + c_{2,10}) \Big), \\
b_3 &= c_{3,1} + M_1^2 M_2^3 r_1 c_{3,2} + 4M_2 M_3 \big(c_{3,3} + M_2 M_3 r_4 (c_{3,4} + x_1 c_{3,5}) \\
&\quad + M_2^2 (c_{3,6} + x_1 c_{3,7}) \big) + M_1 \big(c_{3,8} - 2M_2^2 M_3^2 (c_{3,9} + 2f_3 c_{3,10} + c_{3,11}) \\
&\quad + M_2 M_3^2 r_4 (-x_1 c_{3,12} - 4f_3 c_{3,13}) + M_2^3 (-d_1 c_{3,14} + 2f_3 c_{3,15} - 2x_1^2 c_{3,16}) \big), \\
\frac{b_4}{2} &= c_{4,1} - 2c_{4,2} M_1^2 M_2^3 r_1 + 2M_2 M_3 \big(-c_{4,7} + (c_{4,5} - c_{4,6} d_1) M_2^2 \\
&\quad + (c_{4,3} - c_{4,4} d_1) M_2 M_3 r_4 \big) + M_1 \big((d_1 (c_{4,10} + c_{4,11} f_1) + c_{4,9} f_2 + 3c_{4,8} f_3) M_2^3 \\
&\quad + 2(d_1 (c_{4,20} + 2c_{4,19} f_1) + c_{4,18} f_2 + 3c_{4,17} f_3) M_2^2 M_3^2 \\
&\quad + (d_1 (c_{4,15} + 2c_{4,16} f_1) + c_{4,14} f_2 + 3c_{4,13} f_3) M_2 M_3^2 r_4 - c_{4,12} M_3^3 r_4^2 x_1 y_4 \big), \\
c_{1,1} &= 4M_1^2 M_2^3 r_1 x^2 x_1 (-d_1 M_3 r_4 + (-\chi_2 M_3 + r_4 (1 + 3M_3)) x_1) \\
&\quad + M_1^3 M_2^3 r_1 (d_1 M_3 (d_{-1} - 3x_1) + 2(1 + 2M_3) x_1^2), \\
c_{1,2} &= -M_3^2 r_4^2 x_1 (r_1 + \alpha_2 x_1) + M_2^2 (-4\alpha_2 M_3^2 + r_1 r_4 (2M_3 - r_4 (1 + w_1) x_1)), \\
c_{1,3} &= M_2 M_3 r_4 \big(4\alpha_2 M_3 x_1 + r_1 (2M_3 - r_4 (2 + w_1) x_1) \big), \\
c_{1,4} &= -2M_3^3 r_4^2 w_1 x_1^2 (r_1 + \alpha_2 x_1),
\end{aligned}$$

$$c_{1,5} = \alpha_2 M_3 x_1 (4(w_1 + h_2 x_1) + r_4 y_1) + r_1 (4\chi_2 M_3 x_1 + r_4(-4w_1 x_1 + M_3 y_1)),$$

$$c_{1,6} = d_1 M_3 (4\alpha_2 M_3^2 + r_1 r_4(-2M_3 + r_4 x_1))$$
$$- 2x_1 (8\alpha_2 M_3^3 + r_1 r_4(-4M_3^2 + r_4 x_1 - 2M_3(1 + w_1 - y_2))),$$

$$c_{1,7} = d_1 r_4 (2\alpha_2 M_3 x_1 + r_1 (M_3 - r_4 x_1))$$
$$- 4x_1 (r_1 r_4 (M_3 - y_2) + \alpha_2 M_3 (r_4 x_1 + y_2)),$$

$$c_{2,1} = 2M_1^3 M_2^3 r_1 (f_3 - d_1 f_1 M_3 + 2f_3 M_3),$$

$$c_{2,2} = -4f_3 M_3 r_4 (-2 + 3x^2) + d_1 M_3 r_4 (d_{-1} - 4f_1 + 3f_1 x^2 - 5x_1),$$

$$c_{2,3} = 4f_3 (-\chi_2 M_3 + r_4) x_1 + 4(-\chi_2 M_3 + r_4 + 2M_3 r_4) x_1^2,$$

$$c_{2,4} = -4\alpha_2 M_2 M_3^2 (M_2 - r_4 x_1)$$
$$- r_4 (M_2 r_1 (-2M_1 M_3 + r_4 (M_1 + M_3 + M_1 w_1) x_1) + M_3^2 r_4 x_1 y_4),$$

$$c_{2,5} = -4\alpha_2 M_3^3 + r_1 r_4 (-r_4 (M_3 x_4 + x_1) + 2M_3 (1 + M_3 + w_1 + \chi_2 x_1)),$$

$$c_{2,6} = 4\alpha_2 M_3^3 x_2 + r_1 r_4 (r_4 (x_1^2 + M_3 y_3) - 2M_3 (M_3 x_2 + x_1 (1 + y_5))),$$

$$c_{2,7} = x_1^2 (-2\chi_2 x_1 + r_4 (x_4 + x_1))$$
$$+ f_3 (-2\chi_2 x_1^2 + r_4 (1 + 4x^4 + x_1 - 2x^2 (3 + x_1))),$$

$$c_{2,8} = 2f_3 r_1 r_4 (M_3 x_2 + 2\chi_2 x_1^2 - r_4 y_3)$$
$$+ r_4 x_1 (2r_1 x_1 (M_3 - r_4 x_4 + 2\chi_2 x_1) + d_1 y_6 y_7),$$

$$c_{2,9} = -2r_1 (-2\chi_2 M_3 x_1^3 + r_4 (x_1^2 (M_3 x_4 + 2w_1 (f_3 + x_1)) + f_3 M_3 y_3)),$$

$$c_{2,10} = M_3 x_1 (\alpha_2 (4f_3 w_1 x_1 - 2r_4 x_4 x_1^2 - 2f_3 r_4 y_3 + 4x_1^2 y_5)$$
$$+ y_4 (4f_3 \chi_2 x_1 + d_1 r_4 y_7)),$$

$$c_{3,1} = M_1^3 M_2^3 r_1 (-d_1 f_1 M_3 + 2f_3 (1 + \chi_1 + 2M_3) + 2\chi_1 x_1^2),$$

$$c_{3,2} = -4(-\chi_2 M_3 + r_4 + 3M_3 r_4 + \chi_3 M_3 x^2 - \chi_1 r_4 x^2) x_1^2$$
$$- d_1 (d_{-1} M_3 r_4 + f_1 M_3 r_4 (-2 + x^2) + (-5M_3 r_4 - 4\chi_3 M_3 x^2$$
$$+ 4\chi_1 r_4 x^2) x_1) + 4f_3 (M_3 (-\chi_3 x^2 + \chi_2 x_1) + r_4 (-3M_3 x_1 + y_8)),$$

$$c_{3,3} = -M_3^2 r_4^2 x_1 y_4 (-d_1 \chi_1 x^2 x_1 + f_3 y_8 + x_1^2 y_9),$$

$$c_{3,4} = f_3 (4\alpha_2 M_3 x_1 y_8 + r_1 (-r_4 x_1 (-2 - w_1 + x^2 (2 + 2\chi_1 + w_1 + w_2)) + 2M_3 y_8)),$$

$$c_{3,5} = -d_1 x^2 (2\chi_1 (2\alpha_2 M_3 x_1 + r_1 (M_3 - r_4 x_1)) - r_1 r_4 x_1 w_2)$$
$$+ x_1 (4\alpha_2 M_3 x_1 y_9 + r_1 (r_4 x_1 (w_1 - x^2 w_2 - 2y_9) + 2M_3 y_9)),$$

$$c_{3,6} = -f_3 (4\alpha_2 M_3^2 y_8 + r_1 r_4 (r_4 x_1 (-1 - w_1 + x^2 (1 + \chi_1 + w_1 + w_2)) - 2M_3 y_8)),$$

$$c_{3,7} = d_1 x^2 (\chi_1 (4\alpha_2 M_3^2 + r_1 r_4 (-2M_3 + r_4 x_1)) + r_1 r_4^2 x_1 w_2)$$
$$+ x_1 (-4\alpha_2 M_3^2 y_9 + r_1 r_4 (r_4 x_1 (w_1 - x^2 w_2 - y_9) + 2M_3 y_9)),$$

$$c_{3,8} = -2M_3^3 r_4^2 x_1 (f_3 (-w_1 x_1 + x^2 w_2) - x_1 (w_1 x_1 + x^2 (d + x^2) w_2)) y_4,$$

$$c_{3,9} = 2x_1^2 \big(r_1 r_4 (3M_3 - 2r_4 x_3 + 2\chi_2 x_1 - 2\chi_3 x^2 x_1) $$
$$+ \alpha_2 M_3 (-2(\chi_2 - \chi_3 x^2) x_1 + r_4 (2x_3 + 3x_1))) \big) ,$$

$$c_{3,10} = \alpha_2 M_3 (2x_1 (\chi_3 x^2 - \chi_2 x_1) + r_4 (2 + 4x^4 + 3x_1 - 2x^2 (4 + x_1)))$$
$$+ r_1 r_4 (3M_3 - 2M_3 x^2 - 2\chi_3 x^2 x_1 + 2\chi_2 x_1^2 - 2r_4 y_{10}) ,$$

$$c_{3,11} = d_1 \big(x_1 (\alpha_2 M_3 (r_4 (-x_5 - 4x_1) - 4\chi_3 x^2 x_1)$$
$$+ r_1 r_4 (-4M_3 + r_4 x_5 + 4\chi_3 x^2 x_1)) + d_{-1} M_3 r_4 y_4 + f_1 M_3 r_4 (-2 + x^2) y_4 \big) ,$$

$$c_{3,12} = d_1 \big(\alpha_2 M_3 x_1 (r_4 x_5 + 4x^2 (\chi_3 x_1 + w_2)) + r_1 (4\chi_3 M_3 x^2 x_1$$
$$+ r_4 (M_3 x_5 - 4x^2 x_1 w_2))) - 4x_1 \big(\alpha_2 M_3 x_1 (-w_1 + r_4 x_3 - \chi_2 x_1 + \chi_3 x^2 x_1$$
$$+ x^2 w_2) + r_1 (M_3 (-\chi_2 + \chi_3 x^2) x_1 + r_4 (M_3 x_3 + w_1 x_1 - x^2 x_1 w_2)) \big) ,$$

$$c_{3,13} = \alpha_2 M_3 x_1 (w_1 x_1 - r_4 y_{10} - x^2 (\chi_3 x_1 + w_2))$$
$$- r_1 (\chi_3 M_3 x^2 x_1 + r_4 (w_1 x_1^2 + M_3 y_{10} - x^2 x_1 w_2)) + \chi_2 M_3 x_1^2 y_4 ,$$

$$c_{3,14} = 2d_{-1} M_3^2 (-2\alpha_2 M_3 + r_1 r_4) + 2f_1 M_3^2 (-2\alpha_2 M_3 + r_1 r_4)(-2 + x^2)$$
$$+ x_1 \big(16\alpha_2 M_3^3 + r_1 r_4 (r_4 (M_3 x_5 - 2\chi_1 x^2 x_1)$$
$$+ 4M_3 (-2M_3 + x^2 (\chi_1 + \chi_3 x_1 + w_2)))) ,$$

$$c_{3,15} = 4\alpha_2 M_3^3 (3 - 2x^2) + r_1 r_4 \big(M_3^2 (-6 + 4x^2)$$
$$+ 2M_3 (\chi_1 x^2 - x_1 + \chi_3 x^2 x_1 - w_1 x_1 - \chi_2 x_1^2 + r_4 y_{10} + x^2 w_2) - r_4 x_1 y_8),$$

$$c_{3,16} = -12\alpha_2 M_3^3 + r_1 r_4 \big(2M_3 (3M_3 + w_1 + \chi_2 x_1 - \chi_3 x^2 x_1 - x^2 w_2 - y_9)$$
$$- r_4 (2M_3 x_3 - x_1 y_9) \big) ,$$

$$c_{4,1} = M_1^3 M_2^3 (3f_3 \chi_1 + f_2 (1 + \chi_1 + 2M_3)) r_1 ,$$

$$c_{4,2} = -3f_3 (2\chi_2 M_3 - \chi_3 M_3 x^2 + r_4 (-5M_3 + x_8))$$
$$+ f_2 (M_3 (\chi_3 x^2 + \chi_2 x_7) - r_4 (M_3 (-5 + 3x^2) + x_9))$$
$$+ d_1 (f_1 (M_3 (\chi_2 + \chi_3 - 2\chi_3 x^2) + r_4 (-2M_3 + x_{10})) + \chi_3 M_3 x_1 - \chi_1 r_4 x_1) ,$$

$$c_{4,3} = 3f_3 (4\alpha_2 M_3 x_8 x_1 + r_1 (2M_3 x_8 + r_4 x_1 (2w_1 - 2x_8 - x^2 w_2)))$$
$$+ f_2 (4\alpha_2 M_3 x_9 x_1 + r_1 (2M_3 x_9 - r_4 x_1 (-2(2 + w_1)$$
$$+ x^2 (2 + 2\chi_1 + w_1 + w_2)))) ,$$

$$c_{4,4} = x_1 (-2\chi_1 (M_3 r_1 + 2\alpha_2 M_3 x_1 - r_1 r_4 x_1) + r_1 r_4 x_1 w_2) + f_1 (4\alpha_2 M_3 x_{10} x_1$$
$$+ r_1 (2M_3 x_{10} + r_4 x_1 (2 + 2\chi_1 - 4\chi_1 x^2 + w_1 + x_2 w_2))) ,$$

$$c_{4,5} = 3f_3 (-4\alpha_2 M_3^2 x_8 + r_1 r_4 (2M_3 x_8 - r_4 x_1 (-2w_1 + x_8 + x^2 w_2)))$$
$$+ f_2 (-4\alpha_2 M_3^2 x_9 + r_1 r_4 (2M_3 x_9 - r_4 x_1 (-2(1 + w_1)$$
$$+ x^2 (1 + \chi_1 + w_1 + w_2)))) ,$$

$$c_{4,6} = x_1 (\chi_1 (4\alpha_2 M_3^2 + r_1 r_4 (-2M_3 + r_4 x_1)) + r_1 r_4^2 x_1 w_2)$$
$$+ f_1 (-4\alpha_2 M_3^2 x_{10} + r_1 r_4 (2M_3 x_{10} + r_4 x_1 (-x_{10} + w_1 + x_2 w_2))) ,$$

$$c_{4,7} = M_3^2 r_4^2 x_1(-d1 f_1 x_{10} + 3 f_3 x_8 + f_2 x_9 + d1\chi_1 x_1)y_4\,,$$

$$c_{4,8} = 16\alpha_2 M_3^3 + r_1 r_4(r_4(M_3 x_6 - x_8 x_1)$$
$$+\, 2M_3(-4M_3 - 2w_1 + x_8 - 2\chi_2 x_1 + \chi_3 x^2 x_1 + x^2 w_2))\,,$$

$$c_{4,9} = -8\alpha_2 M_3^3 x_7 + r_1 r_4\big(r_4(-x_9 x_1 + M_3 y_{13})$$
$$+\, 2M_3(\chi_1 x^2 + (1 + 2M_3 + w_1)x_7 + \chi_3 x^2 x_1 - \chi_2 y_{11} + x^2 w_2)\big)\,,$$

$$c_{4,10} = r_1 r_4(d_{-1} M_3 r_4 + x_1(2\chi_1 M_3 - 5M_3 r_4 + 2\chi_3 M_3 x_1 - \chi_1 r_4 x_1 + 2M_3 w_2))\,,$$

$$c_{4,11} = -8\alpha_2 M_3^3 + r_1 r_4\big(4M_3^2 - r_4(x_1 + \chi_1 y_{12})$$
$$+\, 2M_3(2\chi_3 x^4 - x_{10} + r_4 x_{11} + w_1 + \chi_3 x_3 + \chi_2 x_1 + x_2 w_2)\big)\,,$$

$$c_{4,12} = -6 f_3 w_1 + f_2 w_1 x_7 + f_2 x^2 w_2 + 3 f_3 x^2 w_2 + d1(x_1 w_2 + f_1(w_1 + x_2 w_2))\,,$$

$$c_{4,13} = \alpha_2 M_3 x_1(-4w_1 + r_4 x_6 + 2x^2(\chi_3 x_1 + w_2))$$
$$+\, r_1(2\chi_3 M_3 x^2 x_1 + r_4(M_3 x_6 + 4w_1 x_1 - 2x^2 x_1 w_2)) - 4\chi_2 M_3 x_1 y_4\,,$$

$$c_{4,14} = \alpha_2 M_3(-2w_1 y_{11} + r_4 x_1 y_{13} + 2x^2 x_1(\chi_3 x_1 + w_2))$$
$$+\, r_1(2\chi_3 M_3 x^2 x_1 + r_4(M_3 y_{13} - 2x_1(w_1 x_7 + x^2 w_2))) - 2\chi_2 M_3 y_{11} y_4\,,$$

$$c_{4,15} = d_{-1} M_3 r_4 y_4 - x_1(r_1 r_4(5M_3 + 2x_1 w_2) + M_3 x_1(5\alpha_2 r_4 - 2\alpha_2 w_2 - 2\chi_3 y_4))\,,$$

$$c_{4,16} = r_1 r_4(M_3 x_{11} - x_1(w_1 + x_2 w_2))$$
$$+\, M_3(\alpha_2 r_4 x_{11} x_1 + \alpha_2 w_1 x_1 + \alpha_2 y_{12} w_2 + \chi_2 x_1 y_4 + \chi_3 y_{12} y_4)\,,$$

$$c_{4,17} = r_1 r_4(-4M_3 + r_4 x_6 - 4\chi_2 x_1 + 2\chi_3 x^2 x_1)$$
$$-\, \alpha_2 M_3(r_4 x_6 - 4\chi_2 x_1 + 4r_4 x_1 + 2\chi_3 x^2 x_1)\,,$$

$$c_{4,18} = r_1 r_4(2M_3 x_7 + 2\chi_3 x^2 x_1 - 2\chi_2 y_{11} + r_4 y_{13})$$
$$-\, \alpha_2 M_3(2\chi_3 x^2 x_1 + 2r_4 x_2 x_1 - 2\chi_2 y_{11} + r_4 y_{13})\,,$$

$$c_{4,19} = r_1 r_4(M_3 + r_4 x_{11} + \chi_2 x_1 + \chi_3 y_{12})$$
$$-\, \alpha_2 M_3(r_4 x_{11} + \chi_2 x_1 - r_4 x_1 + \chi_3 y_{12})\,,$$

$$c_{4,20} = (d_{-1} r_4 + x_1(-5r_4 + 2\chi_3 x_1))y_6\,,$$

$$M_1 = 2(1 + \chi_1 x^2) + (\chi_2 + x^2\chi_3)(1 - x^2)\,,$$
$$M_2 = 1 + \chi_1 x^2 + (\chi_2 + x^2\chi_3)(1 - x^2)\,,\quad M_3 = 1 + \chi_1 x^2\,,$$
$$r_1 = 1 + \alpha_1 x^2\,,\quad r_4 = \chi_2 + \chi_3 x^2\,,\quad f_1 = x^2 d - 1\,,$$
$$f_2 = -(d + 2)x^4 + (d + 3)x^2 - 1\,,\quad f_3 = (d + 4)(d + 2)x^4 - 6(d + 2)x^2 + 3\,,$$
$$w_1 = 1 + \chi_2 + \chi_3 x^4\,,\quad w_2 = \chi_1 - \chi_2 + \chi_3(1 - 2x^2)\,,$$
$$y_1 = -3 + d + 8x^2\,,\quad y_2 = r_4(1 - 2x^2) - \chi_2(1 - x^2)\,,\quad y_3 = 1 - 6x^2 + 4x^4\,,$$
$$y_4 = r_1 + \alpha_2 x_1\,,\ y_5 = w_1 + \chi_2 x_1\,,\ y_6 = r_1 r_4 - \alpha_2 M_3\,,\ y_7 = 1 - 3x^2 + f_1 x_1\,,$$
$$y_8 = -1 + (1 + \chi_1)x^2\,,\quad y_9 = -1 + \chi_1 x^2\,,\quad y_{10} = 1 - 4x^2 + 2x^4\,,$$

$$y_{11} = 2 - 3x^2 + x^4, \quad y_{12} = 1 - 3x^2 + 2x^4, \quad y_{13} = 3 - 12x^2 + 4x^4,$$

$$d_1 = d + 1, \quad d_{-1} = d - 1,$$

$$x_1 = 1 - x^2, \quad x_2 = 1 - 2x^2, \quad x_3 = 1 - 3x^2, \quad x_4 = 1 - 4x^2,$$

$$x_5 = 1 - 5x^2, \quad x_6 = 3 - 10x^2, \quad x_7 = -2 + x^2, \quad x_8 = \chi_1 x^2 - 2,$$

$$x_9 = (1 + \chi_1)x^2 - 2, \quad x_{10} = \chi_1(2x^2 - 1) - 1, \quad x_{11} = -3 + 5x^2.$$

Appendix II

In this Appendix we shall prove the necessary and sufficient conditions needed for the convergence of integrals (4.2).

Theorem 1. *If the expressions $M_i, i = 1, 2, 3$ (see Appendix I) are nonzero at each point $x \in [-1, 1]$ then they are positive on the whole interval.*

Proof. Expressions $M_i, i = 1, 2, 3$ are continuous functions with respect to x within the interval $x \in [-1, 1]$. On the other hand, $M_3(0) = 1 > 0$. If one supposes that there exists $y \in [-1, 1]$ such that $M_3(y) < 0$ then according to the property of continuity there must exist a point at which the function is equal to zero. But it is in a contradiction with the assumption of the theorem. Thus, $M_3(x) > 0$ for all $x \in [-1, 1]$. Because $M_2(1) = 1 + \chi_1 = M_3(1) > 0$, then using the same arguments we come to the same conclusion, namely, $M_2(x) > 0$ for all $x \in [-1, 1]$. Finally, because $M_1 = M_2 + M_3$, then $M_1(x)$ is also positive for all $x \in [-1, 1]$. Thus, the theorem is proved. ∎

Now we are able to prove the necessary and sufficient conditions of the convergence which are the contents of the following theorem.

Theorem 2 [Necessary and sufficient conditions of the convergence of integrals (4.2)]. *Expressions $M_i, (i = 1, 2, 3)$ are nonzero for each $x \in [-1, 1]$ if and only if the following conditions are satisfied:*

i) $\chi_1 \in (-1, \infty)$;
ii) $\chi_2 \in (-1, \infty)$;
iii) $\chi_3 \in \left(-\left(\sqrt{1 + \chi_1} + \sqrt{1 + \chi_2}\right)^2, \infty \right)$.

Proof. First we prove that if expressions M_i are nonzero for each $x \in [-1, 1]$ then the conditions *i)*, *ii)*, and *iii)* are satisfied. Suppose that the implication is not true, i.e., expressions M_i are nonzero and at the same time some of the conditions *i)*, *ii)* and *iii)* are not satisfied. We shall show in items *a)*, *b)*, and *c)* that when the conditions for parameters χ_1, χ_2, χ_3 are not satisfied then one comes to a contradiction.

a) Let us suppose that $\chi_1 \leq -1$. It is enough to take $\hat{x} = \sqrt{-1/\chi_1}$, and it is evident that $0 < \sqrt{-1/\chi_1} \leq 1$, therefore $\hat{x} \in (0, 1] \subseteq [-1, 1]$. But

$$M_3(\hat{x}) = M_3\left(\sqrt{-\frac{1}{\chi_1}}\right) = 1 + \chi_1\left(\sqrt{-\frac{1}{\chi_1}}\right)^2 = 0,$$

which is in contradiction with the assumption of the theorem.

b) Suppose that $\chi_2 \leq -1$, then $M_2(0) = 1 + \chi_2 \leq 0$, and according to the Theorem 1 it is in contradiction with the assumption of the theorem.

c) In the end, suppose that $\chi_3 \leq -\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2$. Therefore,

$$\chi_3 + \left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2 \leq 0.$$

At the same time, as was already proved in items *a*) and *b*), $\chi_1 > -1$ and $\chi_2 > -1$. Take

$$0 < \hat{x} = \sqrt{\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1} + \sqrt{1+\chi_2}}} < 1, \ \hat{x} \in (0,1) \subseteq [-1,1].$$

Then

$$M_2(\hat{x}) = \left(1 + \chi_1\hat{x}^2\right) + \left(\chi_2 + \chi_3\hat{x}^2\right)\left(1 - \hat{x}^2\right) = 1 + \chi_1\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1} + \sqrt{1+\chi_2}}$$

$$+ \left(\chi_2 + \chi_3\frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1} + \sqrt{1+\chi_2}}\right)\left(1 - \frac{\sqrt{1+\chi_2}}{\sqrt{1+\chi_1} + \sqrt{1+\chi_2}}\right),$$

and after some manipulations we have

$$M_2(x) = \frac{\sqrt{1+\chi_1}\sqrt{1+\chi_2}\left[\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2 + \chi_3\right]}{\left(\sqrt{1+\chi_1} + \sqrt{1+\chi_2}\right)^2} \leq 0.$$

Thus according to Theorem 1 it is again in contradiction with the assumptions of the theorem.

Now we have to prove the second part of the theorem, namely: if the conditions *i*), *ii*), and *iii*) are satisfied then expressions M_i, $i = 1, 2, 3$ are nonzero for each $x \in [-1, 1]$.

a) Suppose that the conditions *i*)–*iii*) are satisfied and, at the same time, there exists a point $\tilde{x}$ from the interval $[-1, 1]$ such that

$$M_3(\tilde{x}, \chi_1) = 1 + \chi_1\tilde{x}^2 = 0.$$

Because $M_3(0, \chi_1) = 1$ then $\tilde{x} \neq 0$ and

$$M_3(\tilde{x}, \chi_1) = 1 + \chi_1\tilde{x}^2 = 0 \overset{(\tilde{x} \neq 0)}{\Rightarrow} \chi_1 = -\frac{1}{\tilde{x}^2} \Rightarrow \chi_1 \leq -1,$$

which is in contradiction with assumption *i*). Thus, $M_3(x) > 0$ for all $x \in [-1, 1]$.

b) As in the previous case, suppose that conditions $i)$–$iii)$ are satisfied and, at the same time, suppose the existence of $\tilde{x} \in [-1, 1]$ such that $M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) = 0$. But

$$M_2(0, \chi_1, \chi_2, \chi_3) = 1 + \chi_2 > 0, \quad M_2(\pm 1, \chi_1, \chi_2, \chi_3) = 1 + \chi_1 > 0.$$

Thus, if $M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) = 0$ then $\tilde{x} \neq 0 \wedge \tilde{x} \neq \pm 1$. As a result $\tilde{x}^2 \in (0, 1)$. Then we have

$$M_2(\tilde{x}, \chi_1, \chi_2, \chi_3) = 1 + \chi_1 \tilde{x}^2 + \left(\chi_2 + \chi_3 \tilde{x}^2\right)\left(1 - \tilde{x}^2\right)$$
$$= \tilde{x}^2\left(1 - \tilde{x}^2\right)\left[\frac{1 + \chi_1}{1 - \tilde{x}^2} + \frac{1 + \chi_2}{\tilde{x}^2} + \chi_3\right] = 0.$$

Because $\tilde{x} \neq 0 \wedge \tilde{x} \neq \pm 1$ then

$$\frac{1 + \chi_1}{1 - \tilde{x}^2} + \frac{1 + \chi_2}{\tilde{x}^2} + \chi_3 = 0,$$

which is equivalent to

$$\chi_3 = -\frac{1 + \chi_1}{1 - \tilde{x}^2} - \frac{1 + \chi_2}{\tilde{x}^2}.$$

Further, the maximum of function $f(t) = -\dfrac{1 + \chi_1}{1 - t} - \dfrac{1 + \chi_2}{t}$ within the interval $t \in (0, 1)$ is obtained at the point $t^\star = \dfrac{\sqrt{1 + \chi_2}}{\sqrt{1 + \chi_1} + \sqrt{1 + \chi_2}}$ and its value is

$$f(t^\star) = -\left(\sqrt{1 + \chi_1} + \sqrt{1 + \chi_2}\right)^2.$$

Therefore $\chi_3 \leq -\left(\sqrt{1 + \chi_1} + \sqrt{1 + \chi_2}\right)^2$, which is in contradiction with assumption $iii)$. As a result $M_2(x) > 0$ for every x from the interval $[-1, 1]$.

c) It is evident that

$$M_1(x, \chi_1, \chi_2, \chi_3) = M_2(x, \chi_1, \chi_2, \chi_3) + M_3(x, \chi_1)$$

and according to $a)$ and $b)$ one obtains

$$M_1(x, \chi_1, \chi_2, \chi_3) = M_2(x, \chi_1, \chi_2, \chi_3) + M_3(x, \chi_1) > 0$$

for all $x \in [-1, 1]$. Thus, the theorem is proved.

∎

References

[1] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil'ev. The problem of infrared divergences and renormalization group in theory of developed turbulence. *Zh. Eks. Teor. Fiz.*, **95**(4), 1272–1288, 1989.

[2] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasiliev. *The Field Theoretic Renormalization Group in Fully Developed Turbulence.* Gordon & Breach, London, 1999.

[3] L. Ts. Adzhemyan, M. Hnatich, D. Horvath and M. Stehlik. Infrared properties of an anisotropically driven MHD turbulence. *Int. J. Mod. Phys. B*, **9**(26), 3401–3420, 1995.

[4] L. Ts. Adzhemyan, A. N. Vasil'ev and Yu. M. Pis'mak. Renormalization-group approach in the theory of turbulence: the dimensions of composite operators. *Theor. Math. Phys.*, **57**(1), 1131–1141, 1983.

[5] R. Bausch, H. K. Janssen and H. Wagner. Renormalization field theory of critical dynamics. *Z. Physik B*, **24**(1), 113–127, 1976.

[6] J. Busa, M. Hnatic, E. Jurcisinova, M. Jurcisin and M. Stehlik. Influence of anisotropy on the scaling regimes in fully developed turbulence. *Acta Phys. Slov.*, **52**(6), 547–552, 2002.

[7] J. Busa, M. Hnatich, J. Honkonen and D. Horvath. Stability of kolmogorov scaling in anisotropically forced turbulence. *Phys. Rev. E*, **55**(1), 381–394, 1997.

[8] C. DeDominicis and P. C. Martin. Energy spectra of certain randomly-stirred fluid. *Phys. Rev. A*, **19**(1), 419–422, 1979.

[9] I. S. Gradshtejn and I. M. Ryzhik. *Tables of Integrals, Series and Products.* Academic, New York, 1965.

[10] E. A. Hayryan, J. Buša, E. E. Donets, I. Pokorný and O. I. Streltsova. Numerical studies of perturbed static solutions decay in the coupled system of Yang-Mills-Dilaton equations with use of MPI technology. *Journal of Mathematical Modeling (in Russian)*, **17**(6), 103–121, 2005.

[11] E. A. Hayryan, E. Jurcisinova, M. Jurcisin, I. Pokorny and M. Stehlik. Stability of kolmogorov scaling in the theory of anisotropically driven developed turbulence. *Communication of JINR E17-2005-208*, 2005.

[12] M. Hnatic, E. Jonyova, M. Jurcisin and M. Stehlik. Stability of scaling regimes in $d \geq 2$ developed turbulence with weak anisotropy. *Phys. Rev. E*, **64**(1), 016312, 2001.

[13] T. L. Kim and A. V. Serd'ukov. Quantum-field renormalization group in theory of developed turbulence: Inclusion of anisotropy and passive admixtur. *Teor. Mat. Fiz.*, **105**(2), 412–422, 1995.

[14] A. N. Kolmogorov. The local structure of turbulence in an incompressible fluid with very large reynolds numbers. *Dokl. Akad. Nauk SSSR*, **30**(2), 301–305, 1941.

[15] P. C. Martin, E. D. Siggia and H. A. Rose. Statistical dynamics of classical systems. *Phys. Rev. A*, **8**(1), 423–437, 1973.

[16] A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics.* MIT Press, Cambridge, MA, 1975. Vol. 2

[17] A. M. Obukhov. On the distribution of energy in the spectrum of turbulent flow. *Dokl. Akad. Nauk SSSR*, **32**(2), 22–24, 1941.

[18] R. Rubinstein and J. M. Barton. Infrared properties of an anisotropically stirred fluid. *Phys. Fluids.*, **30**(10), 2987–2992, 1987.

[19] A. N. Vasil'ev. *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics.* Sankt Peterburg, 1998. in Russian, [English translation: Chapman & Hall/CRC, Boca Roaton, FL, 2004]

[20] V. V. Voevodin and Vl. V. Voevodin. *Parallel Computations.* St.Peterburg, 2002. (in Russian)

[21] H. W. Wyld. Formulation of the theory of turbulence in an incompressible fluid. *Ann. Phys.*, **14**(1), 141–165, 1961.