Mathematical Modelling and Analysis
Volume 12 Number 3, 2007, pages 343–356
 ISSN 1392-6292 print, ISSN 1648-3510 online

NONLINEAR DIFFERENTIAL EQUATIONS WITH MARCHAUD-HADAMARD-TYPE FRACTIONAL DERIVATIVE IN THE WEIGHTED SPACE OF SUMMABLE FUNCTIONS

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Received November 25, 2006; revised April 22, 2007; published online September 15, 2007

Abstract. The paper is devoted to the study of the Cauchy-type problem for the nonlinear differential equation of fractional order $0<\alpha<1$:

$$(\boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y)(x)=f[x,y(x)] \qquad (0<\alpha<1),$$
$$(x^{\mu}\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(0+)=b,\ b\in\mathbb{R},$$

containing the Marchaud-Hadamard-type fractional derivative $(\boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y)(x)$, on the half-axis $\mathbb{R}_+=(0,+\infty)$ in the space $X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)$ defined for $\alpha>0$ by

$$X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)=\left\{y\in X^p_c(\mathbb{R}_+):\ \boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y\in X^p_{c,\,0}(\mathbb{R}_+)\right\},$$

where $X^p_{c,\,0}(\mathbb{R}_+)$ is the subspace of $X^p_c(\mathbb{R}_+)$ of functions $g\in X^p_c(\mathbb{R}_+)$ with compact support on infinity: $g(x)\equiv 0$ for large enough $x>R$. The equivalence of this problem and of the nonlinear Volterra integral equation is established. The existence and uniqueness of the solution $y(x)$ of the above Cauchy-type problem is proved by using the Banach fixed point theorem. Solution in closed form of the above problem for the linear differential equation with $f[x,y(x)]=\lambda y(x)+f(x)$ is constructed. The corresponding assertions for the differential equations with the Marchaud-Hadamard fractional derivative $(\boldsymbol{D}^{\alpha}_{0+}\, y)(x)$ are presented. Examples are given.
Key words: Keywords: differential equation of fractional order, Hadamard-type fractional derivative, existence and uniqueness theorem, Mittag-Leffler function

1. Introduction

Let $\mathcal{D}^{\alpha}_{0+,\,\mu}$ be the Hadamard-type fractional derivative of order $\alpha>0$ on the real half-axis $\mathbb{R}_+=(0,+\infty)$ defined for $x>0$ and $\mu\in\mathbb{R}$ by:

$$(\mathcal{D}^{\alpha}_{0+,\,\mu}\, f)(x) = x^{-\mu}\delta^n x^{\mu}\, (\mathcal{J}^{n-\alpha}_{0+,\,\mu}\, f)(x), \quad \delta = x\frac{d}{dx},\ (n = [\alpha]+1), \qquad (1.1)$$

where $\mathcal{J}^{\alpha}_{0+,\,\mu}$ is the Hadamard-type fractional integral

$$(\mathcal{J}^{\alpha}_{0+,\,\mu}\, f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t} \quad (\alpha > 0,\ x > 0). \qquad (1.2)$$

When $0 < \alpha < 1$ the fractional operator (1.1) takes the form

$$(\mathcal{D}^{\alpha}_{0+,\,\mu}\, f)(x) = x^{1-\mu}\frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_0^x \left(\ln\frac{x}{t}\right)^{-\alpha} f(t)\frac{dt}{t^{1-\mu}} \quad (0 < \alpha < 1). \quad (1.3)$$

Hadamard-type fractional derivatives and integrals (1.1) and (1.2) were introduced in [2]. There was proved the boundedness property for the fractional integrals $\mathcal{J}^{\alpha}_{0+,\,\mu} f$ in the space $X^p_c \equiv X^p_c\,(\mathbb{R}_+)$ $(c \in \mathbb{R},\ 1 \leqslant p \leqslant \infty)$ of Lebegue measurable functions h on $\mathbb{R}_+$ for which $\|h\|_{X^p_c} < \infty$, where

$$\|h\|_{X^p_c} = \left(\int_0^{\infty} |t^c\, h(t)|^p\, \frac{dt}{t}\right)^{1/p} \ (1 \leqslant p < \infty),$$
$$\|h\|_{X^{\infty}_c} = \operatorname*{ess\,sup}_{t>0}[t^c\, |h(t)|].$$

The semigroup property of the Hadamard-type fractional integral (1.2) was proved in [3], Mellin transforms were studied and the formula of fractional integration by parts was established in [4].

Hadamard-type fractional derivative (1.3) was investigated in [7], where its representation was constructed in the form:

$$(\boldsymbol{D}^{\alpha}_{0+,\,\mu} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty} e^{-\mu t}\frac{f(x)-f(xe^{-t})}{t^{1+\alpha}}dt + \mu^{\alpha} f(x)$$
$$= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\ln\frac{x}{t}\right)^{-\alpha-1}[f(x)-f(t)]\,\frac{dt}{t} + \mu^{\alpha} f(x)\ \ (0 < \alpha < 1).$$

Such a form, called Marchaud-Hadamard-type fractional derivative, is more natural on the half-axis $\mathbb{R}_+$ than Hadamard-type fractional derivative $\mathcal{D}^{\alpha}_{0+,\,\mu}$. In particular,

$$\boldsymbol{D}^{\alpha}_{0+,\,\mu} f \equiv \mathcal{D}^{\alpha}_{0+,\,\mu} f, \quad f \in X^p_c. \qquad (1.4)$$

We shall understand Marchaud-Hadamard-type fractional derivative as the convergent integral in the following sense. Let

$$(\boldsymbol{D}^{\alpha}_{0+,\,\mu;\,\varepsilon} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_{\varepsilon}^{\infty} e^{-\mu t}\frac{f(x)-f(xe^{-t})}{t^{1+\alpha}}dt + \mu^{\alpha} f(x), \qquad (1.5)$$

where $0 < \alpha < 1$, $\varepsilon > 0$. Then by the definition

$$\boldsymbol{D}^{\alpha}_{0+,\,\mu} f = \lim_{\substack{\varepsilon \to 0 \\ (X^p_c)}} \boldsymbol{D}^{\alpha}_{0+,\,\mu;\,\varepsilon} f,$$

where the limit is understand in the norm of the space X^p_c. Expression (1.5) is called truncated Marchaud-Hadamard-type fractional derivative.

In this paper we apply the properties of the operators $\mathcal{J}^{\alpha}_{0+,\mu}$ and $\mathcal{D}^{\alpha}_{0+,\,\mu}$ in the space $X^p_c(\mathbb{R}_+)$ to study the problem of the existence and uniqueness of a solution of the Cauchy-type problem for the nonlinear differential equation of fractional order $0 < \alpha < 1$:

$$(\boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y)(x) = f[x, y(x)] \ \ (0 < \alpha < 1) \tag{1.6}$$

on the half-axis $\mathbb{R}_+ = (0, +\infty)$ with the initial condition

$$(x^{\mu} \mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(0+) = b,\ b \in \mathbb{R}. \tag{1.7}$$

The notation $(x^{\mu} \mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(0+)$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, 0+\varepsilon)$, $\varepsilon > 0$, of $x = 0$:

$$(x^{\mu} \mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(0+) = \lim_{x \to 0+} [x^{\mu} (\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)]. \tag{1.8}$$

First we give conditions for the equivalence of such a Cauchy-type problem and of the nonlinear Volterra integral equation

$$y(x) = \frac{b}{\Gamma(\alpha)} x^{-\mu} (\ln x)^{\alpha-1}_{+} + \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{t}{x}\right)^{\mu} \left(\ln \frac{x}{t}\right)^{\alpha-1} f[t, y(t)]\, \frac{dt}{t} \tag{1.9}$$

in the sense that if $y(x) \in X^p_c(\mathbb{R}_+)$ satisfies (1.6) and (1.7), then it satisfies (1.9), and inverse. Then by using the Banach fixed point theorem, we give conditions for the existence and uniqueness of a solution of the Cauchy-type problem (1.6)–(1.7) in the space $X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)$ defined for $\alpha > 0$ by

$$X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+) = \left\{ y \in X^p_c(\mathbb{R}_+) :\ \boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y \in X^p_{c,\,0}(\mathbb{R}_+) \right\}, \tag{1.10}$$

where $X^p_{c,\,0}(\mathbb{R}_+)$ is the subspace of $X^p_c(\mathbb{R}_+)$ of functions $g \in X^p_c(\mathbb{R}_+)$ with compact support on infinity: $g(x) \equiv 0$ for large enough $x > R$. In particular, we consider the Cauchy-type problem (1.6)–(1.7) in the case $\mu = 0$

$$\begin{cases} (\boldsymbol{D}^{\alpha}_{0+}\, y)(x) = f[x, y(x)], \ \ (0 < \alpha < 1), \\ (\mathcal{J}^{1-\alpha}_{0+}\, y)(0+) = b, \ \ b \in \mathbb{R}. \end{cases} \tag{1.11}$$

We also establish solution in closed form of the Cauchy-type problem for the linear differential equation of fractional order

$$\begin{cases} (\boldsymbol{D}^{\alpha}_{0+,\,\mu}\, y)(x) = \lambda y(x) + f(x), \quad (0 < \alpha < 1;\ \lambda \in \mathbb{R}), \\ (x^{\mu} \mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(0+) = b, \quad b \in \mathbb{R}, \end{cases} \tag{1.12}$$

and give some examples in conclusion.

The paper is organized as follows. Section 2 contains preliminary results involving properties of the Marchaud-Hadamard-type fractional derivatives (1) and the Hadamard-type fractional calculus operators (1.2) and (1.3) on the space $X_c^p(\mathbb{R}_+)$. Section 3 is devoted to the equivalence of the Cauchy-type problem (1.6)–(1.7) and of the nonlinear integral equation (1.9). Section 4 deals with the existence and uniqueness of the solution of the Cauchy-type problem (1.6)–(1.7) and (1.11). Solution in closed form of the Cauchy-type problem (1.12) is obtained in Section 5. Examples are given in Section 6.

2. Marchaud-Hadamard-Type Fractional Integro-Differentiation

In this section we present some properties of the Hadamard-type fractional integral (1.2) and the Marchaud-Hadamard-type fractional derivative (1) which will be used later. Let y_+^α $(y \in \mathbb{R},\ \alpha > 0)$ is the truncated power function [9, (1.100)]

$$y_+^\alpha = y^\alpha, \quad y > 0, \quad y_+^\alpha = 0, \quad y < 0. \tag{2.1}$$

The following properties of fractional integration operator $\mathcal{J}_{0+,\,\mu}^\alpha$ and differential operator $\boldsymbol{D}_{0+,\,\mu}^\alpha$ in the space X_c^p are known.

Lemma 1. [2, Theorem 4(a)] *If $1 \leqslant p \leqslant \infty$, $c \in \mathbb{R}$, $\alpha > 0$ and $\mu \in \mathbb{R}$, then for $\mu > c$, the operator $\mathcal{J}_{0+,\,\mu}^\alpha$ is bounded in $X_c^p(\mathbb{R}_+)$ and*

$$\left\|\mathcal{J}_{0+,\,\mu}^\alpha f\right\|_{X_c^p} \leqslant C_1 \|f\|_{X_c^p}, \quad C_1 = [\mu - c]^{-\alpha}. \tag{2.2}$$

Lemma 2. [3, Theorem 1(a)] *Let $1 \leqslant p \leqslant \infty$, and let $c \in \mathbb{R}$, $\alpha > 0, \beta > 0$ and $\mu \in \mathbb{R}$. If $\mu > c$, then the semigroup property*

$$(\mathcal{J}_{0+,\,\mu}^\alpha \mathcal{J}_{0+,\,\mu}^\beta f)(x) = (\mathcal{J}_{0+,\,\mu}^{\alpha+\beta} f)(x) \quad (\alpha > 0, \beta > 0) \tag{2.3}$$

holds for $f(x) \in X_c^p(\mathbb{R}_+)$.

Lemma 3. [6, Theorem 2] *Let $0 < \alpha < 1$, $1 \leqslant p < \infty$, $\mu \geqslant 0$, $c \in \mathbb{R}$ and$\mu > c$. Then for $f(x) \in X_c^p(\mathbb{R}_+)$ there holds*

$$(\boldsymbol{D}_{0+,\mu}^\alpha \mathcal{J}_{0+,\,\mu}^\alpha f)(x) = f(x). \tag{2.4}$$

Hadamard-type fractional integration and differentiation of the truncated power-logarithmic function $x^{-\mu}(\ln x)_+^{\beta-1}$ yield a function of the same kind.

Lemma 4. *Let $0 < \alpha < 1$, $\beta > 0$, $\mu \in \mathbb{C}$. There hold the following formulas:*

$$\left(\mathcal{J}_{0+,\,\mu}^\alpha\, t^{-\mu}(\ln t)_+^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x^{-\mu}(\ln x)_+^{\beta+\alpha-1}, \tag{2.5}$$

$$\left(\mathcal{D}_{0+,\,\mu}^\alpha\, t^{-\mu}(\ln t)_+^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{-\mu}(\ln x)_+^{\beta-\alpha-1}. \tag{2.6}$$

In particular, for $\beta = \alpha$

$$\left(\mathcal{D}_{0+,\,\mu}^\alpha t^{-\mu}(\ln t)_+^{\beta-1}\right)(x) = 0. \tag{2.7}$$

Proof. Using (1.2) with $f(x) = x^{-\mu}(\ln x)_+^{\beta-1}$ and taking into account (2.1) we have

$$\left(\mathcal{J}^{\alpha}_{0+,\,\mu}\, t^{-\mu}(\ln t)_+^{\beta-1}\right)(x) = \frac{x^{-\mu}}{\Gamma(\alpha)} \int\limits_1^x \left(\ln\frac{x}{t}\right)^{\alpha-1} (\ln t)^{\beta-1}\,\frac{dt}{t}.$$

Then making the change of variable $\tau = \ln t/\ln x$, we obtain

$$\left(\mathcal{J}^{\alpha}_{0+,\,\mu}\, t^{-\mu}(\ln t)_+^{\beta-1}\right)(x) = x^{-\mu}(\ln x^{\beta+\alpha-1}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1}\,d\tau,$$

which yields (2.5) in accordance with the known relations for the beta– and gamma–functions [5, 1.5(1) and (1.5.5)]. (2.6) is deduced from (1.1) with $f(x) = x^{-\mu}(\ln x)_+^{\beta-1}$ by the use of the formula (2.5), in which α is replaced by $1-\alpha$, and by direct δ–differentiation. (2.7) follows from (2.6), if we take into account that the gamma function $\Gamma(z)$ has poles of first order at the points $z = 0, -1, -2, \ldots$ [5, 1.1(7)-1.1(8)]. Thus lemma is proved.

Let $AC[a,b]$ be a class of functions absolutely continuous on $[a,b]$ [9, § 1.1]. For $n\in\mathbb{N}$, $\mu\in\mathbb{R}$ and $\delta = xD$ $(D = d/dx)$ in [8] there was denoted by $AC^n_{\delta;\mu}[a,b]$ a class functions $g(x)$, such that $x^{\mu}g(x)$ has δ–derivatives up to order $n-1$ on $[a,b]$ and $\delta^{n-1}[x^{\mu}g(x)] \in AC[a,b]$. In our case on the half-axis $\mathbb{R}_+$ with $0<\alpha<1$ the definition takes the form

$$AC^1_{\delta,\mu}(\mathbb{R}_+) \equiv AC_{\mu}(\mathbb{R}_+) = \{h : \mathbb{R}_+ \to \mathbb{C} \ : x^{\mu}h \in AC(\mathbb{R}_+),\ \mu\in\mathbb{R}\}.$$

Thus, the class $AC_{\mu}(\mathbb{R}_+)$ coincides with the class of Lebegue primitives functions for $x^{\mu}f(x)$:

$$f(x) \in AC_{\mu}(\mathbb{R}_+) \Leftrightarrow x^{\mu}f(x) = c + \int\limits_0^x \varphi(t)\,dt, \quad \varphi(t)\in L(\mathbb{R}_+). \tag{2.8}$$

■

Let consider the composition of fractional integration operators $\mathcal{J}^{\alpha}_{0+,\,\mu}$ and differential operators $\mathcal{D}^{\alpha}_{0+,\,\mu}$.

Theorem 1. *Let* $0<\alpha<1$, $\mu\in\mathbb{R}$, $\mu\geqslant 0$, $c\in\mathbb{R}$, $\mu>c$, $1<p<\dfrac{1}{\alpha}$ *and let* $(\mathcal{J}^{1-\alpha}_{0+,\,\mu}y)(x)$ *be the Hadamard-type fractional integral* (1.2). *If* $y(x)\in X^p_c(\mathbb{R}_+)$ *and* $(\mathcal{J}^{1-\alpha}_{0+,\,\mu}y)(x)\in AC_{\mu}(\mathbb{R}_+)$, *then there holds*

$$(\mathcal{J}^{\alpha}_{0+,\,\mu}\mathcal{D}^{\alpha}_{0+,\,\mu}\, y)(x) = y(x) - \frac{\left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+)}{\Gamma(\alpha)}\, x^{-\mu}\,(\ln x)_+^{\alpha-1}, \tag{2.9}$$

where, by analogy to (1.8), $\left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+)$ *means that the limit is taken at almost all points of the right-sided neighborhood of* $x=0$.

Proof. By (2.8), the function $(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)$ can be represent in the form:

$$
\begin{aligned}
x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x) &= \left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+) + \int_0^x \delta\left(t^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y(t)\right)\frac{dt}{t} \\
&= \left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+) + x^{\mu}\int_0^x \left(\frac{t}{x}\right)^{\mu}(\mathcal{D}^{\alpha}_{0+,\,\mu}\, y)(t)\frac{dt}{t} \\
&= \left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+) + x^{\mu}(\mathcal{J}^{1}_{0+,\,\mu}\mathcal{D}^{\alpha}_{0+,\,\mu}\, y)(x). \qquad (2.10)
\end{aligned}
$$

Using the directly verified equality

$$
\int_0^x \left(\ln\frac{x}{t}\right)^{-\alpha}(\ln t)^{\alpha-1}_{+}\,\frac{dt}{t} = \Gamma(1-\alpha)\Gamma(\alpha), \qquad (2.11)
$$

where $(\ln t)^{\alpha-1}_{+}$ is the truncated power function (2.1), from (2.10) we have

$$
\begin{aligned}
x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x) &= \frac{\left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\, y)(x)\right)(0+)}{\Gamma(1-\alpha)\Gamma(\alpha)} \\
&\times x^{\mu}\int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\ln\frac{x}{t}\right)^{-\alpha} t^{-\mu}\,(\ln t)^{\alpha-1}_{+}\,\frac{dt}{t} + x^{\mu}(\mathcal{J}^{1}_{0+,\,\mu}\mathcal{D}^{\alpha}_{0+,\,\mu} y)(x) \\
&= \frac{\left(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\,\mu} y)(x)\right)(0+)}{\Gamma(\alpha)}\, x^{\mu}\left(\mathcal{J}^{1-\alpha}_{0+,\,\mu} t^{-\mu}\,(\ln t)^{\alpha-1}_{+}\right)(x) \\
&+ x^{\mu}\left(\mathcal{J}^{1-\alpha}_{0+,\,\mu}\mathcal{J}^{\alpha}_{0+,\,\mu}\mathcal{D}^{\alpha}_{0+,\,\mu}\, y\right)(x).
\end{aligned}
$$

Hence

$$
x^{\mu}\mathcal{J}^{1-\alpha}_{0+,\mu}\left(y(x) - (\mathcal{J}^{\alpha}_{0+,\mu}\mathcal{D}^{\alpha}_{0+,\mu} y)(x) - \frac{(x^{\mu}(\mathcal{J}^{1-\alpha}_{0+,\mu} y)(x))(0+)}{\Gamma(\alpha)} x^{-\mu}\left(\ln x\right)^{\alpha-1}_{+}\right) = 0.
$$

In accordance with the theorem on inversion of the Hadamard-type fractional integral (1.2) in X^p_c [6, Theorem 2], we deduce that $\mathcal{J}^{\alpha}_{0+,\,\mu}\varphi = 0$, $\varphi \in X^p_c$, if and only if $\varphi(t) \equiv 0$, thus we obtain formula (2.9). ■

3. Equivalence of the Cauchy-Type Problem and of the Volterra Nonlinear Integral Equation

In this section we prove that the Cauchy-type problem (1.6)–(1.7) and the nonlinear Volterra integral equation (1.9) are equivalent in the sense that if $y(x) \in X^p_c(\mathbb{R}_+)$ satisfies one of these relations then it also satisfies another one.

Theorem 2. *Let* $0 < \alpha < 1$, $\mu \in \mathbb{R}$, $\mu \geqslant 0$, $c \in \mathbb{R}$, $\mu > c$, $1 < p < \dfrac{1}{\alpha}$. *Let* G *be an open set in* $\mathbb{R}$ *and let* $f : \mathbb{R}_+ \times G \to \mathbb{R}$ *be a function such that* $f(x,y) \in X^p_{c,\,0}(\mathbb{R}_+)$ *for any* $y \in G$. *If* $y(x) \in X^p_c(\mathbb{R}_+)$, *then* $y(x)$ *satisfies a.e. the relations* (1.6) *and* (1.7) *if and only if* $y(x)$ *satisfies a.e. the integral equation* (1.9).

Proof. First we prove the necessity. Let function $y(x) \in X^p_c(\mathbb{R}_+)$ satisfies (1.6) and (1.7). If $f(x,y) \in X^p_{c,\,0}(\mathbb{R}_+)$ for any $y \in G$, then $f(x,y)$ also belongs to the space $X^1_\mu(\mathbb{R}_+)$. Indeed, using Holder's inequality we have

$$\|f\|_{X^1_\mu} = \int\limits_0^\infty |x^{\mu-1} f(x)|\,dx = \int\limits_0^\infty |x^c f(x) \frac{1}{x^{1/p}} x^{\mu-c} \frac{1}{x^{1/p'}}|\,dx$$

$$\leqslant \Big(\int\limits_0^\infty |x^c f(x)|^p \frac{dx}{x} \Big)^{1/p} \Big(\int\limits_0^\infty \frac{dx}{x^{(c-\mu)p'+1}} \Big)^{1/p'},$$

where $p' = p/(p-1)$ and $(c-\mu)p' + 1 < 1$, since $0 < \alpha < 1$, $\mu > c$ and $1 < p < \dfrac{1}{\alpha}$. From here and from the definition of the space $X^p_{c,\,0}(\mathbb{R}_+)$ it follows, that $\|f\|_{X^1_\mu} < \infty$, that is $f(x,y) \in X^1_\mu(\mathbb{R}_+)$. So the equation (1.6) means the existence of the fractional derivative $\boldsymbol{D}^\alpha_{0+,\,\mu} \in X^1_\mu(\mathbb{R}_+)$ on the half-axis $\mathbb{R}_+$. Then by (1.1), (1.4) and (2.8) $(\mathcal{J}^{1-\alpha}_{0+,\,\mu} y)(x) \in AC_\mu(\mathbb{R}_+)$, and hence we can apply Theorem 1. Applying the operator $\mathcal{J}^\alpha_{a+,\,\mu}$ to both sides of (1.6), using (2.9), (1.7) and Lemma 1 we obtain equation (1.9), and hence the necessity is proved.

Now we prove the sufficiency. Let $y(x) \in X^p_c(\mathbb{R}_+)$ satisfies a.e. the equation (1.9). Applying the operator $\boldsymbol{D}^\alpha_{0+,\,\mu}$ to the both sides of (1.9) and using (2.4) and (2.11) we come to equation (1.6).

Now we show that the relations in (1.7) are also valid. For this we apply the operator $\mathcal{J}^{1-\alpha}_{0+,\,\mu}$ to both sides of (1.9). Then using (2.3) and (2.11) and taking a limit as $x \to 0+$ a.e. we obtain relation (1.7). Thus sufficiency is proved which completes the proof of the theorem.

When $\mu = 0$, from Theorem 2 we obtain the corresponding results for the Cauchy-type problem (1.11). ■

Theorem 3. *Let* $0 < \alpha < 1$, $c \in \mathbb{R}$, $c < 0$, $1 < p < \dfrac{1}{\alpha}$. *Let* G *be an open set in* $\mathbb{R}$ *and let* $f : \mathbb{R}_+ \times G \to \mathbb{R}$ *be a function such that* $f(x,y) \in X^p_{c,\,0}(\mathbb{R}_+)$ *for any* $y \in G$. *If* $y(x) \in X^p_c(\mathbb{R}_+)$, *then* $y(x)$ *satisfies a.e. the relations* (1.11) *if and only if* $y(x)$ *satisfies a.e. the Volterra integral equation*

$$y(x) = \frac{b}{\Gamma(\alpha)} (\ln x)^{\alpha-1}_+ + \frac{1}{\Gamma(\alpha)} \int_0^x \Big(\ln \frac{x}{t}\Big)^{\alpha-1} f[t, y(t)]\, \frac{dt}{t}.$$

4. Existence and Uniqueness of the Solution of the Cauchy-Type Problem

In this section we establish the existence of a unique solution of the Cauchy-type problem (1.6)–(1.7) in the class $X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)$ under conditions of Theorem 2 and an additional Lipschitzian condition on $f[x, y(x)]$ with respect to the second variable: there exists a constant $A{>}0$ such that, for all $x{\in}\mathbb{R}_+$ and all $y, y_1{\subset}G{\in}\mathbb{R}$, we have

$$|f[x, y(x)] - f[x, y_1(x)]| \leqslant A|y(x) - y_1(x)| \quad (A > 0). \tag{4.1}$$

To prove the existence and uniqueness of a solution of the Cauchy–type problem (1.6)–(1.7) we use the Banach theorem of fixed point of contracting mapping [1].

DEFINITION 1. The mapping $f\,:\,(X, \rho_X){\to}(X, \rho_X)$ is contracting, if there exists a constant $0{\le}\alpha{<}1$ such that for any $x_1, x_2{\in}X$ there holds

$$\rho\,(f(x_1), f(x_2)) \le \alpha\rho(x_1, x_2).$$

Thus, contracting mapping is a mapping satisfying Lipschitzian condition with a constant $\alpha{<}1$. Such a mapping is always continuous and uniformly continuous.

Theorem 4. (Banach) *A contracting mapping has one and only one fixed point in a complete metric space.*

Corollary 1. Successive approximations $x_n{=}f(x_{n-1})$ $(n{\in}N)$ converge to the fixed point a of mapping f for any initial approximation x_0. Moreover, the following error estimate is valid:

$$\rho(x_n, a) \le \frac{\alpha^n}{1-\alpha}\rho(x_0, f(x_0)) \;\; (n{\in}N).$$

Theorem 5. *Let* $0 < \alpha < 1,\ \mu \in \mathbb{R},\ \mu \geqslant 0,\ \ c \in \mathbb{R},\ \mu > c,\ 1 < p < \dfrac{1}{\alpha}$. *Let G is an open set in $\mathbb{R}$ and let $f\,:\,\mathbb{R}_+ \times G \to \mathbb{R}$ be such function that $f(x, y) \in X^{p}_{c,\,0}(\mathbb{R}_+)$ for all $y \in G$ and condition* (4.1) *and the inequality*

$$A(\mu - c)^{-\alpha} < 1 \tag{4.2}$$

are satisfied. Then there exists a unique solution $y(x)$ of the Cauchy-type problem (1.6)–(1.7) *in the space $X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)$.*

Proof. First we prove the existence of a unique solution $y(x) \in X^p_c(\mathbb{R}_+)$. According to Theorem 2 it is sufficient to prove the existence of a unique solution $y(x) \in X^p_c(\mathbb{R}_+)$ of the nonlinear Volterra integral equation (1.9). For this we use the Banach fixed point Theorem 4 for the space $X^p_c(\mathbb{R}_+)$ which is the complete metric space with the distance

$$d(y_1,y_2)=\|y_1-y_2\|_{X^p_c}\equiv\Big(\int\limits_0^\infty |t^c(y_1(t)-y_2(t))|^p\,\frac{dt}{t}\Big)^{1/p}.$$

We rewrite the integral equation (1.9) in the form $y(x)=(Ty)(x)$, where T is the operator in the right-hand side of (1.9):

$$(Ty)\,(x)=y_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^x\left(\frac{t}{x}\right)^\mu\left(\ln\frac{x}{t}\right)^{\alpha-1}f[t,y(t)]\,\frac{dt}{t},\tag{4.3}$$

with

$$y_0(x)=\frac{b}{\Gamma(\alpha)}x^{-\mu}(\ln x)_+^{\alpha-1}.\tag{4.4}$$

Let now show that the operator T maps the space $X^p_c(\mathbb{R}_+)$ into itself. If $y(x)\in X^p_c(\mathbb{R}_+)$, then it is sufficient to prove that $(Ty)\,(x)\in X^p_c(\mathbb{R}_+)$. It follows from (4.4) that $y_0(x)\in X^p_c(\mathbb{R}_+)$. Since $f(x,y(x))\in X^p_{c,\,0}(\mathbb{R}_+)\subset X^p_c(\mathbb{R}_+)$, then by Lemma 1 the integral in the right-hand side of (4.3) also belongs to the space $X^p_c(\mathbb{R}_+)$, and hence, $(Ty)\,(x)\in X^p_c(\mathbb{R}_+)$.

Now applying Lemma 1 and using the Lipschitzian condition (4.1) for all $y_1(x),\ y_2(x)\in X^p_c(\mathbb{R}_+)$, we have:

$$\begin{aligned}&\|(Ty_1)(x)-(Ty_2)(x)\|_{X^p_c(\mathbb{R}_+)}\\&\quad=\left\|\frac{1}{\Gamma(\alpha)}\int_0^x\left(\frac{t}{x}\right)^\mu\left(\ln\frac{x}{t}\right)^{\alpha-1}(f[t,y_1(t)]-f[t,y_2(t)])\frac{dt}{t}\right\|_{X^p_c(\mathbb{R}_+)}\\&\quad\leqslant(\mu-c)^{-\alpha}\,\|f[t,y_1(t)]-f[t,y_2(t)]\|_{X^p_c(\mathbb{R}_+)}\\&\quad\leqslant A(\mu-c)^{-\alpha}\,\|y_1(t)-y_2(t)\|_{X^p_c(\mathbb{R}_+)}\,.\end{aligned}$$

In accordance with (4.2) and Theorem 4 there exists a unique function $y^\star(x)\in X^p_c(\mathbb{R}_+)$ such that $(Ty^\star)(x)=y^\star(x)$, and hence of the integral equations (1.9).

By Theorem 4, the solution y^* is a limit of a convergence sequence $(T^m y_0^*)(x)$:

$$\lim_{m\to\infty}\|(T^m y_0^*)(x)-y^*(x)\|_{X^p_c(\mathbb{R}_+)}=0,\tag{4.5}$$

where $y_0^*(x)$ is any function in $X^p_c(\mathbb{R}_+)$. If $b\neq 0$ in the initial condition (1.7), we can take $y_0^*(x)=y_0(x)$ with $y_0(x)$ defined by (4.4).

By (4.3) the sequence $(T^m y_0^*)(x)$ is defined by recurrent formulas

$$(T^m y_0^*)(x)=y_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^x\left(\frac{t}{x}\right)^\mu\left(\ln\frac{x}{t}\right)^{\alpha-1}f[t,(T^{m-1}y)(t)]\,\frac{dt}{t}\qquad(m=1,2,\ldots).$$

If we denote $y_m(x)=(T^m y_0^*)(x)$, then the last relation takes the form

$$y_m(x)=y_0(x)+\frac{1}{\Gamma(\alpha)}\int_0^x\left(\frac{t}{x}\right)^\mu\left(\ln\frac{x}{t}\right)^{\alpha-1}f[t,y_{m-1}(t)]\,\frac{dt}{t}\quad(m=1,2,\ldots)\tag{4.6}$$

and (4.5) is rewritten as

$$\lim_{m\to\infty} \|y_m(x) - y^*(x)\|_{X_c^p(\mathbb{R}_+)} = 0. \tag{4.7}$$

This means that we really apply the method of successive approximation to find a unique solution $y^*(x)$ of the integral equation (1.9) on $\mathbb{R}_+$. Thus, there exists a unique solution $y(x) \in X_c^p(\mathbb{R}_+)$ of the Volterra integral equation (1.9), and hence of the Cauchy-type problem (1.6)–(1.7).

To complete the proof of theorem we show that such an unique solution $y(x) \in X_c^p(\mathbb{R}_+)$ belongs to the space $X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$. In accordance with (1.10), it is sufficient to prove that $\left(\boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right)(x) \in X_{c,\,0}^{p}(\mathbb{R}_+)$. By (1.6) and (4.1), we have:

$$\left\|\boldsymbol{D}_{0+,\,\mu}^{\alpha} y_m - \boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right\|_{X_c^p} \leqslant \|f[t, y_m] - f[t, y]\|_{X_c^p} \leqslant A\|y_m - y\|_{X_c^p}.$$

Thus, by (4.7)

$$\lim_{m\to\infty} \left\|\boldsymbol{D}_{0+,\,\mu}^{\alpha} y_m - \boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right\|_{X_c^p} = 0,$$

and hence $\left(\boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right)(x) \in X_c^p(\mathbb{R}_+)$. It follows from equation (1.6) and the definition of the space $X_{c,\,0}^{p}(\mathbb{R}_+)$ that $\left(\boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right)(x) \equiv 0$ for large enough $x > R$. Then, $\left(\boldsymbol{D}_{0+,\,\mu}^{\alpha} y\right)(x) \in X_{c,\,0}^{p}(\mathbb{R}_+)$. This completes the proof of theorem.

When $\mu = 0$, from Theorem 5 and Theorem 3 we obtain the corresponding results for the Cauchy-type problem (1.11). ■

Theorem 6. *Let* $0 < \alpha < 1$, $c \in \mathbb{R}$, $c < 0$, $1 < p < \dfrac{1}{\alpha}$. *Let* G *is an open set in* $\mathbb{R}$ *and let* $f : \mathbb{R}_+ \times G \to \mathbb{R}$ *be such a function that* $f(x, y) \in X_{c,\,0}^{p}(\mathbb{R}_+)$ *for all* $y \in G$ *and the condition* (4.1) *and the inequality*

$$A(-c)^{-\alpha} < 1 \tag{4.8}$$

are satisfied. Then there exists a unique solution $y(x)$ *of the Cauchy-type problem* (1.11) *in the space* $X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$.

5. Solution of Cauchy-Type Problem for Linear Fractional Differential Equation

In this section we show that solution in closed form of the Cauchy-type problem (1.12) is expressed in terms of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined for complex $z \in \mathbf{C}$ and positive $\alpha > 0$ and $\beta > 0$ by [1, Section 18.1]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{5.1}$$

By Theorem 2, if $f(x) \in X_{c,0}^{p}(\mathbb{R}_+)$, then solutions of the Cauchy-type problem (1.12) and of the linear Volterra integral equation

$$y(x)=\frac{b}{\Gamma(\alpha)}x^{-\mu}(\ln x)_{+}^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^x\left(\frac{t}{x}\right)^{\mu}\left(\ln\frac{x}{t}\right)^{\alpha-1}[\lambda y(t)+f(t)]\,\frac{dt}{t} \quad (5.2)$$

are equivalent.

Further, the condition (4.1) is fulfilled with $A=|\lambda|$ and the condition (4.2) takes the form

$$|\lambda|(\mu-c)^{-\alpha}<1. \quad (5.3)$$

Thus, if this condition is satisfied, then, in accordance with Theorem 5, the Cauchy-type problem (1.12) has a unique solution $y(x)\in X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$. We find the explicit form of this solution following the proof of Theorem 5.

First we consider the homogeneous Cauchy-type problem ($f(x)\equiv 0$):

$$\begin{cases}(\boldsymbol{D}_{0+,\,\mu}^{\alpha}\,y)(x)=\lambda y(x) \quad (0<\alpha<1;\ \lambda\in\mathbb{R}),\\ (x^{\mu}\mathcal{J}_{0+,\,\mu}^{1-\alpha}\,y)(0+)=b, \quad b\in\mathbb{R},\end{cases} \quad (5.4)$$

corresponding to (1.12). By the definition (1.2), we rewrite the relations (4.6) in the form

$$y_m(x)=y_0(x)+\lambda(\mathcal{J}_{0+,\,\mu}^{\alpha}\,y_{m-1})(x) \quad (m=1,2,\ldots), \quad (5.5)$$

where $y_0(x)$ is defined by (4.4). Using (5.5) with $m=1$ and (4.4) and taking (2.5) into account, we have

$$y_1(x)=y_0(x)+\lambda(\mathcal{J}_{0+,\,\mu}^{\alpha}\,y_0)(x)=b\,x^{-\mu}\left[\frac{1}{\Gamma(\alpha)}\,(\ln x)_{+}^{\alpha-1}+\frac{\lambda}{\Gamma(2\alpha)}(\ln x)_{+}^{2\alpha-1}\right],$$

$$y_2(x)=y_0(x)+\lambda(\mathcal{J}_{0+,\,\mu}^{\alpha}\,y_1)(x)=b\,x^{-\mu}\sum_{k=0}^{2}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\,(\ln x)_{+}^{k\alpha+\alpha-1}.$$

Continuing this process, we find

$$y_m(x)=b\,x^{-\mu}\sum_{k=0}^{m}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}(\ln x)_{+}^{k\alpha+\alpha-1} \quad (m=1,2,\ldots).$$

Taking a limit, when $m\to\infty$, and applying (5.1) we obtain the solution of the Cauchy-type problem (5.4):

$$y(x)=b\,x^{-\mu}(\ln x)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(\ln x)_{+}^{\alpha}\right). \quad (5.6)$$

Now we consider the nonhomogeneous Cauchy-type problem (1.12) In this case the relations (4.6) take the form

$$y_m(x)=y_0(x)+\lambda(\mathcal{J}_{0+,\,\mu}^{\alpha}\,y_{m-1})(x)+(\mathcal{J}_{0+,\,\mu}^{\alpha}\,f)(x) \quad (m=1,2,\ldots).$$

Similarly to the above we deduce

$$y_m(x)=bx^{-\mu}\sum_{k=0}^{m}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}(\ln x)_{+}^{\alpha k+\alpha-1}+\sum_{k=0}^{m-1}\lambda^k\left(\mathcal{J}_{0+,\,\mu}^{\alpha k+\alpha}f\right)(x) \quad (m=1,2,\ldots).$$

Taking a limit, as $m \to \infty$, making the change of order of summations and integration, we obtain the explicit solution of the problem (1.12) in the form

$$y(x) = bx^{-\mu}(\ln x)_+^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(\ln x)_+^{\alpha}\right) + \int_0^x \left(\frac{t}{x}\right)^{\mu} \left(\ln\frac{x}{t}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\ln\frac{x}{t}\right)^{\alpha}\right) f(t)\,\frac{dt}{t}. \quad (5.7)$$

It is clear tat solution (5.7) belongs to the space $X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$. It is directly verified that $y(x)$ in (5.7) yields the explicit solution of the integral equation (5.2) and hence of the boundary value problem (1.12). Since by the theory of Volterra integral equations of the second kind (for example, see [9]) a solution of the Volterra integral equation (5.2) is unique then condition (5.3) can be omitted. From here we deduce the result.

Theorem 7. *Let* $0 < \alpha < 1,\ \mu \in \mathbb{R},\ \lambda \in \mathbb{R},\ \mu \geqslant 0,\ c \in \mathbb{R},\ \mu > c,\ 1 < p < \frac{1}{\alpha}$ *and let* $f(x) \in X_{c,\,0}^{p}(\mathbb{R}_+)$. *Then the Cauchy-type problem* (1.12) *is solvable in the space* $X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$. *Its unique solution is given by* (5.7), *and the solution of the corresponding homogeneous Cauchy-type problem* (5.4) *has the form* (5.6).

Theorem 7 with $\mu = 0$ yields the following assertion.

Theorem 8. *Let* $0 < \alpha < 1,\ \lambda \in \mathbb{R},\ c \in \mathbb{R},\ c < 0,\ 1 < p < \frac{1}{\alpha}$ *and let* $f(x) \in X_{c,\,0}^{p}(\mathbb{R}_+)$. *Then the Cauchy-type problem*

$$\begin{cases} (\boldsymbol{D}_{0+}^{\alpha}\, y)(x) = \lambda y(x) + f(x) \quad (0 < \alpha < 1;\ \lambda \in \mathbb{R}), \\ (x^{\mu} \mathcal{J}_{0+,}^{1-\alpha}\, y)(0+) = b, \quad b \in \mathbb{R}, \end{cases} \quad (5.8)$$

is solvable in the space $X_{c,\,0}^{p,\,\alpha}(\mathbb{R}_+)$, *and its unique solution is given by*

$$y(x) = b(\ln x)_+^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(\ln x)_+^{\alpha}\right) + \int_0^x \left(\ln\frac{x}{t}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\ln\frac{x}{t}\right)^{\alpha}\right) f(t)\,\frac{dt}{t}.$$

The solution of the corresponding homogeneous Cauchy-type problem for the equation

$$(\boldsymbol{D}_{0+}^{\alpha}\, y)(x) = \lambda y(x) \quad (0 < \alpha < 1;\ \lambda \in \mathbb{R}),$$

with the initial conditions (5.8) *has the form*

$$y(x) = b(\ln x)_+^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(\ln x)_+^{\alpha}\right).$$

6. Examples

In this section we give examples of solution $y(x)$ of the Cauchy-type problem for linear differential equations of fractional order (1.12).

Example 1. The Cauchy-type problem for the inhomogeneous linear differential equation of fractional order 1/2:

$$(\boldsymbol{D}^{1/2}_{0+,\,\mu}\, y)(x) = \lambda y(x) + f(x), \quad (x^{\mu}\mathcal{J}^{1/2}_{0+,\,\mu} y)(0+) = b, \quad b \in \mathbb{R},$$

has the unique solution $y(x) \in X^{p,\,\alpha}_{c,\,0}(\mathbb{R}_+)$ given by

$$y(x) = bx^{-\mu}(\ln x)_+^{-1/2} E_{1/2,1/2}\left(\lambda\,(\ln x)_+^{1/2}\right) + \int_0^x \left(\frac{t}{x}\right)^{\mu}\left(\ln\frac{x}{t}\right)^{-1/2} E_{1/2,1/2}\left(\lambda\left(\ln\frac{x}{t}\right)^{1/2}\right) f(t)\frac{dt}{t}.$$

The solution of the corresponding homogeneous Cauchy-type problem

$$(\boldsymbol{D}^{1/2}_{0+,\,\mu}\, y)(x) = \lambda y(x), \quad (x^{\mu}\mathcal{J}^{1/2}_{0+,\,\mu} y)(0+) = b, \quad b \in \mathbb{R},$$

has the form

$$y(x) = bx^{-\mu}(\ln x)_+^{-1/2} E_{1/2,1/2}\left(\lambda\,(\ln x)_+^{1/2}\right),$$

where the Mittag-Leffler function $E_{1/2,1/2}(z)$ is given by (5.1).

Example 2. We consider the linear fractional differential equation of fractional order $0 < \alpha < 1$

$$(\boldsymbol{D}^{\alpha}_{0+}\, y)(x) = \lambda y^m(x) \quad (\lambda \in \mathbb{R},\ \lambda \neq 0). \tag{6.1}$$

It is directly verified that this equation has the exact solution

$$y(x) = \frac{\Gamma(1-\alpha)}{\lambda\Gamma(1-2\alpha)}(\ln x)_+^{-\alpha} \quad (0 < \alpha < 1) \tag{6.2}$$

and this solution belongs to the space $X^{p}_{c,\,0}(\mathbb{R}_+)$. In this case the right-hand side of the equation (6.1) takes the form

$$f[x, y(x)] = \frac{1}{\lambda}\left[\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right]^2 (\ln x)_+^{-2\alpha}$$

and this function is, generally speaking, does not belong to the space $X^{p}_{c,\,0}(\mathbb{R}_+)$.

If we suppose that $2\alpha < 1$, then the right-hand side of the equation (6.1) belongs to the space $X^{p}_{c,\,0}(\mathbb{R}_+)$ and, since $\alpha+\beta < 2\alpha+\beta$, then $\alpha+\beta < 1$ and the equation (6.1) has the exact solution (6.2) which belongs to $X^{p}_{c,\,0}(\mathbb{R}_+)$.

Acknowledgments

The present investigation was supported in part by Belarusian Fundamental Research Fund (projects F05MC-050 and F06R-106).

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