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ON THE NAVIER-STOKES EQUATION WITH SLIP BOUNDARY CONDITIONS OF FRICTION TYPE [1]

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Abstract. In this paper we deal with the boundary value problem for the stationary flow for Newtonian and incompressible fluid governed by the Navier-Stokes equation with slip boundary conditions of friction type, mostly by means of variational inequalities. Among others, theorems concerning existence and uniqueness of weak solutions are presented.

Key words: Navier-Stokes equations, variational inequality, slip boundary conditions, fixed point theorem

1. Introduction

In hydrodynamics and in mathematics, extensive study has been done so far for the motion of incompressible fluid which is governed by the Stokes or Navier-Stokes equation (for example, see [7, 12]). As to the boundary condition, almost all of these works have dealt with the non-slip condition to the surface of a rigid body, namely, with the Dirichlet boundary condition. This approximation is consistent with the nature of such fluids and walls. However, there exist some flow phenomena, modelling of which might require introduction of slip and/or leak boundary conditions in reality or apparently (or metaphorically). Examples are flow through a drain or canal with its bottom covered by sherbet of mud and pebbles, flow of melted iron coming out from a smelting furnace, flow through a net or sieve, flow through a filter, and water flow in a purification plant etc.

Furthermore, among these phenomena there are those cases where the non-trivial movements, say leak or slip, take place only when magnitude of

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the stress at boundary surpasses a threshold. Also, the slip boundary condition was frequently used in free boundary problems containing dynamic or static contact lines, see [2, 8, 9, 11].

In this paper, we restrict our consideration to the flow governed by the Navier-Stokes equation with slip boundary conditions of friction type where the transition from the trivial adhesive state to a non-trivial movement on the boundary depends on the magnitude of the total stress there. One reason of such restriction is our intention to focus on the characteristic difficulties caused solely by the slip boundary conditions of friction type and also the influence of inertia terms to generalize the results obtained by H. Fujita and H. Kawarada in [5]. The study of the Navier-Stokes equation with leak boundary conditions of friction type will be considered in a forthcoming paper [10].

The paper is structured as follows. In Section 2, we describe our problem, including the definition of the slip boundary conditions of friction type. Some preliminaries concerning the functional space framework and the bilinear-trilinear form are introduced in Section 3. In section 4, the weak formulation in variational terms of inequality of the second kind and main results are stated. The final Section 5 is devoted to the proof of the main theorem concerning the existence and uniqueness of weak solutions.

2. Description of the Problem

We consider fluid motions in a bounded domain Ω in $\mathbb{R}^N$ ($N = 2$ or 3). We suppose that the boundary $\partial\Omega = \Gamma$ of Ω is composed of two separate portions (connected compact components of Γ) Γ_0 and Γ_1.

As mentioned in Section 1, throughout the present paper we deal with the stationary flow governed by the Navier-Stokes equation which is written in a familiar form as follows.

$$\begin{cases} -\nu\Delta u + \rho(u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div}(u) = 0. \end{cases} \tag{2.1}$$

Here, $u = (u_i)_{i=1,\dots,N}$ is the velocity field, ρ the density, p the pressure, f the external force. The positive constant ν stands for the kinetic viscosity.

The boundary condition is prescribed on the part of the boundary where the fluid adheres to the wall:

$$u = 0 \quad \text{on } \Gamma_0. \tag{2.2}$$

In the remaining part of the boundary we assume a slip boundary condition of friction type on Γ_1 (see [4]):

$$u_n = 0, \quad -\sigma_\tau \in g\partial\big|u_\tau\big| \ \text{ on } \Gamma_1. \tag{2.3}$$

Here, g is a given positive functions, $u_n \equiv u \cdot n$ and $u_\tau \equiv u - u_n n$ are the normal and tangential components of the velocity, respectively, where

$n = (n_i)_{i=1,\dots,N}$ is the outward unit normal on the boundary Γ_1, $\sigma_\tau = \sigma_\tau(u)$ denotes tangential components of the stress vector (the precise definitions will be recalled in Section 3); and finally $\partial|z|$ with $z \in \mathbb{R}^N$ denotes a graph

$$\partial|z| = \begin{cases} \dfrac{z}{|z|}, & \text{if } z \neq 0, \\ \{a \in \mathbb{R}^N, \quad |a| \leq 1\}, & \text{if } z = 0. \end{cases} \tag{2.4}$$

Remark 1. Notice that the second condition of (2.3) is equivalent to

$$|\sigma_\tau| \leq g, \quad \sigma_\tau \cdot u_\tau + g|u_\tau| = 0 \quad \text{on } \Gamma_1; \tag{2.5}$$

which implies that, for arbitrary smooth v,

$$\sigma_\tau \cdot (v_\tau - u_\tau) + g|v_\tau - u_\tau| \geq 0 \quad \text{on } \Gamma_1.$$

The purpose of this paper is concerned with the slip boundary value problem of friction type, which is composed of (2.1), (2.2) and (2.3). This problem appears in modelling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks [4].

3. Preliminaries

3.1. Notation

The deformation tensor $e(u) = (e_{ij}(u))$ and stress tensor $S(u,p) = (S_{ij}(u,p))$ associated with a velocity field $u = (u_i)$ and pressure p are denoted by

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ and } S_{ij}(u,p) = -p\delta_{ij} + 2\nu e_{ij}(u),$$

respectively, where δ_{ij} denotes Kroneker's delta. The stress vector $\sigma(u,p)$ is defined by $\sigma(u,p) = S \cdot n$ of which the i-th component is $\sum_{j=1}^N S_{ij}(u,p)n_j$.

In general, the normal and tangential components of a vector field u are given as $u_n = u \cdot n$ and $u_\tau = u - u_n n$, respectively. In particular,

$$\sigma_n(u,p) = \sigma(u,p) \cdot n \quad \text{and} \quad \sigma_\tau(u) = \sigma(u,p) - \sigma_n(u,p)n$$

are the normal and tangential components of the stress vector, respectively.

We will use the $L^2(\Omega)$ space and usual Sobolev space $H^1(\Omega)$. We put

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \quad \int_\Omega q\,dx = 0\}.$$

We write $\|.\|_{1,\Omega}$ instead of $\|.\|_{H^1(\Omega)}$.

We also use the Sobolev space $H^s(\Gamma_0)$ defined on the boundary Γ_0, where $s \in \mathbb{R}$. We write $\|.\|_{s,\Gamma_0} = \|.\|_{H^s(\Gamma_0)}$. $H^0(\Gamma_0)$ is understood as $L^2(\Gamma_0)$. The surface element of Γ_0 is denoted by ds, that is

$$\|\eta\|^2_{0,\Gamma_0} = \int_{\Gamma_0} |\eta|^2 \, ds.$$

We write $(.,.) = (.,.)_{L^2(\Omega)}$ and $(.,.)_{\Gamma_0} = (.,.)_{L^2(\Gamma_0)}$.

Let Tr be the trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma_0)$. Then the trace Tr v on Γ_0 of $v \in H^1(\Omega)$ is denoted by $v|_{\Gamma_0}$. If it is clear from the context, we will not distinguish v from $v|_{\Gamma_0}$. The meaning of $v|_{\Gamma_1}$ is similar.

In general, for a Hilbert space X, the adjoint space is denoted by X^*, and X^N denotes the set of vector $v = (v_1, \dots, v_N), v_j \in X$. For vector functions, we use same symbol to indicate their inner product and norm; $(.,.)_X = (.,.)_{X^N}$ and $\|.\|_X = \|.\|_{X^N}$. We use closed subspaces of $\big(H^1(\Omega)\big)^N$:

$$U = \Big\{ v \in \Big(H^1(\Omega)\Big)^N, \;\; v|_{\Gamma_1} = 0 \Big\}, \quad V = \Big\{ v \in U, \;\; v_n|_{\Gamma_1} = 0 \Big\},$$
$$V_d = \Big\{ v \in V, \;\; \operatorname{div}(v) = 0 \text{ in } \Omega \Big\}.$$

The norm $\|.\|_1$ is equivalent to Dirichlet's norm $\|\nabla.\|$ in U by Poincaré's inequality. We shall not emphasize this in what follows.

Let ψ be a proper ($\psi \not\equiv \infty$) lower semi-continuous convex function defined on $\mathbb{R}^N$. Then, for any $z \in \mathbb{R}^N$, $\partial\psi(z)$ denotes the set

$$\partial\psi(z) = \big\{ h \in \mathbb{R}^N; \quad \psi(z') - \psi(z) \geq h \cdot (z' - z), \quad \forall z' \in \mathbb{R}^N \big\},$$

which is called the subdifferential of ψ at z. It is easy to see that the right-hand side of (2.4) coincides with $\partial\psi(z)$, when $\psi(z) = |z|$ for $z \in \mathbb{R}^N$.

The symbol C_i $(i = 1, \dots)$ denotes various generic constant depending only on Ω.

3.2. Bilinear and trilinear forms

We introduce a bilinear form on $U \times U$ defined as

$$a(u, v) = 2 \int_\Omega e_{ij}(u) e_{ij}(v) \, dx, \quad \forall (u, v) \in U \times U.$$

Here and hereafter the summation convention is employed. Clearly a is continuous on $U \times U$:

$$\big|a(u, v)\big| \leq C_1 \|u\|_1 \, \|v\|_1, \quad \forall (u, v) \in U \times U. \tag{3.1}$$

a is coercive on $U \times U$, that is:

$$a(v, v) \geq C_2 \|v\|_1^2, \quad \forall v \in U. \tag{3.2}$$

In fact, (3.2) is a consequence of Korn's inequality (for example, see [3]).

Now, we introduce a trilinear form on $U \times U \times U$ defined as

$$b(u, v, w) = \big((u \cdot \nabla)v, w\big) = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad \forall (u, v, w) \in U \times U \times U,$$

here b is continuous on $U \times U \times U$, that is:

$$\big|b(u,v,w)\big| \le C_3 \|u\|_1 \, \|v\|_1 \, \|w\|_1, \quad \forall (u,v,w) \in U \times U \times U \tag{3.3}$$

and anti-symmetrical on $V_d \times V_d \times V_d$:

$$b(u,v,w) + b(u,w,v) = 0, \quad \forall (u,v,w) \in V_d \times V_d \times V_d. \tag{3.4}$$

We shall also use a continuous bilinear form on $U \times L^2(\Omega)$

$$c(v,q) = -\int_\Omega q \,\mathrm{div}(v)\, dx, \quad \forall (v,q) \in U \times L^2(\Omega).$$

3.3. Green's formula

If a smooth vector field u and a smooth scalar field p solve (2.1), then by integration by parts,

$$\nu a(u,\varphi) + \rho b(u,u,\varphi) + c(\varphi,p) = \int_{\partial\Omega} \sigma \cdot \varphi \, ds + (f,\varphi), \quad \forall \varphi \in (H^1(\Omega))^N.$$

In particular,

$$\nu a(u,\varphi) + \rho b(u,u,\varphi) + c(\varphi,p) = \int_{\Gamma_1} \sigma_\tau \cdot \varphi_\tau \, ds + (f,\varphi), \quad \forall \varphi \in U. \tag{3.5}$$

Variational inequality (4.2) which will appear in the subsequent section is based on this identity and the definition of subdifferential.

4. Weak Formulation and Main Theorem

We introduce a friction functional as

$$j(\eta) = \int_{\Gamma_1} g|\eta| \, ds, \quad \forall \eta \in \left(H^{\frac{1}{2}}(\Gamma_1)\right)^N. \tag{4.1}$$

We now state the variational formulation of (2.1)-(2.3).

Problem 1. Find $u \in V$ and $p \in L^2(\Omega)$ satisfying $\forall v \in V$

$$\nu a(u, v-u) + \rho b(u,u,v-u) + c(v-u,p) + j(v_\tau) - j(u_\tau) \ge (f, v-u), \tag{4.2}$$

$$c(u,q) = 0, \quad \forall q \in L^2(\Omega). \tag{4.3}$$

Since the functional j is not differentiable, a variational inequality appears.

In order to state the main result of this paper, some assumptions are presented. We suppose that the following assumptions hold:

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset, \quad \Gamma_0 \ne \emptyset; \tag{4.4}$$

$$g \in H^{\frac{1}{2}}(\Gamma_1), \quad g > 0 \text{ a.e. on } \Gamma_1; \tag{4.5}$$

$$f \in \left(L^2(\Omega)\right)^N. \tag{4.6}$$

Let $\mathcal{B}(0,R)$ be the ball of V_d defined by

$$\mathcal{B}(0,R) = \{\xi \in V_d; \quad \|\xi\|_1 \leq R\},$$

where R is a positive constant that we will specify in the following theorem.

Theorem 1. *If we assume* (4.4)–(4.6) *and*

$$\nu > \rho R C_3 / C_2, \tag{4.7}$$

Then there exists a solution $(u,p) \in V \times L^2(\Omega)$ *of Problem 1. Moreover,* u *is unique in* $\mathcal{B}(0,R)$ *for all*

$$R \geq \|f\|/(\nu C_2) \tag{4.8}$$

and p *is also unique up to an additive constant. In particular,* p *such that* $(p,1) = 0$ *is unique.*

The proof of this theorem is based on the reduction of Problem 1 to a problem for a velocity. Then, the result of the existence and uniqueness of the velocity is shown by application to the Banach fixed point theorem. To give the pressure, we use the equivalence results between the problems employed.

5. Proof of Theorem 1

5.1. Existence and uniqueness of u

When we restrict test functions in (4.2) to divergence free functions, we obtain another weak formulation which does not involve p.

Problem 2. Find $u \in V_d$ such that

$$\nu a(u, v-u) + \rho b(u,u,v-u) + j(v_\tau) - j(u_\tau) \geq (f, v-u), \quad \forall v \in V_d.$$

We now consider the following auxiliary problem:

Problem 3. Given $l \in V_d$, find $u^l \in V_d$ such that

$$\nu a(u^l, v-u^l) + \rho b(l, u^l, v-u^l) + j(v_\tau) - j(u^l_\tau) \geq (f, v-u^l), \quad \forall v \in V_d, \tag{5.1}$$

For this problem we have the following result.

Proposition 1. *If we assume* (4.4)–(4.6)*, then there exists a unique solution* $u^l \in V_d$ *of Problem 3.*

Proof. From (3.1), (3.2), (3.3) and (3.4), we have

$$\big|\nu a(u,v) + \rho b(l,u,v)\big| \leq \big(\nu C_1 + \rho C_3 \|l\|_1\big)\|u\|_1\,\|v\|_1, \quad \forall (u,v) \in V_d \times V_d,$$

$$\big|\nu a(u,u) + \rho b(l,u,u)\big| = \big|\nu a(u,u)\big| \geq \nu C_2 \|u\|_1^2, \quad \forall u \in V_d.$$

Then the bilinear form $(u,v) \longmapsto \nu a(u,v) + \rho b(l,u,v)$ is continuous and coercive on $V_d \times V_d$. Moreover, j is a proper, convex, and lower semi-continuous functional on V_d. Then the existence and uniqueness of $u^l \in V_d$ satisfying the variational inequality of the second kind (5.1) is well known and follows from [1, 6]. ■

To solve Problem 2, we use the Banach fixed point theorem. For this, we introduce the mapping $\Lambda : V_d \longrightarrow V_d$ defined by

$$l \longmapsto \Lambda(l) = u^l,$$

where $u_l \in \mathcal{B}(0, R)$ is the unique solution of Problem 3.

Proposition 2. *Under the same assumptions as in Theorem* 1*, there exists a unique solution* $u \in \mathcal{B}(0, R)$ *of Problem 2 for all* R *defined by* (4.8)*.*

Proof. We will determine R such that Λ sends $\mathcal{B}(0, R)$ in $\mathcal{B}(0, R)$. Let $l \in \mathcal{B}(0, R)$ and $u^l = \Lambda(l) \in V_d$ such that (5.1) hold. Taking $v = 0 \in V_d$ in (5.1)), we have

$$\nu a(u^l, u^l) + \rho b(l, u^l, u^l) + j(u^l_\tau) \leq (f, u^l).$$

From (3.4) and the fact that $j(u^l_\tau) \geq 0$, we obtain

$$\nu a(u^l, u^l) \leq (f, u^l).$$

Using (3.2) and Cauchy-Schwarz's inequality, we deduce that

$$\|\Lambda(l)\|_1 = \|u^l\|_1 \leq \|f\|/(\nu C_2).$$

It is thus enough that R satisfied (4.8) so that $\Lambda(\mathcal{B}(0, R)) \subset \mathcal{B}(0, R)$. Let us show that the mapping Λ is strictly contracting on $\mathcal{B}(0, R)$ for a particular choice of the viscosity ν. Let $l_1, l_2 \in \mathcal{B}(0, R)$ and

$$u^{l_1} = \Lambda(l_1), \quad u^{l_2} = \Lambda(l_2) \in \mathcal{B}(0, R)$$

such that

$$\nu a(u^{l_1}, v - u^{l_1}) + \rho b(l_1, u^{l_1}, v - u^{l_1}) + j(v_\tau) - j(u^{l_1}_\tau) \geq (f, v - u^{l_1}), \quad \forall v \in V_d,$$

$$\nu a(u^{l_2}, w - u^{l_2}) + \rho b(l_2, u^{l_2}, w - u^{l_2}) + j(w_\tau) - j(u^{l_2}_\tau) \geq (f, w - u^{l_2}), \forall w \in V_d.$$

Taking $v = u^{l_2}$ and $w = u^{l_1}$, we obtain

$$\nu a(u^{l_1}, u^{l_2} - u^{l_1}) + \rho b(l_1, u^{l_1}, u^{l_2} - u^{l_1}) + j(u^{l_2}_\tau) - j(u^{l_1}_\tau) \geq (f, u^{l_2} - u^{l_1}),$$

$$\nu a(u^{l_2}, u^{l_1} - u^{l_2}) + \rho b(l_2, u^{l_2}, u^{l_1} - u^{l_2}) + j(u^{l_1}_\tau) - j(u^{l_2}_\tau) \geq (f, u^{l_1} - u^{l_2}).$$

By addition and from (3.4), we have

$$\begin{aligned} \nu a(u^{l_1} - u^{l_2}, u^{l_1} - u^{l_2}) &\leq \rho\big[b(l_1, u^{l_1}, u^{l_2} - u^{l_1}) - b(l_2, u^{l_2}, u^{l_1} - u^{l_2})\big] \\ &= \rho b(l_1 - l_2, u^{l_1}, u^{l_2} - u^{l_1}). \end{aligned}$$

Using (3.2) and (3.3), we obtain

$$\big\|\Lambda(l_1)-\Lambda(l_2)\big\|_1 = \|u^{l_1}-u^{l_2}\|_1 \le \frac{\rho R C_3}{\nu C_2}\|l_1-l_2\|_1.$$

Therefore the mapping Λ is Lipschitz continuous on $\mathcal{B}(0,R)$, and it will be strictly contracting if $\rho R C_3/(\nu C_2) < 1$. So if condition (4.7) is realized, then by the Banach fixed point theorem, we have the existence and uniqueness of a solution $u \in \mathcal{B}(0,R)$ of Problem 3. ■

5.2. Existence of p

We introduce the following problem:

Problem 4. Find $u \in V_d$ and $p \in L^2(\Omega)$ such that

$$\nu a(u,\varphi) + \rho b(u,u,\varphi) + c(\varphi,p) = (f,\varphi), \quad \forall \varphi \in \big(H_0^1(\Omega)\big)^N, \tag{5.2}$$

u satisfies the slip boundary conditions of friction type (2.5).

We have the following proposition.

Proposition 3. *Problem 1 and Problem 4 are equivalent.*

Proof. Let $(u,p) \in V_d \times L^2(\Omega)$ be a solution of Problem 1. Then substituting into (4.2) $v = u \pm \varphi$ with $\varphi \in (H_0^1(\Omega))^N$, we have

$$\nu a(u,\pm\varphi) + \rho b(u,u,\pm\varphi) + c(\pm\varphi,p) \ge (f,\pm\varphi),$$

thus (5.2) is satisfied. Now we will verify that u satisfies the slip boundary conditions of friction type (2.5). Using Green's formula, we can rewrite (4.2) as follows

$$\int_{\Gamma_1} \sigma_\tau \cdot (v_\tau - u_\tau)\, ds + j(v_\tau) - j(u_\tau) \ge 0, \quad \forall v \in V. \tag{5.3}$$

Putting $v = u + \varphi$ with $\varphi \in V$ and substituting it into (5.3), we obtain

$$\int_{\Gamma_1} \sigma_\tau \cdot \varphi_\tau\, ds + \int_{\Gamma_1} g\big(|u_\tau + \varphi_\tau| - |u_\tau|\big)\, ds \ge 0. \tag{5.4}$$

It follows by means of an elementary property of $|.|$ that

$$-\int_{\Gamma_1} \sigma_\tau \cdot \varphi_\tau\, ds \le \int_{\Gamma_1} g|\varphi_\tau|\, ds. \tag{5.5}$$

In view of the inequality (5.5) with φ replaced by $-\varphi$ and of the original inequality (5.5), we have eventually

$$\left|\int_{\Gamma_1} \sigma_\tau \cdot \varphi_\tau\, ds\right| \le \int_{\Gamma_1} g|\varphi_\tau|\, ds.$$

This implies that functional σ_τ on $H^{\frac{1}{2}}(\Gamma_1)$ can be extended by continuity to a bounded functional on the Banach space

$$\mathcal{X} \equiv L^1_g(\Gamma_1) = \left\{ \xi; \quad \int_{\Gamma_1} g|\xi|\, ds < +\infty \right\} \quad \text{with} \quad \|\xi\|_{\mathcal{X}} = \int_{\Gamma_1} g|\xi|\, ds$$

and that its functional norm ≤ 1. Since the dual space $\mathcal{X}^*$ of $\mathcal{X}$ can be identified with the Banach space

$$\mathcal{X}^* \equiv L^\infty_g(\Gamma_1) = \left\{ \xi; \quad \operatorname*{ess.\,sup}_{\Gamma_1} \frac{|\xi(s)|}{g(s)} < +\infty \right\}$$

with

$$\|\xi\|_{\mathcal{X}^*} = \operatorname*{ess.\,sup}_{\Gamma_1} \frac{|\xi(s)|}{g(s)},$$

we have $\sigma_\tau \in L^\infty_g(\Gamma_1)$ with its norm ≤ 1, namely, we have $|\sigma_\tau| \leq g$ almost everywhere on Γ_1, obtaining the first relation of (2.5). Then, putting $\varphi = -u$ in (5.4), we have

$$-\int_{\Gamma_1} \sigma_\tau . u_\tau\, ds - \int_{\Gamma_1} g|u_\tau|\, ds \geq 0,$$

by using the first relation of (2.5), we get

$$\int_{\Gamma_1} \Big(\sigma_\tau . u_\tau + g|u_\tau| \Big)\, ds = 0$$

and hence $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$ almost everywhere on Γ_1 as desired.

Let $(u, p) \in V_d \times L^2(\Omega)$ be a solution of Problem 4. It suffices to verify (4.2). This is immediate in view of (2.5) and (3.5), as

$$\begin{aligned}
&\nu a(u, v-u) + \rho b(u, u, v-u) + c(v-u, p) + j(v_\tau) - j(u_\tau) - (f, v-u) \\
&\quad = \int_{\Gamma_1} \sigma_\tau \cdot (v_\tau - u_\tau)\, ds + j(v_\tau) - j(u_\tau) \\
&\quad = \int_{\Gamma_1} (\sigma_\tau \cdot v_\tau + g|v_\tau|)\, ds - \int_{\Gamma_1} (\sigma_\tau \cdot u_\tau + g|u_\tau|)\, ds \\
&\quad = \int_{\Gamma_1} \Big(\sigma_\tau \cdot v_\tau + g|v_\tau| \Big)\, ds \geq 0.
\end{aligned}$$

■

Proposition 4. *Let u be a solution of Problem 2. Then there exists $p \in L^2(\Omega)$ such that (u, p) solves Problem 1. Moreover, p is unique except for an additive constant. In particular, p such that $(p, 1) = 0$ is unique.*

Proof. Let u be a solution of Problem 2, we prove as in Proposition 3 that u is the solution of the following problem:

Problem 5. Find $u \in V_d$ such that

$$\nu a(u, \varphi) + \rho b(u, u, \varphi) = (f, \varphi), \quad \forall \varphi \in (H^1_{0,d}(\Omega))^N,$$

and u satisfies the slip boundary conditions of friction type (2.5). Here,

$$H^1_{0,d}(\Omega) = \Big\{ v \in H^1_0(\Omega); \quad \operatorname{div}(v) = 0 \text{ in } \Omega \Big\}.$$

In particular, u satisfies

$$\int_{\Omega}\Big[-2\nu\frac{\partial}{\partial x_j}(e_{ij}(u))+\rho u_j\frac{\partial u_i}{\partial x_j}-f_i\Big]\varphi_i\,dx=0,$$

$$\forall\varphi\in\Big\{v\in\Big(\mathcal{D}(\Omega)\Big)^N;\quad \mathrm{div}(v)=0\text{ in }\Omega\Big\}.$$

Then it is well-known (see [12]) that there exists $p\in H^{-1}(\Omega)$ such that

$$-2\nu\frac{\partial}{\partial x_j}\big(e_{ij}(u)\big)+\rho u_j\frac{\partial u_i}{\partial x_j}-f_i=\frac{\partial p}{\partial x_i}\text{ a. e. in }\Omega,\ i=1,\dots,N. \tag{5.6}$$

Multiplying (5.6) by any $\varphi\in(H_0^1(\Omega))^N$ and using Green's formula, we show easily that (u,p) is solution of Problem 4. According to Proposition 3 we deduce that (u,p) is also solution of Problem 1. ■

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