Mathematical Modelling and Analysis
Volume 12 Number 4, 2007, pages 399–408
 ISSN 1392-6292 print, ISSN 1648-3510 online

A SPLITTING TYPE ALGORITHM FOR NUMERICAL SOLUTION OF PDES OF FRACTIONAL ORDER

N. ABRASHINA-ZHADAEVA and N. ROMANOVA

Belarusian State University

Fr.Skaryny ave, 4, Minsk, Belarus

E-mail: zhadaeva_natalja@mail.ru

Received August 8, 2007; revised October 17, 2007; published online December 15, 2007

Abstract. Fractional order diffusion equations are generalizations of classical diffusion equations, treating super-diffusive flow processes. In this paper, we examine a splitting type numerical methods to solve a class of two-dimensional initial-boundary value fractional diffusive equations. Stability, consistency and convergence of the methods are investigated. It is shown that both schemes are unconditionally stable. A numerical example is presented.
Key words: fractional partial differential equation, finite difference approximation, splitting scheme, stability analysis

1. Introduction

Recently a lot of attention is given to formulation and analysis of mathematical models described by PDEs of a fractional order. Such models can be used when the classical diffusion equation is inadequate to model real situations where a particle plume spreads faster than the classical model predicts. Thus fractional type diffusion models are a generalization of the classical models of the parabolic type. The theory of PDEs with fractional derivatives has a long history and many analytical methods are developed for integration of such equations. But these methods are not effective when real-world applications are investigated. Then numerical methods should be used.

In this paper we present splitting integration schemes, which are based on a well-known method of additive finite-difference schemes (see [3, 12, 15]). Additive schemes are widely used in construction of new efficient numerical algorithms targeted for numerical solution of mathematical models describing important applications in technology and industry.

In many mathematical models of physics, biochemistry, geophysics we deal with an *anomalous* diffusion, which is simulated by derivatives of the solution of the fractional (non-integer) order, see e.g. [8, 13]. A number of such processes is constantly increasing, we mention only the contaminant transport in ground water, diffusion of water through the membrane of cells in biology or description of the behaviour of animals [7]. In 2000 Kirchner have demonstrated examples of sub-diffusion in hydrology. These examples proved that the velocity of movement of contaminants in the underground water was much faster than the predictions given by the classical Fick law (the porous media was strongly non-homogeneous). He proposed a modified model of the diffusion which correctly described a time required to clean-up an environment after ecological catostrophies or to reduce the level of pollutants from chemical plants to non-dangerous one.

In [9] a new theory is developed which is based on the assumption that the jumps of particles have power-law probability and the standard deviation is infinite. Here the difference between the classical diffusion and the anomalous diffusion is investigated and it is shown that a classical model should be replaced by the fractional diffusion model in order to simulate accurately the transport of contaminants in the underground water in high non-homogeneous porous media. In addition some results on the sub-diffusion of fluctuations in protein-systems are presented in [9]. Here the distance between donors and acceptors (one-cell proteins) constantly changes. The obtained model of the motion is totally different from the Brownian motion.

In [2] a phenomena of sub-diffusion is described for the diffusion of proteins through the cell membrane.

We note that mathematical apparatus for fractional integration and differentiation is studied for a quite long time, but analytical methods are not very effective for the analysis of many models describing real world problems [8, 13]. This paper is devoted to development of numerical algorithms for solution of such problems and it is a modification and development of methods proposed in [4, 5, 6].

2. Problem Formulation

Let us examine a two-sided fractional diffusion equation

$$\frac{\partial u}{\partial t} = C^1\Big[(1-p_1)\frac{\partial^\alpha u}{\partial_- x_1^\alpha} + p_1\frac{\partial^\alpha u}{\partial_+ x_1^\alpha}\Big] + C^2\Big[(1-p_2)\frac{\partial^\alpha u}{\partial_- x_2^\beta} + p_2\frac{\partial^\beta u}{\partial_+ x_2^\beta}\Big] + f, \quad (2.1)$$

where $u = u(x,t)$, $C^i = C^i(x) > 0, (i = 1,2), f = f(x,t)$,

$$x = (x_1, x_2) \in P = \{x_{1L} \le x_1 \le x_{1R}, \;\; x_{2L} \le x_2 \le x_{2R}\},$$

and $\dfrac{\partial^\alpha u}{\partial_+ x_{1,2}^\alpha}$, $\dfrac{\partial^\alpha u}{\partial_- x_{1,2}^\beta}$ are left-handed (+) and right–handed (−) fractional derivatives, $p_i \in [0,1]$, $i = 1,2$. Let us denote $C^i(1-p_i) = c^i_-(x)$, $C^i p_i = c^i_+(x)$, $i = 1,2$. Homogeneous equation (2.1) with constant coefficients defines a transit density of the stability operator for the Levy processes. The independent stability of each component is of order α, β, these constants are asymmetrically defined by p_1, p_2 (see [6, 14]). Such processes present a stochastic model for the anomal diffusion when the clustering of independent jumps in each coordinate is taken into account [14]. The weights p_1, p_2 define probabilities of the jump in the positive directions x_1, x_2, respectively, and $(1-p_1)$, $(1-p_2)$ define probabilities of the jump in the negative directions.

The Dirichlet boundary conditions are defined on the boundary of P

$$u(x,t) = 0, \quad x \in \partial P, \tag{2.2}$$

and the following initial condition is given:

$$u(x,0) = \varphi(x). \tag{2.3}$$

It is well-known, that fractional derivatives can be defined in different ways [13]. In our paper the left–handed and the right–handed fractional derivatives are the Riemann–Liouville fractional derivatives of order α defined as

$$(D^\alpha_{L+}v)(x,t) = \frac{\partial^\alpha v}{\partial_+ x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_L^x \frac{v(\xi,t)\,d\xi}{(x-\xi)^{\alpha+1-n}}, \tag{2.4}$$

$$(D^\alpha_{R-}v)(x,t) = \frac{\partial^\alpha v}{\partial_- x^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_x^R \frac{v(\xi,t)\,d\xi}{(\xi-x)^{\alpha+1-n}}, \tag{2.5}$$

where $v(x,t)$ is a given function defined on $[L,R]\times[0,T]$, n is an integer such that $n-1<\alpha\le n$, and $\Gamma(z)$ is the gamma function. If $\alpha = k$ is an integer number, then definitions (2.4), (2.5) reduce to the standard derivatives, that is

$$(D^k_{L+}v)(x,t) = \frac{\partial^k v}{\partial x^k}, \quad (D^k_{R-}v)(x,t) = (-1)^k \frac{\partial^k v}{\partial x^k} = \frac{\partial^k v}{\partial (-x)^k}.$$

It is easy to see that if $\alpha = \beta = 2$ and setting $c^i(x) = c^i_-(x) + c^i_+(x)$, $i = 1,2$, equation (2.1) becomes the classical diffusion equation

$$\frac{\partial u}{\partial t} = c^1 \frac{\partial^2 u}{\partial x_1^2} + c^2 \frac{\partial^2 u}{\partial x_2^2} + f.$$

In the case of $\alpha = \beta = 1$ and setting $c^i = -c^i_-(x) + c^i_+(x)$, $i = 1,2$, equation (2.1) gives the following hyperbolic transport equation

$$\frac{\partial u}{\partial t} = c^1 \frac{\partial u}{\partial x_1} + c^2 \frac{\partial u}{\partial x_2} + f.$$

3. Scheme of Approximate Factorization

Let us define in P a uniform space grid

$$\omega_h = \big\{(x_{1i}, x_{2j}) : x_{1i} = x_{1L} + ih_1,\ x_{2j} = x_{2L} + jh_2,\ h_k = \frac{x_{kR} - x_{kL}}{N_k},\ k = 1,2\big\}$$

and in the interval $0 \le t \le T$ a uniform time grid

$$\omega_h = \{t_n: \ \ t_n = n\tau, \ n = 0, 1, \ldots, N, \ \ t_N = T\}.$$

Let us denote discrete functions:

$$y^n = y^n_{ij} = y(X_{ij}, t_n) = y(x_{1i}, x_{2j}, t_n),$$
$$c^k_{+ij} = c^k_+(X_{ij}), \ \ c^k_{-ij} = c^k_-(X_{ij}), \ k = 1, 2, \ \ f^n = f^n_{ij} = f(X_{ij}, t_n).$$

For the approximation of left-handed fractional derivative (2.4) we use an one-shift Grünwald formula [6]

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{h_1^\alpha} \lim_{N_1 \to \infty} \sum_{k=0}^{N_1} g_{\alpha,k}\, u\big(x^1 - (k-1)h_1,\, x^2,\, t\big),$$

where g_k are the Grünwald weights

$$g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \binom{\alpha}{k}, \quad h_1 = \frac{x_1 - x_{1L}}{N_1}, \quad 1 < \alpha \le 2.$$

The accuracy of the truncated formula is estimated as

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{1}{h_1^\alpha} \sum_{k=0}^{N_1} g_{\alpha k} u(x_1 - (k-1)h_1, x_2, t) + \mathcal{O}(h_1).$$

Let us define the finite-difference operators:

$$\hat{A}_1 y^n_{ij} = \frac{C^1_{ij}}{h_1^\alpha}\Big[(1-p_1)\sum_{k=0}^{i+1} g_{\alpha k} y^n_{i-k+1,j} + p_1 \sum_{k=0}^{N_1-i+1} g_{\alpha k} y^n_{i+k-1,j}\Big],$$

$$\hat{A}_2 y^n_{ij} = \frac{C^2_{ij}}{h_2^\beta}\Big[(1-p_2)\sum_{k=0}^{j+1} g_{\beta k} y^n_{i,j-k+1} + p_2 \sum_{k=0}^{N_2-j+1} g_{\beta k} y^n_{i,j+k-1}\Big].$$

We approximate differential problem (2.1)–(2.3) by the following finite-difference scheme, which is written in a canonical form [12]

$$\hat{B}\frac{y^{n+1} - y^n}{\tau} = \hat{A} y^n + f, \tag{3.1}$$

where $\hat{A} = \hat{A}_1 + \hat{A}_2$, $\hat{B} = (E - \tau\hat{A})$. For sufficiently smooth solutions scheme (3.1) approximates fractional differential equation (2.1) with the accuracy $\mathcal{O}(\tau + h_1 + h_2)$.

As in the case $\alpha = \beta = 2$ for two-dimensional parabolic problems, it is very important to define an efficient algorithm to solve (3.1) at each time level. Splitting algorithms are very popular tools in this area (see [3, 12, 15]).

Taking into account that for $u \in W^1_r(R^2)$, $(x_1, x_2) \in R^2$, $r > \alpha = \beta + 3$, the mixed derivative can be estimated by [5]

$$\frac{\partial^\beta}{\partial {x_2}^\beta}\frac{\partial^\alpha u(x_1,x_2,t)}{\partial {x_1}^\alpha} = h_1^{-\alpha} h_2^{-\beta}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{(k+l)}\binom{\alpha}{k}\binom{\beta}{l} \times u\bigl(x^1-(k-1)h_1,\, x^2-(l-1)h_2,\, t\bigr) + O(h_1+h_2),$$

we change operator $\hat{B}$ in (3.1) by the factorized operator

$$B = (E-\tau\hat{A}_1)(E-\tau\hat{A}_2) = \hat{B} + \tau^2\hat{A}_1\hat{A}_2.$$

It is obvious that the approximation error of the factorized finite-difference scheme

$$(E-\tau\hat{A}_1)(E-\tau\hat{A}_2)\,\frac{y^{n+1}-y^n}{\tau} = (\hat{A}_1+\hat{A}_2)y^n + f^{n+1}, \tag{3.2}$$

with $f^{n+1} = f + \mathcal{O}(\tau + h_1 + h_2)$, is of the same order as of the basic scheme (3.1). Each factorized scheme can be implemented in a few different ways, when a multidimensional (or a multi-process) problem is reduced to a sequence of simple (in most cases one-dimensional) problems. We will use a simple splitting algorithm:

$$\begin{cases} \dfrac{y_{ij}^{n+1/2}-y_{ij}^{n}}{\tau} = A_1 y_{ij}^{n+1/2} + f^{n+1}, \\ \dfrac{y_{ij}^{n+1}-y_{ij}^{n+1/2}}{\tau} = A_2 y_{ij}^{n+1}, \end{cases} \tag{3.3}$$

with zero Dirichlet boundary conditions for $y_{ij}^{n+1/2}$.

The order of approximation with respect to time can be increased by using the symmetric version the scheme

$$(E-\frac{\tau}{2}\hat{A}_1)(E-\frac{\tau}{2}\hat{A}_2)y^{n+1} = (E+\frac{\tau}{2}\hat{A}_1)(E+\frac{\tau}{2}\hat{A}_2)y^n + \tau f^n, \tag{3.4}$$

where $f^n = f + \mathcal{O}(\tau^2 + h_1 + h_2)$. Since $\tau\hat{A}_1\hat{A}_2(y^{n+1}-y^n)$ is of order $\mathcal{O}(\tau^2 + h_1 + h_2)$, the same estimate is also valid for the global approximation error of (3.4).

Let us consider an implementation algorithm of the factorized finite-difference scheme (3.4), which coincides with the classical method of alternating directions [15]:

$$(E-\frac{\tau}{2}\hat{A}_1)y^{n+1/2} = (E+\frac{\tau}{2}\hat{A}_2)y^n + \frac{\tau}{2}f^n, \tag{3.5}$$

$$(E-\frac{\tau}{2}\hat{A}_2)y^{n+1} = (E+\frac{\tau}{2}\hat{A}_1)y^{n+1/2} + \frac{\tau}{2}f^n,$$

$$y^0 = \varphi_{ij}, \quad x_{ij}\in\omega_h.$$

The boundary conditions for $y^{n+1/2}$ are written in accordance to [12] and the order of accuracy $\mathcal{O}(\tau^2 + h_1 + h_2)$ is preserved. We also note that the the right-hand side approximations in (3.4) and (3.5) are different, thus these schemes are not equivalent.

4. Stability Analysis

It is well-known, that if boundary conditions on the fractional time step are compatible with the basic factorized scheme, then the accuracy and stability of the discrete scheme depends only on the properties of the main factorized scheme [12, 15]. Thus for stability analysis we can investigate any compatible implementation algorithm of schemes (3.2) and (3.4).

In order to get a matrix representation of algorithm (3.3), we put explicit expressions of $\hat{A}_1$, $\hat{A}_2$ into these equations. Let us introduce the following notation

$$d^1_{ij} = c^1_{+ij}\frac{\tau}{h_1^\alpha}, \quad e^1_{ij} = c^1_{-ij}\frac{\tau}{h_1^\alpha}, \quad d^2_{ij} = c^2_{+ij}\frac{\tau}{h_2^\beta}, \quad e^2_{ij} = c^2_{-ij}\frac{\tau}{h_2^\beta},$$

then we rewrite algorithm (3.3) as:

$$-\left(e^1_{ij}g_{\alpha 0} + d^1_{ij}g_{\alpha 2}\right)y^{n+1/2}_{i-1,j} + \left(1-(d^1_{ij}+e^1_{ij})g_{\alpha 1}\right)y^{n+1/2}_{ij} - \left(e^1_{ij}g_{\alpha 2} + d^1_{ij}g_{\alpha 0}\right)$$
$$\times\, y^{n+1/2}_{i+1,j} - d^1_{ij}\sum_{k=3}^{i+1} g_{\alpha k}y^{n+1/2}_{i-k+1,j} - e^1_{ij}\sum_{k=3}^{N_1-i+1} g_{\alpha k}y^{n+1/2}_{i+k-1,j} = y^n_{ij} + \tau f^{n+1}_{ij}, \quad (4.1)$$
$$y^{n+1/2}_{0j} = 0, \quad y^{n+1/2}_{N_1 j} = 0, \quad y^0_{ij} = \varphi_{ij}, \quad i = \overline{1, N_1-1}, \quad j = \overline{1, N_2-1},$$

$$-\left(e^2_{ij}g_{\beta 0} + d^2_{ij}g_{\beta 2}\right)y^{n+1}_{i,j-1} + \left(1-(d^2_{ij}+e^2_{ij})g_{\beta 1}\right)y^{n+1}_{ij} - (e^2_{ij}g_{\beta 2} + d^2_{ij}g_{\beta 0})y^{n+1}_{i,j+1}$$
$$- d^2_{ij}\sum_{k=3}^{j+1} g_{\beta k}y^{n+1/2}_{i,j-k+1} - e^2_{ij}\sum_{k=3}^{N_1-i+1} g_{\beta k}y^{n+1/2}_{i,j+k-1} = y^{n+1/2}_{ij}, \quad (4.2)$$
$$y^{n+1}_{i0} = 0, \quad y^{n+1}_{i,N_2} = 0, \quad i = \overline{0, N_1}, \quad \overline{0, N_2}.$$

Lemma 1. *One-dimensional problems* (4.1) *and* (4.2) *are unconditionally stable for* $1 < \alpha, \beta < 2$.

Proof. Let us consider problem (4.1) for a fixed $j = j_0$, $j_0 = \overline{1, N_2-1}$. We apply a matrix stability analysis for the linear system of equations

$$A_1^{j_0} Y^{n+1/2}_{j_0} = R^{n+1}_{j_0}$$

arising from the finite-difference scheme (4.1), where

$$Y^{n+1/2}_{j_0} = \left[y^{n+1/2}_{1j_0}, y^{n+1/2}_{2j_0}, \ldots, y^{n+1/2}_{N_1-1j_0}\right]^T,$$
$$R^{n+1}_{j_0} = \left[r^{n+1}_{1j_0}, r^{n+1}_{2j_0}, \ldots, r^{n+1}_{N_1-1,j_0}\right]^T, \quad r^{n+1}_{ij_0} = y^n_{ij_0} + \tau f^{n+1}_{ij_0},$$

$A_1^{j_0} = (a^1_{im})$ is the matrix of coefficients, its dimension $(N_1-1)\times(N_1-1)$, the coefficients are defined by $a^1_{00} = 1$, $a^1_{0m} = 0$ for $m = 1, \ldots, N_1$, and $a^1_{N_1N_1} = 1$, $a^1_{N_1m} = 0$ for $m = 0, \ldots, N_1-1$, and

$$a_{im}^1 = \begin{cases} -(d_{ij_0}^1 g_{\alpha 2} + e_{ij_0}^1 g_{\alpha 0}), & m = i-1, \\ 1 - (d_{ij_0}^1 + e_{ij_0}^1) g_{\alpha 1}, & m = i, \\ -(d_{ij_0}^1 g_{\alpha 0} + e_{ij_0}^1 g_{\alpha 2}), & m = i+1, \\ -d_{ij_0}^1 (g_{\alpha i-m+1}, & m < i-1, \\ -e_{ij_0}^1 g_{\alpha m-i+1}, & m > i+1. \end{cases} \tag{4.3}$$

Note that $g_{\alpha 1} = -\alpha$, $g_{\alpha i} > 0$, $\sum_{i=0}^{\infty} g_{\alpha i} = 0$. Thus we have that $g_{\alpha 1}$ satisfies the inequality

$$-g_{\alpha 1} > \sum_{k=0, k \neq 1}^{N_1} g_{\alpha k}.$$

It follows from the estimates given above that

$$a_{ii}^1 = 1 - (d_{ij_0}^1 + e_{ij_0}^1) g_{\alpha 1} = 1 + (d_{ij_0}^1 + e_{ij_0}^1)\alpha,$$

$$r_i = \sum_{m=0, m\neq i}^{N_1} |a_{im}^1| = \sum_{m=0, m\neq i}^{i+1} d_{ij_0}^1 g_{\alpha m} + \sum_{m=0, m\neq i}^{N_1-i+1} e_{ij_0}^1 g_{\alpha m} < (d_{ij_0}^1 + e_{ij_0}^1)\alpha.$$

According to the Greschgorin theorem [3] the eigenvalues of the matrix $A_1^{j_0}$ lie in the union of the circles centered at a_{kk}^1:

$$|z - a_{kk}^1| \le r_k, \quad r_k = \sum_{i=1, i\neq k}^{n} |a_{ki}^1|, \quad k = 1, \ldots, n.$$

This implies that the eigenvalues of the matrix $A_1^{j_0}$ are all no less than one in magnitude. Hence the spectral radius of the inverse matrix

$$\rho\big((A_1^{j_0})^{-1}\big) < 1, \quad j_0 = 1, \ldots, N_2 - 1.$$

Thus the error in Y_{j0}^n is not magnified and scheme (4.1) is unconditionally stable. A similar result is valid for the second splitting step (4.2). The lemma is proved. ■

We write scheme (3.2) in the following form

$$A_1 A_2 Y^{n+1} = Y^n + F^{n+1},$$

where $A_j = E - \tau \hat{A}_j$, $j = 1, 2$ and

$$Y^n = \big[y_{11}^n, y_{21}^n, \ldots, y_{N_1-11}^n, y_{12}^n, y_{22}^n, \ldots, y_{1N_2-1}^n, y_{2N_2-1}^n, \ldots, y_{N_1-1N_2-1}^n\big]^{\mathrm{T}},$$

F^{n+1} consists of f_{ij}^{n+1} and it also includes terms from boundary conditions. To illustrate the matrix pattern of A_j we note that

$$A_1 = \mathrm{diag}\big(A_1^1, A_1^2, \ldots, A_1^{N_2-1}\big),$$

with $(N_1 - 1) \times (N_1 - 1)$ blocks $A_1^{j_0}$, $j_0 = 1, \ldots, N_2 - 1$. Matrix A_2 is defined similarly.

Theorem 1. *If A_1 and A_2 commute, i.e. the equality $A_1A_2 = A_2A_1$ is satisfied, then scheme* (3.2) *with $1 < \alpha, \beta < 2$ is unconditionally stable.*

Proof. We consider the stability of (3.2) with respect to the error in the initial condition. Let δ^0 be the error of Y^0 in system (4.3), then the error of Y^n is defined by

$$\delta^n = \big(A_1A_2\big)^{-n}\delta^0.$$

Using the commutativity property we get that

$$\delta^n = (A_1^{-1})^n\,(A_2^{-1})^n\,\delta^0.$$

It is well-known (see [1]) that in order to have $(A^{-1})^n \to 0$, it is necessary and sufficient that $\rho(A^{-1}) < 1$. By using the result of Lemma 1 we get the estimate

$$(A_1^{-1})^n(A_2^{-1})^n \to 0, \quad n \to \infty.$$

The theorem is proved. ■

Remark 1. From the commutativity of A_1 and A_2 we get that this property is also satisfied for $(E - \tau\hat{A}_1)$ and $(E - \tau\hat{A}_2)$. It is well-known that such a condition is typical for AD schemes with $\alpha = \beta = 2$.

Theorem 2. *The solution of splitting scheme* (3.3) *unconditionally converges to the solution of problem* (2.1) *and the convergence rate is $\mathcal{O}(\tau + h_1 + h_2)$.*

Proof. The proof follows from the Lax theorem and the unconditional stability and the consistency of splitting scheme (3.3). ■

Remark 2. It is possible to use a staggered grid when the solution is approximated at grid nodes and the fluxes are approximated at locations offset from grid points in their respective directions.

5. Splitting Scheme of the Second-order Accuracy

Similarly to the analysis given above, it is possible to investigate the symmetric splitting scheme (3.4). It approximates the differential equation with the second-order accuracy in time. For simplicity of presentation let us assume that $p_1 = p_2 = 1/2$, and denote $c_{ij}^1 = \frac{C_{ij}^1\tau}{2h_1^\alpha}$, $c_{ij}^2 = \frac{C_{ij}^2\tau}{2h_2^\beta}$. We rewrite (3.5) in the following form :

$$\begin{aligned}
-c_{ij}^1(g_{\alpha0} + g_{\alpha2})y_{i-1}^{n+1/2} &+ (1 - 2c_{ij}^1 g_{\alpha1})y_{ij}^{n+1/2} - c_{ij}^1(g_{\alpha2} + g_{\alpha0})y_{i+1j}^{n+1/2} \\
&- c_{ij}^1\sum_{k=3}^{i+1} g_{\alpha k}y_{i-k+1j}^{n+1/2} - c_{ij}^1\sum_{k=3}^{N_1-i+1} g_{\alpha k}y_{i+k-1j}^{n+1/2} \\
&= y_{ij}^n + c_{ij}^2\sum_{l=0}^{j+1} g_{\beta l}y_{ij-l+1}^n + c_{ij}^2\sum_{l=3}^{N_2-j+1} g_{\beta l}y_{ij+l-1}^n + \frac{\tau}{2}f_{ij}^n,
\end{aligned}$$

where $i = \overline{1, N_1 - 1}$, $j = \overline{1, N_2 - 1}$.

Let us denote $A_1^{j_0} = (a_{im}^1)$ a matrix of dimension $(N_1 - 1) \times (N_1 - 1)$, here j_0 is fixed, $j_0 = \overline{1, N_2}$. The coefficients of $A_1^{j_0}$ are defined by

$$a_{im}^1 = \begin{cases} 2c_{ij_0}^1 g_{\alpha 1}, & m = i, \\ -c_{ij_0}^1 (g_{\alpha 2} + g_{\alpha 0}), & m = i - 1, \\ -c_{ij_0}^1 (g_{\alpha 0} + g_{\alpha 2}), & m = i + 1, \\ -c_{ij_0}^1 g_{\alpha, i-m+1}, & m < i - 1, \\ -c_{ij_0}^1 g_{\alpha, m-i+1}, & m > i + 1, \end{cases}$$

$a_{00}^1 = 1$, $a_{0m}^1 = 0$, $m = 1, \ldots, N_1$, $a_{N_1 N_1}^1 = 1$, $a_{N_1 m}^1 = 0$, $m = 0, \ldots, N_1 - 1$. To illustrate the matrix pattern of A_1 we note that

$$A_1 = \mathrm{diag}\big(A_1^1,\, A_1^2,\, \ldots,\, A_1^{N_2 - 1}\big),$$

with $(N_1 - 1) \times (N_1 - 1)$ blocks $A_1^{j_0}$, $j_0 = 1, \ldots, N_2 - 1$. Matrix A_2 is defined similarly.

Theorem 3. *If A_1 and A_2 commute, i.e. the equality $A_1 A_2 = A_2 A_1$ is satisfied, then scheme* (3.4) *with $1 < \alpha, \beta < 2$ is unconditionally stable.*

The proof is similar to one presented in the previuos section.

By using the stability and consistency estimates (the boundary conditions must take into account a special form of solutions at the intermediate time-steps and the influence of source term) we get from the Lax theorem that the solution of (3.4) converges to the solution of the fractional differential problem (2.1) and the error is estimated by $\mathcal{O}(\tau^2 + h_1 + h_2)$.

Results of computational experiments are presented in [10, 11], they confirm theoretical results.

References

[1] D.K. Fadeev and V.N. Fadeeva. *Computational methods of linear algebra.* Freeman, London, 1963.

[2] T. Fujiwara, K. Ritchie, H. Murakoshi, K. Jacobson and A. Kusumi. Phospholipids undergo hop diffusion in compartmentalized cell membrane. *The Jour. of Cell Biology*, **157**(6), 1071–1081, 2002.

[3] G.I. Marchuk. *Methods of numerical mathematics. Applications of Mathematics*, volume 2. Springer Verlag, New York, Heidelberg, Berlin, 1973.

[4] M.M. Meerschaert, J. Mortensen and H.-P. Scheffler. Vector Grünwald formula for fractional derivatives. *J. Fract. Calc. Appl. Anal.*, **7**, 61–81, 2004.

[5] M.M. Meerschaert, H. Scheffer and C. Tadjeran. Finite difference methods for two-dimensional fractional dispersion equation. *J. of Comp. Phys.*, **211**, 246–261, 2006.

[6] M.M. Meerschaert and C. Tadjeran. Finite difference approximations for fractional advection-dispersion flow equations. *J. of Comp. and Appl. Mathem.*, **172**, 65–77, 2004.

[7] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A.*, **37**(31), 161–208, 2004.

[8] K. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations.* Wiley, New York, 1993.

[9] W. Min, B.P. English, G. Luo, B.J. Cherayil, S.C. Kou and X.S. Xie. Fluctuating enzymes: lessons from single-molecule studies. *Accounts of Chemical Research*, **38**(12), 923–931, 2005.

[10] N.S. Romanova. Divectorization decomposition model in scientific and applied problems. *J. Works of BSTU. Ser.6*, **15**, 28–32, 2007. (in Russian)

[11] N.S. Romanova. Divectorization of the numerical decomposition model for fractional order partial differential equations. *J. Vesti Instition Sovremen. Znanii.*, 2, 107–111, 2007. (in Russian)

[12] A.A. Samarskii. *The theory of difference schemes.* Marcel Dekker, Inc., New York–Basel, 2001.

[13] S.G. Samko, A.A. Kilbas and O.I. Marichev. *Fractional integrals and derivatives: theory and applications.* Gordon and Breach, London, 1993.

[14] R. Schumer, D.A. Benson, M.M. Meerschaert and S.W. Wheatcraft. Eulerian derivation of the fractional advection-dispersion equation. *J. Contamin. Hydrol.*, **48**, 69–88, 2001.

[15] N.N. Yanenko. *The method of fractional steps. The solution of problems of mathematical physics in several variables.* Springer-Verlag, New York, 1971.