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Cubic Spline Histopolation*

Evely Kirsiaed[a], Peeter Oja[a] and Gul Wali Shah[a]

[a] *University of Tartu, Institute of Mathematics and Statistics*
J. Liivi 2, Tartu
E-mail(*corresp.*): evely.kirsiaed@ut.ee
E-mail: peeter.oja@ut.ee
E-mail: gulwali@ut.ee

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Abstract. Cubic spline histopolation with arbitrary placement of histogram knots and spline knots between them is studied. Classical boundary conditions are used. Histopolating spline is represented with the help of second moments and particular integrals. The systems determining these parameters are investigated in different cases where diagonal dominance in matrices takes place or may be absent.

Keywords: histopolation, cubic spline, existence of histopolant.

AMS Subject Classification: 65D07; 41A15.

1 Introduction

The histopolation problem is more practical than the interpolation problem as, e.g., the statistical information is rather given in form of histograms. On the other hand, for any given histopolation problem an equivalent interpolation problem could be formulated and the derivative of the interpolant is, in fact, the histopolant [19]. We treat in this paper the histopolation problem with cubic splines. However, instead of that, it is possible to solve the corresponding interpolation problem with quartic splines and afterwards calculate its derivative. This additional step in practice accompanies with additional (at least round-off) errors at calculations. Because of that it is preferable to have direct algorithms for finding the cubic spline histopolant. In other words, instead of implicit theory via quartic spline interpolation we develop explicit theory of cubic spline histopolation. We consider most common boundary conditions like given values of the spline and its first and second derivatives in endpoints of given interval.

A wide class at solving differential and integral equations is projection methods. The collocation method is the interpolation projection method and the

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subdomain method is the histopolating projection method. The application of these methods requires detailed description of interpolation and histopolation processes, respectively. It is remarkable that, for the boundary value problem with second order linear differential equation, on uniform mesh, the collocation with cubic splines has the rate $\mathcal{O}(h^2)$ [10, 17, 18] but the subdomain method has $\mathcal{O}(h^4)$ [15, 18]. The subdomain method is very natural if, e.g., the free term function in differential equation is given approximately via mean values on subintervals. The same idea works well in case of Volterra integral equations [5]. These circumstances are a great motivation to give a special attention to histopolation problem with cubic splines.

The histopolation with splines is studied in many papers under different names like area matching interpolation [2, 3, 6], interpolation in the mean [4, 6, 20], interpolation of mean values [11], histospline [20]. The spline histopolation on biinfinite knot sequence is treated in [20]. There are several papers by quartic spline interpolation, e.g., [11,12,13] but the interpolation problem which is equivalent to the histopolation with cubic splines is not treated in them in such extent as we do in current paper.

We confirm with numerical examples the known fact that the cubic spline interpolant and also the cubic spline histopolant (which is, in fact, an interpolant in intermediate points) do not preserve geometrical properties. One way to overcome this disadvantage is to use rational or combined splines [7, 8]. But rational interpolating or histopolating splines do not exist for any data [7, 16], cubic spline interpolants or histopolants exist always. Another idea to preserve geometrical properties is to add some auxiliary spline knots (see, e.g., [14] to preserve monotonicity). In [8] monotonicity is preserved without auxiliary knots by using quadratic and rational spline pieces. The convexity preserving combined spline theory similar to [8] should use cubic spline histopolation which we develop in this paper.

2 The histopolation problem

Let x_i be given points on an interval $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$ and let z_i, $i = 1, \ldots, n$, be given real numbers (histogram heights). Denote $h_i = x_i - x_{i-1}$, $i = 1, \ldots, n$. We consider the problem of finding a function $S : [a, b] \to \mathbb{R}$ such that

$$\int_{x_{i-1}}^{x_i} S(x)dx = z_i h_i, \quad i = 1, \ldots, n \tag{2.1}$$

and S is a cubic spline from the class C^2. Conditions (2.1) are called histopolation conditions.

Since a cubic spline with knots x_i, $i = 0, \ldots, n$, has $n + 3$ free parameters (dimension of the cubic spline space is $n + 3$), and this could not be well combined with (2.1), we choose cubic spline knots as

$$\xi_1 = x_0, \quad \xi_i \in (x_{i-1}, x_i), i = 2, \ldots, n-1, \quad \xi_n = x_n.$$

Then the cubic spline has $n+2$ free parameters. We add to the histopolation conditions two boundary conditions from

$$S(a)=\alpha, \quad S'(a)=\alpha, \quad S''(a)=\alpha,$$
$$S(b)=\beta, \quad S'(b)=\beta, \quad S''(b)=\beta$$

at different endpoints a and b.

3 Representation of the histopolant

Several representations of cubic spline could be considered, but the one which uses second moments and particular integrals is appropriate. Thus, on the interval $[\xi_i, \xi_{i+1}]$ we use four parameters to represent the spline:

$$M_i = S''(\xi_i), \quad M_{i+1} = S''(\xi_{i+1}), \quad \lambda_i = \int_{\xi_i}^{x_i} S(x)dx, \quad \rho_i = \int_{x_i}^{\xi_{i+1}} S(x)dx.$$

Denote $\varepsilon_i = x_i - \xi_i$, $\eta_i = \xi_{i+1} - x_i$, $\delta_i = \varepsilon_i + \eta_i$, $i = 1, \dots, n-1$ (see Figure 1).

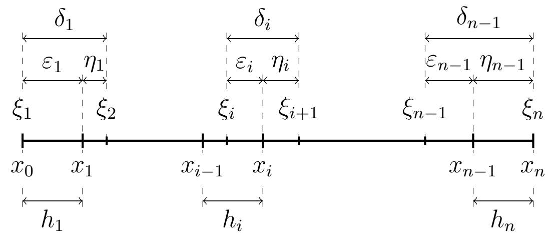

Figure 1. Additional parameters.

Then the spline could be written as

$$S(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3, \quad x \in [\xi_i, \xi_{i+1}],$$
$$i = 1, \dots, n-1, \quad (3.1)$$

where

$$c_i = \frac{M_i\eta_i + M_{i+1}\varepsilon_i}{2\delta_i}, \quad d_i = \frac{M_{i+1} - M_i}{6\delta_i},$$
$$a_i = \frac{1}{\delta_i}\left(\lambda_i\frac{\eta_i}{\varepsilon_i} + \rho_i\frac{\varepsilon_i}{\eta_i}\right) - \frac{c_i}{3}\varepsilon_i\eta_i - \frac{d_i}{4}(\eta_i - \varepsilon_i)\varepsilon_i\eta_i,$$
$$b_i = \frac{2}{\delta_i}\left(\frac{\rho_i}{\eta_i} - \frac{\lambda_i}{\varepsilon_i}\right) - 2\left(\frac{c_i}{3}(\eta_i - \varepsilon_i) + \frac{d_i}{4}(\eta_i^2 - \eta_i\varepsilon_i + \varepsilon_i^2)\right).$$

To determine the parameters M_i, $i = 1, \dots, n$, λ_i, ρ_i, $i = 1, \dots, n-1$, we use smoothness conditions

$$S(\xi_i - 0) = S(\xi_i + 0), \quad i = 2, \dots, n-1, \quad (3.2)$$

$$S'(\xi_i - 0) = S'(\xi_i + 0), \quad i = 2, \dots, n-1, \tag{3.3}$$

histopolation conditions

$$\rho_{i-1} + \lambda_i = z_i h_i, \quad i = 1, \dots, n \tag{3.4}$$

with $\rho_0 = 0$, $\lambda_n = 0$, and two boundary conditions. Equations (3.2) and (3.3) take the form, respectively,

$$\begin{aligned} &-\frac{\eta_{i-1}}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}}\left(2 + \frac{\varepsilon_{i-1}}{\eta_{i-1}}\right)\rho_{i-1} - \frac{1}{\delta_i}\left(2 + \frac{\eta_i}{\varepsilon_i}\right)\lambda_i + \frac{\varepsilon_i}{\delta_i\eta_i}\rho_i \\ &\quad = \frac{1}{24}\Big[-\eta_{i-1}(\varepsilon_{i-1} + 2\eta_{i-1})M_{i-1} \\ &\qquad + \big(-\eta_{i-1}(3\varepsilon_{i-1} + 2\eta_{i-1}) + \varepsilon_i(2\varepsilon_i + 3\eta_i)\big)M_i \\ &\qquad + \varepsilon_i(2\varepsilon_i + \eta_i)M_{i+1}\Big], \quad i = 2, \dots, n-1, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &-\frac{1}{\delta_{i-1}\varepsilon_{i-1}}\lambda_{i-1} + \frac{1}{\delta_{i-1}\eta_{i-1}}\rho_{i-1} + \frac{1}{\delta_i\varepsilon_i}\lambda_i - \frac{1}{\delta_i\eta_i}\rho_i \\ &\quad = -\frac{1}{24}\left[\frac{\delta_{i-1}^2 + \eta_{i-1}\delta_{i-1} + \eta_{i-1}^2}{\delta_{i-1}}M_{i-1}\right. \\ &\qquad + \left(\frac{3\delta_{i-1}^2 + 2\eta_{i-1}\delta_{i-1} + \eta_{i-1}\varepsilon_{i-1}}{\delta_{i-1}} + \frac{3\delta_i^2 + 2\varepsilon_i\delta_i + \varepsilon_i\eta_i}{\delta_i}\right)M_i \\ &\qquad \left. + \frac{\delta_i^2 + \varepsilon_i\delta_i + \varepsilon_i^2}{\delta_i}M_{i+1}\right], \quad i = 2, \dots, n-1. \end{aligned} \tag{3.6}$$

Observe that these equations are linear (homogeneous) with respect to the unknowns λ_{i-1}, ρ_{i-1}, λ_i, ρ_i, M_{i-1}, M_i, M_{i+1}.

4 Systems defining spline parameters

In total, we have to determine $3n-2$ unknowns $M_1, \dots, M_n$, $\lambda_1, \dots, \lambda_{n-1}$, $\rho_1, \dots, \rho_{n-1}$ from the system of $3n-2$ equations: (3.4), (3.5), (3.6) and two boundary conditions. This system is of undetermined form to study. We take 9 equations (3.4,$i-1$), (3.5,$i-1$), (3.6,$i-1$), (3.4,i), (3.5,i), (3.6,i), (3.4,$i+1$), (3.5,$i+1$), (3.6,$i+1$) containing eight unknowns λ_{i-2}, ρ_{i-2}, λ_{i-1}, ρ_{i-1}, λ_i, ρ_i, λ_{i+1}, ρ_{i+1}. These λ_j, ρ_j could be eliminated by using the linear combination of equations with coefficients indicated below:

$$(3.4, i-1) \quad -\frac{h_i + h_{i+1}}{h_{i-1}}, \quad (3.5, i-1) \quad \frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2},$$

$$(3.6, i-1) \quad -\frac{h_i + h_{i+1}}{h_{i-1}}\eta_{i-2}^2, \quad (3.4, i) \quad \frac{h_{i-1} + 2h_i + h_{i+1}}{h_i},$$

$$(3.5, i) \quad \frac{\varepsilon_i(h_i + h_{i+1}) - \eta_{i-1}(h_{i-1} + h_i)}{h_i},$$

$$(3.6, i) \quad \frac{\varepsilon_i^2(h_i + h_{i+1}) + \eta_{i-1}^2(h_{i-1} + h_i)}{h_i} - (h_{i-1} + h_i)(h_i + h_{i+1}),$$

$$(3.4, i+1) \quad -\frac{h_{i-1}+h_i}{h_{i+1}}, \quad (3.5, i+1) \quad -\frac{h_{i-1}+h_i}{h_{i+1}}\varepsilon_{i+1},$$

$$(3.6, i+1) \quad -\frac{h_{i-1}+h_i}{h_{i+1}}\varepsilon_{i+1}^2.$$

For $i = 3, \ldots, n-2$ we obtain the equation

$$c_{i,i-2}M_{i-2} + c_{i,i-1}M_{i-1} + c_{ii}M_i + c_{i,i+1}M_{i+1} + c_{i,i+2}M_{i+2} = D_i, \tag{4.1}$$

where

$$D_i = (h_i + h_{i+1})z_{i-1} - (h_{i-1} + 2h_i + h_{i+1})z_i + (h_{i-1} + h_i)z_{i+1}, \tag{4.2}$$

$$c_{i,i-2} = \frac{1}{24}\frac{\eta_{i-2}^4(h_i + h_{i+1})}{\delta_{i-2}h_{i-1}}, \tag{4.3}$$

$$\begin{aligned} c_{i,i-1} = \frac{1}{24}\Bigg(\Bigg(&\eta_{i-2}(3\eta_{i-2} + 2\varepsilon_{i-1} + 3\eta_{i-1}) + (\varepsilon_{i-1} + 2\eta_{i-1})(h_{i-1} + \eta_{i-1}) \\ &+ \frac{\eta_{i-2}^2}{h_{i-1}}\left(\frac{\varepsilon_{i-2}\eta_{i-2}}{\delta_{i-2}} + \frac{\varepsilon_{i-1}\eta_{i-1}}{\delta_{i-1}}\right)\Bigg)(h_i + h_{i+1}) \\ &+ \frac{\eta_{i-1}^3\varepsilon_i(h_{i-1} + h_i + h_{i+1})}{\delta_{i-1}h_i} + \frac{\eta_{i-1}^2(h_{i-1} + \eta_{i-1})(\varepsilon_i + h_{i+1})}{\delta_{i-1}}\Bigg), \end{aligned} \tag{4.4}$$

$$\begin{aligned} c_{ii} = \frac{1}{24}\Bigg(\Bigg(&\eta_{i-2}(2\varepsilon_{i-1} + \eta_{i-1}) + (3\varepsilon_{i-1} + 2\eta_{i-1})(h_{i-1} + \eta_{i-1}) \\ &+ \frac{\eta_{i-2}^2\varepsilon_{i-1}^2}{\delta_{i-1}h_{i-1}}\Bigg)(h_i + h_{i+1}) \\ &+ \left(\varepsilon_{i+1}(\varepsilon_i + 2\eta_i) + (2\varepsilon_i + 3\eta_i)(\varepsilon_i + h_{i+1}) + \frac{\eta_i^2\varepsilon_{i+1}^2}{\delta_i h_{i+1}}\right)(h_{i-1} + h_i) \\ &+ \left(\left(3 + \frac{\varepsilon_{i-1}}{\delta_{i-1}}\right)\eta_{i-1} + \left(3 + \frac{\eta_i}{\delta_i}\right)\varepsilon_i\right)\Bigg((h_{i-1} + \eta_{i-1})(\varepsilon_i + h_{i+1}) \\ &+ \eta_{i-1}\varepsilon_i\frac{h_{i-1} + h_i + h_{i+1}}{h_i}\Bigg)\Bigg), \end{aligned} \tag{4.5}$$

$$\begin{aligned} c_{i,i+1} = \frac{1}{24}\Bigg(\Bigg(&\varepsilon_{i+1}(3\varepsilon_i + 2\eta_i + 3\varepsilon_{i+1}) + (2\varepsilon_i + \eta_i)(\varepsilon_i + h_{i+1}) \\ &+ \frac{\varepsilon_{i+1}^2}{h_{i+1}}\left(\frac{\varepsilon_i\eta_i}{\delta_i} + \frac{\varepsilon_{i+1}\eta_{i+1}}{\delta_{i+1}}\right)\Bigg)(h_{i-1} + h_i) \\ &+ \frac{\eta_{i-1}\varepsilon_i^3(h_{i-1} + h_i + h_{i+1})}{\delta_i h_i} + \frac{\varepsilon_i^2(h_{i-1} + \eta_{i-1})(\varepsilon_i + h_{i+1})}{\delta_i}\Bigg), \end{aligned} \tag{4.6}$$

$$c_{i,i+2} = \frac{1}{24}\frac{\varepsilon_{i+1}^4(h_{i-1} + h_i)}{\delta_{i+1}h_{i+1}}. \tag{4.7}$$

Let us notice certain symmetry in equation (4.1). There are symmetric pairs of parameters: $h_{i-1} \leftrightarrow h_{i+1}$, $\delta_{i-2} \leftrightarrow \delta_{i+1}$, $\delta_{i-1} \leftrightarrow \delta_i$, $\eta_{i-2} \leftrightarrow \varepsilon_{i+1}$, $\varepsilon_{i-1} \leftrightarrow \eta_i$,

$\eta_{i-1} \leftrightarrow \varepsilon_i$. Then we see the symmetry between $c_{i,i-2}$ and $c_{i,i+2}$, $c_{i,i-1}$ and $c_{i,i+1}$, inside c_{ii}. However, all coefficients (4.3)–(4.7) are positive.

In case of $i = 2$ we take seven equations (3.4,1), (3.4,2), (3.5,2), (3.6,2), (3.4,3), (3.5,3), (3.6,3) to eliminate six unknowns λ_1, ρ_1, λ_2, ρ_2, λ_3, ρ_3. The coefficients of the appropriate linear combination are as in general case. This leads to the equation

$$c_{21}M_1 + c_{22}M_2 + c_{23}M_3 + c_{24}M_4 = D_2,$$

where D_2 is determined by (4.2), c_{23} and c_{24} by (4.6) and (4.7), respectively. There is certain difference in c_{21} and c_{22} compared to (4.4) and (4.5), but they could be calculated similarly to the general case taking into account also the configuration of the intervals near the endpoint a.

Similar situation takes place in case of $i = n - 1$.

The simplest boundary equation here is $S''(a) = \alpha$ or $M_1 = \alpha$. The other possible boundary conditions, e.g., $S(a) = \alpha$ and $S'(a) = \alpha$, require the calculation of $S(\xi_1 + 0)$ and $S'(\xi_1 + 0)$ as it was done at transformation of (3.2) and (3.3). This should be followed by the elimination of appearing parameters λ_j, ρ_j. Both cases give us the equation

$$c_{11}M_1 + c_{12}M_2 + c_{13}M_3 = D_1 \tag{4.8}$$

with certain expression D_1 depending on α and given histogram parameters. However, (4.8) includes $M_1 = \alpha$.

The boundary conditions at the endpoint b could be treated similarly.

Thus, the spline parameters $M_1, \ldots, M_n$ are determined by the five-diagonal system

$$\begin{cases} c_{11}M_1 + c_{12}M_2 + c_{13}M_3 = D_1, \\ c_{21}M_1 + c_{22}M_2 + c_{23}M_3 + c_{24}M_4 = D_2, \\ c_{i,i-2}M_{i-2} + c_{i,i-1}M_{i-1} + c_{ii}M_i + c_{i,i+1}M_{i+1} + c_{i,i+2}M_{i+2} = D_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 3, \ldots, n-2, \\ c_{n-1,n-3}M_{n-3} + c_{n-1,n-2}M_{n-2} + c_{n-1,n-1}M_{n-1} + c_{n-1,n}M_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = D_{n-1}, \\ c_{n,n-2}M_{n-2} + c_{n,n-1}M_{n-1} + c_{nn}M_n = D_n. \end{cases} \tag{4.9}$$

Solving this, the system consisting of all equations (3.5), (3.6) allows to determine the parameters λ_j, ρ_j. Its unique solvability is shown in [9]. Note that the values λ_1 and ρ_{n-1} are known due to the histopolation conditions (3.4,1) and (3.4,n). We discuss the solvability of (4.9) in next section.

5 Existence and uniqueness of the solution

It is clear that the unique solvability of system (4.9) is equivalent to the existence of unique solution to the histopolation problem. Let us start with particular cases.

Consider the case of spline knots as $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \dots, n-1$. Then $\eta_{i-1} = \varepsilon_i = h_i/2$, $i = 2, \dots, n-1$, $\varepsilon_1 = h_1$, $\eta_{n-1} = h_n$. The coefficients in (4.1) are (we write them also keeping symmetrical structure)

$$
\begin{aligned}
c_{i,i-2} &= \frac{1}{192}(h_i + h_{i+1})\frac{h_{i-1}^3}{h_{i-2} + h_{i-1}},\\
c_{i,i-1} &= \frac{1}{192}\left((h_i{+}h_{i+1})\left(14h_{i-1}^2{+}17h_{i-1}h_i{+}6h_i^2{+}\frac{h_{i-2}h_{i-1}^2}{h_{i-2} + h_{i-1}}\right){+}h_i^2h_{i+1}\right),\\
c_{ii} &= \frac{1}{192}\Big((h_i + h_{i+1})(17h_{i-1}^2 + 30h_{i-1}h_i + 10h_i^2)\\
&\quad + (h_{i-1} + h_i)(10h_i^2 + 30h_ih_{i+1} + 17h_{i+1}^2) + 2h_{i-1}h_ih_{i+1}\Big),\\
c_{i,i+1} &= \frac{1}{192}\left((h_{i-1}{+}h_i)\left(6h_i^2{+}17h_ih_{i+1}{+}14h_{i+1}^2{+}\frac{h_{i+1}^2h_{i+2}}{h_{i+1}{+}h_{i+2}}\right){+}h_{i-1}h_i^2\right),\\
c_{i,i+2} &= \frac{1}{192}(h_{i-1} + h_i)\frac{h_{i+1}^3}{h_{i+1} + h_{i+2}}.
\end{aligned}
$$

We see here the diagonal dominance in rows as

$$
\begin{aligned}
&c_{ii} - (c_{i,i-2} + c_{i,i-1} + c_{i,i+1} + c_{i,i+2})\\
&\qquad = \frac{1}{192}\Big((h_i + h_{i+1})(2h_{i-1}^2 + 13h_{i-1}h_i + 3h_i^2)\\
&\qquad\quad + (h_{i-1} + h_i)(3h_i^2 + 13h_ih_{i+1} + 2h_{i+1}^2) + 2h_i^3 + 2h_{i-1}h_ih_{i+1}\Big).
\end{aligned}
$$

Similar calculations give the diagonal dominance in near-boundary equations which yields the unique solvability of (4.9) in this case.

In case of uniform mesh with $h_i = h$, $i = 1, \dots, n$, and $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \dots, n-1$, the interior equations of (4.9) are

$$
\begin{cases}
\dfrac{h^3}{192}(52M_1 + 255M_2 + 76M_3 + M_4) = D_2,\\
\dfrac{h^3}{576}(2M_1 + 229M_2 + 690M_3 + 228M_4 + 3M_5) = D_3,\\
\dfrac{h^3}{192}(M_{i-2}{+}76M_{i-1}{+}230M_i{+}76M_{i+1}{+}M_{i+2}) = D_i, \quad i = 4, \dots, n-3,\\
\dfrac{h^3}{576}(3M_{n-4} + 228M_{n-3} + 690M_{n-2} + 229M_{n-1} + 2M_n) = D_{n-2},\\
\dfrac{h^3}{192}(M_{n-3} + 76M_{n-2} + 255M_{n-1} + 52M_n) = D_{n-1}.
\end{cases}
$$

The boundary condition $S(a) = \alpha$ gives the equation

$$\frac{1}{1152}(386M_1 + 379M_2 + 3M_3) = \frac{1}{h^2}(2\alpha - 3z_1 + z_2),$$

$S(b) = \beta$ gives

$$\frac{1}{1152}(3M_{n-2} + 379M_{n-1} + 386M_n) = \frac{1}{h^2}(z_{n-1} - 3z_n + 2\beta),$$

$S'(a) = \alpha$ leads to

$$\frac{1}{1152}(706M_1 + 443M_2 + 3M_3) = \frac{1}{h^2}(z_2 - z_1 - \alpha h),$$

$S'(b) = \beta$ to

$$\frac{1}{1152}(3M_{n-2} + 443M_{n-1} + 706M_n) = \frac{1}{h^2}(h\beta + z_{n-1} - z_n).$$

In general case, there may be no diagonal dominance in equations (4.1). Let us prove that. Consider in coefficients (4.3)–(4.7) the situation where $\eta_{i-2} = \text{const} > 0$ and other used parameters η_j, ε_j are equal to $\varepsilon \to 0$. Then $c_{i,i-2}$ is of order $\eta_{i-2}^2\varepsilon$ but c_{ii} has the order $\eta_{i-2}\varepsilon^2$.

The unique solvability of system (4.9) follows from the next result.

Proposition 1. *The histopolation problem posed in Section 2 has the unique solution.*

Proof. It is sufficient to prove that the corresponding homogeneous problem has only trivial solution. Suppose a cubic spline S satisfies

$$\int_{x_{i-1}}^{x_i} S(x)dx = 0, \quad i = 1, \dots, n \tag{5.1}$$

and two of the boundary conditions $S(a) = 0$, $S(b) = 0$, $S'(a) = 0$, $S'(b) = 0$, $S''(a) = 0$, $S''(b) = 0$ at different endpoints a and b. By (5.1) it exists $\eta_i \in (x_{i-1}, x_i)$ such that $S(\eta_i) = 0$, $i = 1, \dots, n$.

If $S(a) = S(b) = 0$ then there are $\overline{\eta}_i \in (\eta_{i-1}, \eta_i)$, $i = 2, \dots, n$, $\overline{\eta}_1 \in (a, \eta_1)$, $\overline{\eta}_{n+1} \in (\eta_n, b)$ such that $S'(\overline{\eta}_i) = 0$, $i = 1, \dots, n+1$. Therefore, there are $\overline{\overline{\eta}}_i \in (\overline{\eta}_i, \overline{\eta}_{i+1})$, $i = 1, \dots, n$, such that $S''(\overline{\overline{\eta}}_i) = 0$. Consequently, an interval $[\xi_k, \xi_{k+1}]$ contains two (distinct) zeros of S'' which means that $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$.

If $S'(a) = S'(b) = 0$ then again there are $n+1$ zeros of S' in $[x_0, x_n]$ and n zeros of S'' in (x_0, x_n). If $S''(a) = S''(b) = 0$ then S'' has n zeros in $[x_0, x_n]$. Using different kind boundary conditions at different endpoints we also arrive at the situation with $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$.

Let us make some observations about the situation of $S''(x) = 0$, $x \in [\xi_k, \xi_{k+1}]$. Then S is at most first degree polynomial on $[\xi_k, \xi_{k+1}]$. If S keeps the sign in $[\xi_k, x_k]$ then due to $\int_{x_{k-1}}^{x_k} S(x)dx = 0$ we have $\eta_k \in (x_{k-1}, \xi_k)$ with $S(\eta_k) = 0$. We call this case suitable for the left. If S keeps the sign in $[x_k, \xi_{k+1}]$ then S has a zero in (ξ_{k+1}, x_{k+1}) and this case is called suitable for the right. If, e.g., $k = 1$, then S has a zero in (x_0, x_1), S keeps the sign in $[x_1, \xi_2]$, due to $\int_{x_1}^{x_2} S(x)dx = 0$ there is a zero of S in $[\xi_2, x_2)$ and this case is suitable for the right. Similarly, $k = n-1$ (i.e., $k+1 = n$) is a case suitable for the left.

Consider now the case of $[\xi_k, \xi_{k+1}]$ suitable for the left. The interval $[a, \xi_k]$ contains $k-1$ subintervals $[\xi_1, \xi_2], \dots, [\xi_{k-1}, \xi_k]$. We know that $S(\eta_i) = 0$, $\eta_i \in (x_{i-1}, x_i)$, $i = 1, \dots, k-1$, and $S(\eta_k) = 0$, $\eta_k \in (x_{k-1}, \xi_k]$. We have:

1) case $S(a) = 0$, then S has $k+1$ zeros $a, \eta_1, \ldots, \eta_k$, S' has k zeros, S'' has $k-1$ zeros in (a, ξ_k) and $S''(\xi_k) = 0$;

2) case $S'(a) = 0$, then S has k zeros $\eta_1, \ldots, \eta_k$, S' has $k-1$ zeros in (a, ξ_k) and $S'(a) = 0$, S'' has $k-1$ zeros in (a, ξ_k) and $S''(\xi_k) = 0$;

3) case $S''(a) = 0$, then S has k zeros in $(a, \xi_k]$, S' has $k-1$ zeros, S'' has $k-2$ zeros in (a, ξ_k) and $S''(a) = 0$, $S''(\xi_k) = 0$.

Anyway, S'' has k zeros in $k-1$ subintervals and, thus, S'' is again equal to zero in some of them.

Observe that receiving $S''(x) = 0$, $x \in [\xi_{k-1}, \xi_{k+1}]$, first degree polynomial S on $[\xi_{k-1}, \xi_{k+1}]$, due to $\int_{x_{k-1}}^{x_k} S(x)dx = 0$, has a zero in (x_{k-1}, x_k) and keeps the sign in $[\xi_{k-1}, x_{k-1}]$, consequently, $[\xi_{k-1}, \xi_k]$ is suitable for the left. At the same time, S keeps the sign in $[x_k, \xi_{k+1}]$ and $[\xi_k, \xi_{k+1}]$ is suitable for the right.

Presented reasonings allow to assert that during the process there are always adjacent subintervals $[\xi_j, \xi_{j+1}], \ldots, [\xi_{k-1}, \xi_k]$ where the nullity of S'' is not yet established, but $[\xi_{j-1}, \xi_j]$ is suitable for the right and $[\xi_k, \xi_{k+1}]$ is suitable for the left which yields that S'' is equal to zero on one of them. However, it may be as well $j = 1$ or $k = n$. The process ends at $S''(x) = 0$, $x \in [a, b]$, and then $S(x) = 0$, $x \in [a, b]$, by histopolation and boundary conditions. Naturally, suppose that $n \geq 2$ if we use $S''(a) = \alpha$ and $S''(b) = \beta$. $\square$

6 Another representation

Consider the histopolation problem posed in Section 2. A classical representation of cubic spline is the use of $S_i = S(\xi_i)$, $M_i = S''(\xi_i)$, $i = 1, \ldots, n$.
Any cubic spline satisfies the internal equations (continuity of S' at knots ξ_i)

$$\frac{\delta_{i-1}}{\delta_{i-1}+\delta_i} M_{i-1} + 2M_i + \frac{\delta_i}{\delta_{i-1}+\delta_i} M_{i+1} = 6\frac{\frac{S_{i+1}-S_i}{\delta_i} - \frac{S_i - S_{i-1}}{\delta_{i-1}}}{\delta_{i-1}+\delta_i}, \quad i = 2, \ldots, n-1. \tag{6.1}$$

For definiteness, add boundary conditions $M_1 = \alpha$ (first equation) and $M_n = \beta$ (last equation). We obtain the system

$$AM = BS + d, \tag{6.2}$$

where $M = (M_1, \ldots, M_n)$, $S = (S_1, \ldots, S_n)$, first and last rows of B are zero rows, $d = (\alpha, 0, \ldots, 0, \beta)$. The matrix A has diagonal dominance in rows which gives its invertibility. Note that the diagonal dominance of A in rows takes place also in case of other boundary conditions.

Basing on (3.1) we have

$$S_i = a_i - b_i\varepsilon_i + c_i\varepsilon_i^2 - d_i\varepsilon_i^3,$$
$$S_{i+1} = a_i + b_i\eta_i + c_i\eta_i^2 + d_i\eta_i^3.$$

From them we obtain

$$a_i = \frac{\eta_i S_i + \varepsilon_i S_{i+1}}{\delta_i} - c_i\varepsilon_i\eta_i + d_i(\varepsilon_i - \eta_i)\varepsilon_i\eta_i,$$

$$b_i = \frac{S_{i+1} - S_i}{\delta_i} + c_i(\varepsilon_i - \eta_i) - d_i(\eta_i^2 - \varepsilon_i\eta_i + \varepsilon_i^2).$$

The coefficients c_i and d_i were expressed via M_i and M_{i+1} in Section 3. Using (3.1) the histopolation conditions could be written

$$a_{i-1}\eta_{i-1} + \frac{b_{i-1}}{2}\eta_{i-1}^2 + \frac{c_{i-1}}{3}\eta_{i-1}^3 + \frac{d_{i-1}}{4}\eta_{i-1}^4 + a_i\varepsilon_i - \frac{b_i}{2}\varepsilon_i^2 + \frac{c_i}{3}\varepsilon_i^3 - \frac{d_i}{4}\varepsilon_i^4 = z_i h_i$$

or

$$\begin{aligned}
&\frac{\eta_{i-1}^2}{2\delta_{i-1}} S_{i-1} + \left(\frac{\varepsilon_{i-1}\eta_{i-1}}{\delta_{i-1}} + \frac{\eta_{i-1}^2}{2\delta_{i-1}} + \frac{\varepsilon_i\eta_i}{\delta_i} + \frac{\varepsilon_i^2}{2\delta_i}\right) S_i + \frac{\varepsilon_i^2}{2\delta_i} S_{i+1} \\
&- \frac{\eta_{i-1}^2}{24\delta_{i-1}}(2\varepsilon_{i-1}^2 + 4\varepsilon_{i-1}\eta_{i-1} + \eta_{i-1}^2)M_{i-1} \\
&- \left(\frac{\eta_{i-1}^2}{24\delta_{i-1}}(4\varepsilon_{i-1}^2 + 4\varepsilon_{i-1}\eta_{i-1} + \eta_{i-1}^2) + \frac{\varepsilon_i^2}{24\delta_i}(\varepsilon_i^2 + 4\varepsilon_i\eta_i + 4\eta_i^2)\right) M_i \\
&- \frac{\varepsilon_i^2}{24\delta_i}(\varepsilon_i^2 + 4\varepsilon_i\eta_i + 2\eta_i^2)M_{i+1} = z_i h_i, \quad i = 2, \ldots, n-1.
\end{aligned} \tag{6.3}$$

Near the boundary we get

$$\begin{aligned}
&\left(\frac{\eta_1 h_1}{\delta_1} + \frac{h_1^2}{2\delta_1}\right) S_1 + \frac{h_1^2}{2\delta_1} S_2 \\
&- \frac{h_1^2}{24\delta_1}(4\eta_1^2 + 4\eta_1 h_1 + h_1^2)M_1 - \frac{h_1^2}{24\delta_1}(2\eta_1^2 + 4\eta_1 h_1 + h_1^2)M_2 = z_1 h_1
\end{aligned} \tag{6.4}$$

with the counterpart containing $z_n h_n$. These equations together form the system

$$CS = DM + Ez. \tag{6.5}$$

Note that in matrices C and D the diagonal dominates in rows, E is diagonal matrix with entries h_i and $z = (z_1, \ldots, z_n)$. Clearly, to construct the cubic spline histopolant it is necessary and sufficient to solve the system (6.2), (6.5). An opportunity to solve it is the following. Take, e.g., a guess value $M^0 = (M_1, M_2^0, \ldots, M_{n-1}^0, M_n)$, $M_i^0 = D_i/2h_i^3$, $i = 2, \ldots, n-1$ (note that, in uniform grid case, $D_i/2$ is close to $h^3 f''(x_i)$ if the values z_i are determined as in Section 7), then find S^0 from $CS^0 = DM^0 + Ez$, M^1 from $AM^1 = BS^0 + d$, S^1 from $CS^1 = DM^1 + Ez$, in general, the iteration process is $AM^k = BS^{k-1} + d$, $CS^k = DM^k + Ez$, $k = 1, 2, \ldots$. It may be deduced here also the process

$$M^k = A^{-1}BC^{-1}DM^{k-1} + A^{-1}BC^{-1}Ez + A^{-1}d$$

and the convergence is defined by the spectrum of $A^{-1}BC^{-1}D$. Another opportunity is to take a guess value $S^0 = (S_1^0, \ldots, S_n^0)$, e.g., $S_i^0 = z_i$, $i = 1, \ldots, n$, then find M^0 from $AM^0 = BS^0 + d$, in general, $AM^{k-1} = BS^{k-1} + d$, $CS^k = DM^{k-1} + Ez$, $k = 1, 2, \ldots$. This process could be described as

$$S^k = C^{-1}DA^{-1}BS^{k-1} + C^{-1}DA^{-1}d + C^{-1}Ez.$$

It is immediate to check that the eigenvalues of $C^{-1}DA^{-1}B$ and $A^{-1}BC^{-1}D$ coincide.

We will give examples of spectrum in some particular cases in next section.

Let us consider now the uniform mesh with central spline knots, i.e., $h_i = h$, $i = 1, \dots, n$, $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \dots, n-1$. Equation (6.1) is well known in treatments about cubic splines, it should be taken into account that $\delta_1 = \delta_{n-1} = 3h/2$, $\delta_i = h$, $i = 2, \dots, n-2$. Equation (6.3) is

$$S_{i-1} + 6S_i + S_{i+1} = \frac{h^2}{48}(7M_{i-1} + 18M_i + 7M_{i+1}) + 8z_i, \quad i = 3, \dots, n-2,$$
$$2S_1 + 19S_2 + 3S_3 = \frac{h^2}{48}(34M_1 + 77M_2 + 21M_3) + 24z_2$$

and (6.4) is now

$$2S_1 + S_2 = \frac{h^2}{24}(8M_1 + 7M_2) + 3z_1.$$

7 Numerical tests

We histopolated the function $f(x) = 1/x^2$, $x \in [-2, -0.1]$, on uniform grid for $n = 8$ and central spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \dots, n-1$. Histogram heights were computed as $z_i = \frac{1}{h}\int_{x_{i-1}}^{x_i} f(x)dx$, $i = 1, \dots, n$. Resulting histopolants S are given in Figures 2–4.

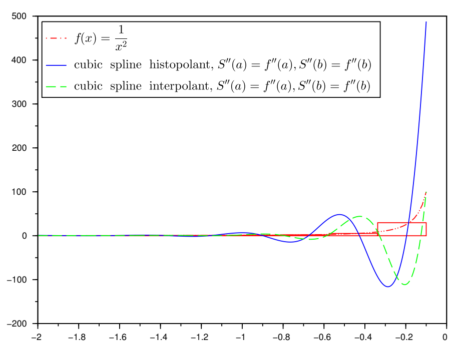

Figure 2. Cubic spline histopolant and interpolant for $n = 8$

In Figure 2 the histopolant with boundary conditions $S''(a) = f''(a)$, $S''(b) = f''(b)$; in Figure 3 the histopolant with boundary conditions $S'(a) = f'(a)$, $S'(b) = f'(b)$; in Figure 4 the histopolant with boundary conditions $S(a) = f(a)$, $S(b) = f(b)$. In comparison also cubic spline interpolants are given satisfying interpolation conditions $S(x_i) = f(x_i)$, $i = 0, \dots, n$.

Considering the representation used in Section 6 we tested the dependence of eigenvalues of matrix $A^{-1}BC^{-1}D$ on grid points x_i and spline knots ξ_i. Again the case $n = 8$ is analysed for different meshes.

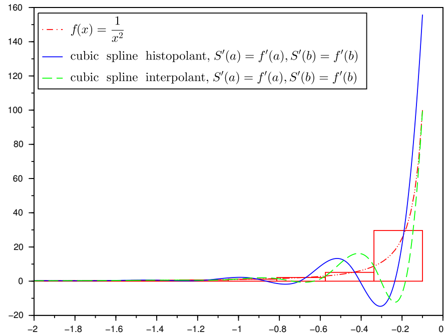

Figure 3. Cubic spline histopolant and interpolant for $n = 8$

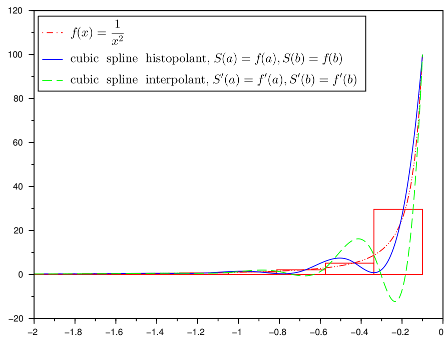

Figure 4. Cubic spline histopolant and interpolant for $n = 8$

1) Uniform grid $x_i = a + ih$, $i = 0, \ldots, n$, and central spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, \ldots, n-1$, give the maximal by modulus eigenvalue $|\lambda_{\max}| = 0.271$.

2) For uniform histogram grid $x_i = a + ih$, $i = 0, \ldots, n$, spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, 3, 6, 7$, $\xi_4 = 0.1x_3 + 0.9x_4$, $\xi_5 = 0.9x_4 + 0.1x_5$ (ξ_4 and ξ_5 are close to x_4) it holds $|\lambda_{\max}| = 2.388$.

3) For uniform grid $x_i = a + ih$, $i = 0, \ldots, n$, spline knots $\xi_i = (x_{i-1} + x_i)/2$, $i = 2, 3, 6, 7$, $\xi_4 = 0.9x_3 + 0.1x_4$, $\xi_5 = 0.1x_4 + 0.9x_5$ (ξ_4 and ξ_5 are close to x_3 and x_5, respectively) it holds $|\lambda_{\max}| = 0.803$.

4) Take $h = (b-a)/n$, $h_i = 0.1h$, $i = 1, 3, 5, 7$, $h_i = 1.9h$, $i = 2, 4, 6, 8$, and central spline knots $\xi_i = (x_{i-1}+x_i)/2$, $i = 2, \ldots, n-1$, then $|\lambda_{\max}| = 0.241$.

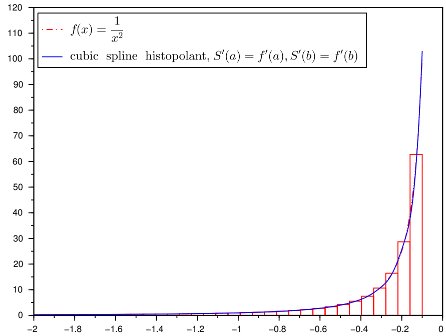

Figure 5. Cubic spline histopolant for $n = 32$

8 Concluding remarks

Construction of histopolating cubic spline could be done using second derivatives M_i and particular integrals λ_i, ρ_i. The crucial moment here is the solution of system (4.9). In case of diagonal dominance in the matrix of (4.9) standard methods (e.g., Gaussian elimination) are stable. In absence of diagonal dominance it may be that other methods should be applied. One way to continue is to solve the system determining parameters λ_i, ρ_i. Another natural way is to solve system (6.5) where the matrix C has diagonal dominance. An opportunity is to use an iteration process described in Section 6 to determine either second derivatives or spline values and then the others by (6.2) or (6.5) with a matrix having diagonal dominance. We have seen in Section 7 that the convergence may be slow or be absent at all. In return, in the presence of convergence, the calculations at iteration are stable.

Numerical tests with the function $1/x^2$ confirmed the known fact that polynomial splines (at interpolation or histopolation) do not preserve geometrical properties like positivity, monotonicity, convexity. However, increasing the number n of knots, the cubic spline histopolant (and cubic spline interpolant) occurs to have these properties because of the uniform convergence of values, first and second derivatives (see Figure 5). For the cubic spline histopolant this follows from the uniform convergence (see, e.g., [1]) of first, second and third derivatives of interpolating quartic splines in equivalent problem as described in Introduction.

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