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THE FIRST BASIC BOUNDARY VALUE PROBLEM OF RIEMANN'S TYPE FOR BIANALYTICAL FUNCTIONS IN A PLANE WITH SLOTS

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Abstract. The paper is devoted to the investigation of one of the basic boundary value problems of Riemann's type for bianalytical functions. In the course of work there was made out a constructive method for solution of the problem given in a plane with slots. There was also found out that the solution of the problem under consideration consists of consequent solutions of two Riemann's boundary value problems for analytical functions in a plane with slots. Besides, a picture of solvability of the problem is being searched and its Noether is identified.
Key words: bianalytical function, boundary value problem, plane with slots, index

1. Statement of the problem

Let us exclude from a full complex plane segments of a real axis $L_m = [a_m, b_m]$ ($m = 1, 2..., n$), and let D be the remaining domain. The boundary L of the domain D is understood as the thrown out segments (slits). Thus, $L = \bigcup\limits_{m=1}^{n} L_m$ and $D = \bar{C}\backslash L$. Further we shall use terms and definitions accepted in [3].

As it is known, a function $F(z) = U(x, y) + iV(x, y)$ is called bianalytical in the domain D if it belongs to the class $C^2(D)$ and satisfies in D the condition

$$\frac{\partial^2 F(z)}{\partial \bar{z}^2} = 0,$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial x}\right)$$

is the Cauchy–Riemann operator in D.

DEFINITION 1. A bianalytical function $F(z)$, which is defined in domain D, belongs to the class $A_2(D)\bigcap\, I^{(2)}(L)$ if it can be prolonged to the contour L together with the partial derivatives $\dfrac{\partial^{\alpha+\beta}F(z)}{\partial z^{\alpha}\partial\bar{z}^{\beta}}$ ($\alpha=0,1;\beta=0,1$) so that boundary values of this function and all specified derivatives satisfy the Holder condition everywhere, may be except for points a_m, b_m ($m=1,\dots,n$), where the reversion at infinity of the integrable order is possible, when $\alpha+\beta<2$.

Let us consider the following boundary value problem.

It is required to find all bianalytical functions $F(z)$, belonging to the class $A_2(D)\bigcap\, I^{(2)}(L)$, vanishing on infinity, limited near the extremities of the contour and satisfying at all internal points of L the following boundary conditions:

$$\frac{\partial F^+(t)}{\partial x}=G_1(t)\frac{\partial F^-(t)}{\partial x}+g_1(t), \tag{1.1}$$

$$\frac{\partial F^+(t)}{\partial y}=G_2(t)\frac{\partial F^-(t)}{\partial y}+ig_2(t), \tag{1.2}$$

where

$$\frac{\partial F^+(t)}{\partial x}=\lim_{z\to t, Imz>0}F(z),\quad \frac{\partial F^-(t)}{\partial x}=\lim_{z\to t, Imz<0}F(z)$$

$$\frac{\partial F^+(t)}{\partial y}=\lim_{z\to t, Imz>0}F(z),\quad \frac{\partial F^-(t)}{\partial y}=\lim_{z\to t, Imz<0}F(z)$$

and $G_k(t), g_k(t)$ ($k=1,2$) are given on L functions of the class $H^{(3-k)}(L)$, and $G_k(t)\neq 0$ on L.

Here, in equality (1.2), the factor i at $g_2(t)$ is put for convenience of notation. The formulated problem is called the first basic boundary value problem of Riemann's type for bianalytical functions in the plane with slots or, shortly, the problem $R_{1,2}$. The appropriate homogeneous problem ($g_1(t)\equiv g_2(t)\equiv 0$) will be denoted as problem $R^0_{1,2}$.

Let us notice, that the problem $R_{1,2}$ represents one of the basic boundary value problems of Riemann's type for bianalytical functions. It was formulated in the well-known monograph of Gakhov (see, for example, [1], p. 316). In case of arbitrary smooth closed loops the considered problem was explicitly investigated in the work of Rasulov (see, for example, [3]).

In the present work for the first time we investigate a more general problem $R_{1,2}$.

2. On the solution of the problem $R_{1,2}$

It is known (see, for example, [1, 3]), that any vanishing on infinity bianalytical function $F(z)$ with a line of saltuses L can be represented as:

$$F(z) = \varphi_0(z) + \bar{z}\varphi_1(z), \tag{2.1}$$

where $\varphi_k(z)$ are analytical functions in domain D (analytical components of the bianalytical function), for which the following conditions are fulfilled:

$$\Pi\{\varphi_k, \infty\} \geq 1 + k, \quad k = 0, 1;$$

here $\Pi\{\varphi_k, \infty\}$ means the order of the function $\varphi_k(z)$ at the point $z = \infty$.

Let us search for the solution of the problem $R_{1,2}$ given by:

$$F(z) = f_0(z) + (\bar{z} - z)f_1(z). \tag{2.2}$$

Then the functions $f_k(z)$ ($k = 0, 1$) will be connected with analytical components of the required bianalytical function $F(z)$ by formulas:

$$\varphi_0(z) = f_0(z) - zf_1(z), \quad \varphi_1(z) = f_1(z)\,. \tag{2.3}$$

As known (see, for example, [1], p. 301)

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right),$$

then taking into account (2.2) and the fact that the equality $\bar{t} = t$ is fulfilled on L, the boundary conditions (1.1) and (1.2) can be written as:

$$\Phi_0^+(t) = G_1(t)\Phi_0^-(t) + g_1(t), \tag{2.4}$$

$$f_1^+(t) = G_2(t)f_1^-(t) + Q_2(t), \tag{2.5}$$

where $\Phi_0(z) = \dfrac{df_0(z)}{dz},\ Q_2(t) = \dfrac{1}{2}\left(\Phi_0^+(t) - G_2(t)\Phi_0^-(t) - g_2(t)\right)$.

The equalities (2.4) and (2.5) represent boundary conditions of usual Riemann's problems for analytical functions in a plane with slots (see, for example, [1] or [2]). Thus, as a matter of fact, the solution of the initial problem $R_{1,2}$ is reduced to sequential solution of two auxiliary problems of Riemann (2.4) and (2.5) in classes of analytical functions in domain D with a line of saltuses L. But as in the problem $R_{1,2}$ we search for the solutions, limited close to extremities of the contour and vanishing on infinity. There arises the necessity to define classes of analytical solutions of auxiliary problems (2.4) and (2.5). Therefore, at first we shall find out in what classes it is necessary to search for solutions of boundary value problems (2.4) and (2.5).

From equalities (2.3) we can see, that the functions $\Phi_0(z)$ and $f_1(z)$ on infinity should have zero not below than the second order. Let us study the behaviour of the function $F(z)$ near the extremities of the contour L. Let c be any of extremities, then $\bar{c} = c$. We have the following series of inequalities:

$$\begin{aligned}
|F(z)| &= |f_0(z) + (\bar{z} - z)f_1(z)| \leq |f_0(z)| + |f_1(z)||\bar{z} - z| \\
&= |f_0(z)| + |f_1(z)||\bar{z} - \bar{c} + c - z| \\
&\leq |f_0(z)| + |f_1(z)||\bar{z} - \bar{c}| + |f_1(z)||z - c| \\
&= |f_0(z)| + 2|f_1(z)||z - c|\,.
\end{aligned} \tag{2.6}$$

For function $F(z)$ to be limited close to the extremities of the contour L, it is *necessary* and *sufficient*, that the functions $\Phi_0(z)$ and $f_1(z)$ satisfy the estimates:

$$|\Phi_0(z)| \leq \frac{const}{|z-c|^{\alpha_0}}, \qquad |f_1(z)| \leq \frac{const}{|z-c|^{\alpha_1}}, \qquad 0 \leq \alpha_0, \alpha_1 < 1. \tag{2.7}$$

Really, if functions $\Phi_0(z)$ and $f_1(z)$ satisfy the condition (2.7), then the required bianalytical function $F(z)$ will be limited in a neighbourhood of c.

And if the function $F(z)$ of the class $A_2(D) \bigcap I^{(2)}(L)$ is limited close to the extremity c, then the functions $\Phi_0(z)$ and $f_1(z)$ satisfy (2.7) (otherwise all solutions of the problem $R_{1,2}$ will not be found).

Thus it is required to find a solution of boundary value problems (2.4) and (2.5), belonging to a class of functions, having on infinity a zero of the second order and infinity of the integrable order on extremities of the contour L.

Let us solve the boundary value Riemann problem (2.4). Let

$$G_1(a_m) = r_{a,1m} e^{i\theta_{1m}}, \quad 0 \leq \theta_{1m} < 2\pi,$$
$$G_1(b_m) = r_{b,1m} e^{i(\theta_{1m} + \Delta\theta_{1m})}, \quad \Delta\theta_{1m} = [argG_1(t)]_{L_m},$$

then, following Gakhov (see, for example, [1], p. 448), we define integer numbers κ_{1m} by the following formulas:

$$\kappa_{1m} = \left[\frac{\theta_{1m} + \Delta\theta_{1m}}{2\pi}\right] + 1 .$$

The index of the problem (2.4) is represented by the following formula

$$\kappa_1 = \sum_{m=1}^{n} \kappa_{1m} .$$

Hence, if $\kappa_1 \geq 2$, a general solution of problem (2.4) is set by the formula (see, for example, [1, 2]):

$$\Phi_0(z) = X_1(z) \left(\frac{1}{2\pi i} \int\limits_L \frac{g_1(\tau)}{X_1^+(\tau)} \frac{d\tau}{\tau - z} + P_{\kappa_1 - 2}(z) \right), \tag{2.8}$$

where $X_1(z)$ is a canonical function of the problem (2.4), $P_{\kappa_1-2}(z)$ is the polynomial of a degree not higher than $(\kappa_1 - 2)$ with arbitrary complex coefficients.

In case when $\kappa_1 \leq 1$, the solution of problem (2.4) also will be expressed by formula (2.8) with only one modification, that $P_{\kappa_1-2}(z) \equiv 0$, and if $\kappa_1 \leq 0$ then $|\kappa_1| + 1$ conditions of solvability should be satisfied:

$$\int\limits_L \frac{g_1(\tau)}{X_1^+(\tau)} \tau^{k-1} d\tau = 0, \qquad k = 1, \ldots, |\kappa_1| + 1. \tag{2.9}$$

Using the function $\Phi_0(z) = \dfrac{df_0(z)}{dz}$, after integration we obtain

$$f_0(z) = \int_{\gamma} \Phi_0(\zeta) d\zeta,$$

where γ is an arbitrary smooth curve, completely laying in the domain D and connecting the infinite point with arbitrary point z of the domain D.

Let us take in expression (2.8) $z \to t \in L$. Then, using the formulas of Sokhotzky-Plemelj (see, for example, [1, 2]), we get

$$\Phi_0^+(t) = X_1^+(t) \left(\frac{1}{2\pi i} \int_L \frac{g_1(\tau)}{X_1^+(\tau)} \frac{d\tau}{\tau - t} + P_{\kappa_1 - 2}(t) \right) + \frac{g_1(t)}{2}, \tag{2.10}$$

$$\Phi_0^-(t) = X_1^-(t) \left(\frac{1}{2\pi i} \int_L \frac{g_1(\tau)}{X_1^+(\tau)} \frac{d\tau}{\tau - t} + P_{\kappa_1 - 2}(t) \right) - \frac{g_1(t)}{2G_1(t)}, \tag{2.11}$$

here in equality (2.11) we have taken into account, that $\dfrac{X_1^+(t)}{X_1^-(t)} = G_1(t)$.

Remark 1. Functions $\Phi_0^{\pm}(t)$, given by formulas (2.10) and (2.11) satisfy the Holder condition everywhere on L, may be except the extremities, where they may have a singularity of the integrable order.

Now we will solve the boundary value Riemann problem (2.5). Let

$$G_2(a_m) = r_{a,2m} e^{i\theta_{2m}}, \quad 0 \le \theta_{2m} < 2\pi,$$
$$G_2(b_m) = r_{b,2m} e^{i(\theta_{2m} + \Delta\theta_{2m})}, \quad \Delta\theta_{2m} = [argG_2(t)]_{L_m}.$$

Then following Gakhov (see, for example., [1], p. 448) we define integer numbers:

$$\kappa_{2m} = \left[\frac{\theta_{2m} + \Delta\theta_{2m}}{2\pi} \right] + 1. \tag{2.12}$$

The index of the problem (2.5) is represented by the following formula

$$\kappa_2 = \sum_{m=1}^{n} \kappa_{2m}. \tag{2.13}$$

As it is known (see [1, 2]), if $\kappa_2 \ge 2$, a general solution of problem (2.5) is represented by formula:

$$f_1(z) = X_2(z) \left(\frac{1}{2\pi i} \int_L \frac{Q_2(\tau)}{X_2^+(\tau)} \frac{d\tau}{\tau - z} + P_{\kappa_2 - 2}(z) \right), \tag{2.14}$$

where $X_2(z)$ is a canonical function of problem (2.5), $P_{\kappa_2-2}(z)$ is a polynomial of a degree not higher than $(\kappa_2 - 2)$ with arbitrary complex coefficients.

If $\kappa_2 \le 1$, the solution of problem (2.5) can be expressed by formula (2.14) with only one modification, that $P_{\kappa_2-2}(z) \equiv 0$, and if $\kappa_2 \le 0$, then $|\kappa_2| + 1$ conditions of a solvability should be satisfied:

$$\int\limits_L \frac{Q_2(\tau)}{X_2^+(\tau)}\tau^{k-1}d\tau = 0, \qquad k = 1, ..., |\kappa_2| + 1\,. \tag{2.15}$$

Remark 2. Generally speaking, absolute term $Q_2(t)$ of problem (2.5) satisfies the Holder condition everywhere on L, except for, possibly, the extremities of the contour, where it may have the integrable singularity. But, as we search for the solution of the boundary value problem (2.5) in the class of functions, having infinity of the integrable order on the extremities of the contour L, the density of the integral in the formula (2.14) has the singularity on the extremities of the contour, which is not higher than the integrable one.

After finding functions $f_0(z)$ and $f_1(z)$, we use formulas (2.3) and restore analytical components of the required bianalytical function. Then using formula (2.1) we restore the bianalytical function $F(z)$ itself. Thus, the following basic result is valid.

Theorem 1. *Let*

$$L = \bigcup_{m=1}^{n}[a_m, b_m], \quad D = \bar{C}\backslash L\,.$$

Then the solution of the problem $R_{1,2}$ is reduced to a sequential solution of the two scalar Riemann problems (2.4) *and* (2.5) *in classes of analytical functions in the plane with slots, having a zero of the second order on infinity and infinity of the integrable order on the extremities of the contour L. The problem $R_{1,2}$ is solvable if and only if problems* (2.4) *and* (2.5) *are simultaneously solvable in the specified classes of functions.*

3. Investigation of a solvability of the problem $R_{1,2}$

As the solution of the problem $R_{1,2}$ is reduced to the sequential solution of the boundary value Riemann problems (2.4) and (2.5), solvability conditions of the problem $R_{1,2}$ can be developed from the solvability conditions of boundary value problems (2.4) and (2.5).

The number $\kappa = \kappa_1 + \kappa_2$ is called *the index* of the problem $R_{1,2}$, and numbers κ_1 and κ_2 are its *private indexes*. For a full investigation of the solvability of the problem $R_{1,2}$ it is necessary to consider 9 cases.

Case 1. Let $\kappa_1 \geq 2$, $\kappa_2 \geq 2$.

In this case boundary value problems (2.4) and (2.5) are solvable and have $\kappa_1 - 1$ and $\kappa_2 - 1$ linearly independent solutions, respectively. Thus, in this case problem $R_{1,2}$ is solvable and by virtue of formulas (2.1) and (2.3) its general solution linearly depends on $\kappa_1 + \kappa_2 - 2$ arbitrary complex constants.

Case 2. Let $\kappa_1 \geq 2$, $\kappa_2 = 1$.

In this case boundary value problem (2.4) is solvable and has $\kappa_1 - 1$ linearly independent solutions, and boundary value problem (2.5) also is solvable and has a unique solution. Hence, in this case problem $R_{1,2}$ is solvable and its general solution linearly depends on $\kappa_1 - 1$ arbitrary complex constants.

Case 3. Let $\kappa_1 \geq 2$, $\kappa_2 \leq 0$.

In this case boundary value problem (2.4) is solvable and has $\kappa_1 - 1$ linearly independent solutions, and the boundary value problem (2.5) has a unique solution if $|\kappa_2| + 1$ conditions of the solvability (2.15) are satisfied.

Remark 3. We shall notice, that some conditions (2.15) can be satisfied at the expense of a choice of values of arbitrary constants, which are included in expression $Q_2(t)$.

Thus, in this case problem $R_{1,2}$ is solvable if $|\kappa_2|+1$ conditions of the solvability (2.15) are satisfied and its general solution linearly depends on l arbitrary complex constants, where $0 \leq l \leq \kappa_1 - 1$.

Case 4. Let $\kappa_1 = 1$, $\kappa_2 \geq 2$.

In this case boundary value problem (2.4) is solvable and has a unique solution, and the boundary value problem (2.5) is solvable and has $\kappa_2 - 1$ linearly independent solutions. Thus, in this case problem $R_{1,2}$ is solvable and its common solution linearly depends on $\kappa_2 - 1$ arbitrary complex constants.

Case 5. Let $\kappa_1 = 1$, $\kappa_2 = 1$.

In this case boundary value problems (2.4) and (2.5) are solvable and each of them has a unique solution. Hence, by virtue of the Theorem 1, the problem $R_{1,2}$ is solvable and has a unique solution.

Case 6. Let $\kappa_1 = 1$, $\kappa_2 \leq 0$.

In this case boundary value problem (2.4) is solvable and has a unique solution, and the boundary value problem (2.5) has a unique solution if $|\kappa_2| + 1$ conditions of the solvability (2.15) are satisfied. Thus, problem $R_{1,2}$ is solvable and has a unique solution.

Case 7. Let $\kappa_1 \leq 0$, $\kappa_2 \geq 2$.

In this case boundary value problem (2.4) has a unique solution if $|\kappa_1| + 1$ conditions of the solvability (2.9) are satisfied. Boundary value problem (2.5) is solvable and has $\kappa_2 - 1$ linearly independent solutions. Thus, in this case problem $R_{1,2}$ is solvable if $|\kappa_1| + 1$ conditions of the solvability (2.9) are satisfied and its general solution linearly depends on $\kappa_2 - 1$ arbitrary complex constants.

Case 8. Let $\kappa_1 \leq 0$, $\kappa_2 = 1$.

In this case boundary value problem (2.4) has a unique solution if $|\kappa_1| + 1$ conditions of the solvability (2.9) are satisfied. Boundary value problem (2.5) is solvable and has a unique solution. Thus, problem $R_{1,2}$ is solvable if $|\kappa_1| + 1$ conditions of the solvability (2.9) are satisfied and it has a unique solution.

Case 9. Let $\kappa_1 \leq 0$, $\kappa_2 \leq 0$.

In this case boundary value problem (2.4) has a unique solution if $|\kappa_1| + 1$ conditions of the solvability (2.9) are satisfied, and boundary value problem (2.5) has a unique solution if $|\kappa_2| + 1$ conditions of the solvability (2.15) are satisfied. Hence, in this case problem $R_{1,2}$ is solvable if $|\kappa_1| + 1$ conditions (2.9) and $|\kappa_2| + 1$ conditions (2.15) are satisfied and it has a unique solution.

From the analysis given above we obtain the following statement.

Theorem 2. *For any values of the index $\kappa = \kappa_1 + \kappa_2$ the number p of conditions of a solvability of inhomogeneous problem $R_{1,2}$ and the number l of linearly independent solutions corresponding to homogeneous problem $R^0_{1,2}$ are finite, i.e. the problem $R_{1,2}$ satisfies the Noether conditions.*

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Apie pirmojo pagrindinio kraštinio Rimano tipo uždavinio bianalizinėms funkcijoms plokštumoje su įtrūkiais sprendimą

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Šiame darbe tyrinėjamas uždavinys, kai ieškoma dalimis bianalizinių funkcijų, nykstančių begalybėje, apribotų greta kontūro trūkio taškų ir šiame kontūre tenkinančių dvi kraštines sąlygas. Parodoma, kad sprendžiamas uždavinys suvedamas į sprendimą dviejų Rimano uždavinių analizinėms funkcijoms.