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FUNCTIONALS WITH VALUES IN THE NON-ARCHIMEDEAN FIELD OF LAURENT SERIES AND THEIR APPLICATIONS TO THE EQUATIONS OF ELASTICITY THEORY. II

M. RADYNA

Department of Informatics Belorussian State University of Informatics and Radioelectronics

P. Brovki 6, Minsk, 220013, Belarus

E-mail: kolya@im.bas-net.by

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ABSTRACT

Functionals with values in Non-Archimedean field of Laurent series applied to the definition of generalized solution (in the form of shock wave) of the Hopf equation and equations of elasticity theory. Calculation method for the profile of shock wave is proposed. It is shown that there is a possibility to find out some of the solutions of this system using the Newton iteration method. Examples and numerical tests are considered.

Key words: generalized functions, distributions, conservation law, Hopf equation, equations of elasticity theory, soliton, shock wave

1. INTRODUCTION

Here we develop ideas which were proposed in the Part I [2]. Namely in [2], we gave the definition of the special solutions of some conservation laws in the sense of $\mathbf{R}\langle\varepsilon\rangle$-distributions and considered the method for the numerical calculation of the smooth shocks of the Hopf equation.

Here we are going to consider the equations of elasticity theory. We present some examples and numerical tests.

The following concept is used in the Part I.

Definition 1.1. The function $v \in I$ (or $w \in J$) is a solution of the Hopf equation up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any

$t \in [0, T]$

$$\int_{-\infty}^{+\infty} \{v_t(t,x,\varepsilon) + v(t,x,\varepsilon)v_x(t,x,\varepsilon)\}\,\psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{1.1}$$

$$\int_{-\infty}^{+\infty} \{w_t(t,x,\varepsilon) + w(t,x,\varepsilon)w_x(t,x,\varepsilon)\}\,\psi(x)dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \tag{1.2}$$

for every $\psi \in \mathcal{S}(\mathbf{R})$. In case when p is equal to $+\infty$ the function $v(t,x,\varepsilon)$ (or $w(t,x,\varepsilon)$) exactly satisfies the Hopf equation in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

We consider two sets of smooth functions, depending on a small parameter $\varepsilon \in (0,1]$. We take all functions $v(t,x,\varepsilon)$ which have the type

$$v(t,x,\varepsilon) = l_0 + \Delta l \varphi\left(\frac{x-ct}{\varepsilon}\right),$$

here $l_0, \Delta l, c$ are real numbers, $\Delta l \neq 0$ and $\varphi \in \mathcal{S}(\mathbf{R})$, $\int_{-\infty}^{+\infty} \varphi(y)dy = 1$. We denote this set of functions by I and call it as a set of infinitely narrow solitons.

We also take all functions $w(t,x,\varepsilon)$ which have the type

$$w(t,x,\varepsilon) = h_0 + \Delta h H\left(\frac{x-at}{\varepsilon}\right),$$

$h_0, \Delta h, a$ are real numbers, $\Delta h \neq 0$ and $H(x) = \int_{-\infty}^{x} \theta(y)dy$, $\int_{-\infty}^{+\infty} \theta(y)dy = 1$ and $\theta \in \mathcal{S}(\mathbf{R})$. We denote this set of functions by J and call it as a set of shock waves.

2. CALCULATIONS OF THE MICROSCOPIC PROFILES OF THE SHOCK WAVE SOLUTIONS OF THE HOPF EQUATION IN THE SENSE OF $\mathbf{R}\langle\varepsilon\rangle$–DISTRIBUTIONS

In this case we seek a solution of the Hopf equation in the set J :

$$w(t,x,\varepsilon) = h_0 + \Delta h K\left(\frac{x-at}{\varepsilon}\right),$$

$h_0, \Delta h, a$ are real numbers, $\Delta h \neq 0$ and

$$K(x) = \int_{-\infty}^{x} \theta(y)dy, \quad \int_{-\infty}^{+\infty} \theta(y)dy = 1, \quad \theta \in \mathcal{S}(\mathbf{R}).$$

Let us put $w(t,x,\varepsilon)$ into the integral expression (1.2) and use the following formulas

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left\{K\left(\frac{x-at}{\varepsilon}\right)\right\}\psi(x)dx = \sum_{k=0}^{+\infty}(-a)\varepsilon^k m_k \frac{\psi^{(k)}(at)}{k!},$$

$$\int_{-\infty}^{+\infty} K\left(\frac{x-at}{\varepsilon}\right)\frac{\partial}{\partial x}\left\{K\left(\frac{x-at}{\varepsilon}\right)\right\}\psi(x)dx = \sum_{k=0}^{+\infty}\varepsilon^k r_k \frac{\psi^{(k)}(at)}{k!},$$

here we denote

$$m_k(\theta) = \int_{-\infty}^{+\infty} y^k\theta(y)dy,\ r_k(\theta) = \int_{-\infty}^{+\infty} x^k\theta(x)\left(\int_{-\infty}^{x}\theta(y)dy\right)dx,\ k \geq 0.$$

Thus, we get

$$\int_{-\infty}^{+\infty}\{w_t + ww_x\}\psi dx = \sum_{k=0}^{+\infty}\left\{(\Delta h)^2 r_k - \Delta h(a-h_0)m_k\right\}\varepsilon^k\frac{\psi^{(k)}(at)}{k!}. \tag{2.1}$$

From the last expression we have conditions for the function $\theta(x)$

$$r_k(\theta) - \frac{a-h_0}{\Delta h}m_k(\theta) = 0,\ k \geq 0. \tag{2.2}$$

From the first equation ($k=0$) we have

$$\frac{a-h_0}{\Delta h} = \int_{-\infty}^{+\infty}\theta(x)\left(\int_{-\infty}^{x}\theta(y)dy\right)dx = \frac{1}{2}.$$

Therefore, we can rewrite (2.2) as

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^k\theta(x)dx = \int_{-\infty}^{+\infty} x^k\theta(x)\left(\int_{-\infty}^{x}\theta(y)dy\right)dx,\ k \geq 0. \tag{2.3}$$

Using the same method one can prove that there exists a function $\theta(x) \in \mathcal{S}(\mathbf{R})$ which satisfies the following conditions

$$\frac{1}{2}\int_{-\infty}^{+\infty} x^k\theta(x)dx = \int_{-\infty}^{+\infty} x^k\theta(x)\left(\int_{-\infty}^{x}\theta(y)dy\right)dx \quad k = 0,1,2\ldots n. \tag{2.4}$$

Thus, we can formulate the following result.

Theorem 2.1. *For any integer p there is a shock wave type solution of the Hopf equation (in the sense of the definition* (1.1)*) up to e^{-p} with respect to the norm $|\cdot|_\nu$:*

$$w(t,x,\varepsilon) = h_0 + \Delta h K\left(\frac{x-at}{\varepsilon}\right),$$

$h_0, \Delta h, a$ are real numbers, $\Delta h \neq 0$ and $K(x) = \int\limits_{-\infty}^{x} \theta(y)dy$, $\int\limits_{-\infty}^{+\infty} \theta(y)dy = 1$,

$\theta \in \mathcal{S}(\mathbf{R})$. *Moreover,*

$$\frac{a-h_0}{\Delta h} = \frac{1}{2}. \tag{2.5}$$

Let us note that the condition (2.5) is the *Rankine — Hugoniot* condition for the velocity of a shock wave.

As in previous section we seek function $\theta(x)$ in the following form:

$$\varphi(x) = a_0 h_0(x) + a_1 h_1(x) + \ldots + a_n h_n(x),$$

where $h_k(x)$ are Hermite functions. Calculations in case $p = 7$ give the following "profile" $K(x)$ for the shock wave $w(t,x,\varepsilon) = K\left(\frac{x-at}{\varepsilon}\right)$ (where $h_0 = 0, \Delta h = 1$).

$$K(x) = \int\limits_{-\infty}^{x} \left\{ \frac{c_0}{\sqrt[4]{\pi}} + \frac{c_2(4\tau^2-2)}{\sqrt{2^2 2!}\sqrt[4]{\pi}} + \frac{c_4(16\tau^4 - 48\tau^2 + 12)}{\sqrt{2^4 4!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau, \tag{2.6}$$

where $c_0 = 0.79617$, $c_2 = -0.53004$, $c_4 = 0.17923$, $c = 1/2$ is a velocity of the shock wave (see Fig. 1). Numbers c_0, c_2, c_4 were found approximately.

Note that the function $K(x)$ is not unique. There is a different function $K_1(x)$ which satisfies the mentioned above conditions. It has the following type

$$\begin{aligned} K_1(x) = &\int\limits_{-\infty}^{x} \left\{ \frac{c_0}{\sqrt[4]{\pi}} + \frac{c_1 2\tau}{\sqrt{2^1 1!}\sqrt[4]{\pi}} + \frac{c_2(4\tau^2-2)}{\sqrt{2^2 2!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau \\ + &\int\limits_{-\infty}^{x} \left\{ \frac{c_3(8\tau^3 - 12\tau)}{\sqrt{2^3 3!}\sqrt[4]{\pi}} + \frac{c_4(16\tau^4 - 48\tau^2 + 12)}{\sqrt{2^4 4!}\sqrt[4]{\pi}} \right\} e^{-\tau^2/2} d\tau, \end{aligned} \tag{2.7}$$

where $c_0 = 0.18357$, $c_1 = -0.73567$, $c_2 = 0.74733$, $c_3 = 0.15327$ $c_4 = -0.29539$, $c = 1/2$ is a velocity of the shock wave (see Fig. 2). Coefficients c_0, c_1, c_2, c_3, c_4 were found approximately by the Newton iteration method.

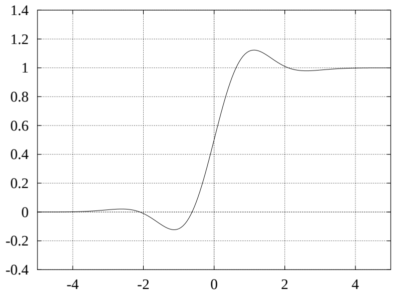

Figure 1. Graph of the function $K(x)$.

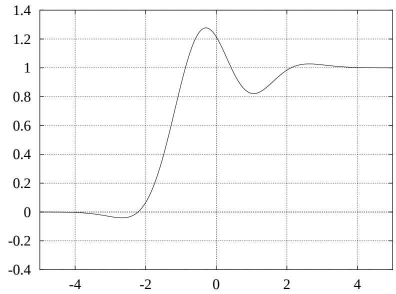

Figure 2. Graph of the function $K_1(x)$.

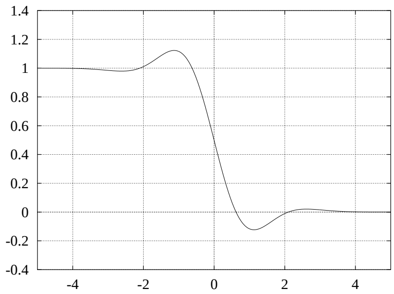

Figure 3. First shock profile $1 - K(x)$.

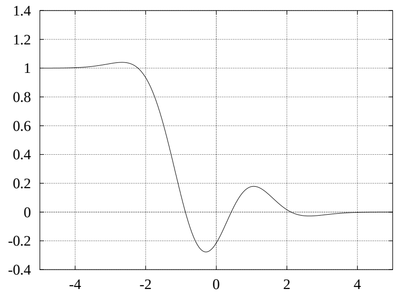

Figure 4. Second shock profile $1 - K_1(x)$.

Taking in account the *Rankine — Hugoniot* condition (2.5) we get shock profiles (Fig.3,4).

Here we describe how it is possible to find coefficients $c_0, c_1, \ldots, c_n$ by the Newton iteration method for the following system of nonlinear equations

$$P(\vec{c}) = A\vec{c} - 2\sum_{k=0}^{n}(S(k)\vec{c}, \vec{c})\vec{e}_k = 0, \;\; \vec{c} = (c_0, c_1, \ldots, c_n).$$

Vector $\vec{e}_k = (e_0, e_1, \ldots, e_n)$ is such that $e_k = 1$ and $e_j = 0$ for all $j \neq k$. $S(k)$ are matrices with elements

$$S_{ij}(k) = \int\limits_{-\infty}^{+\infty} x^k h_i(x) \int\limits_{-\infty}^{x} h_j(y)\, dy\, dx, \;\; i, j, k = 0, 1, 2 \ldots n.$$

Matrix A haelements

$$A_{ij} = \int\limits_{-\infty}^{+\infty} x^i h_j(x)\, dx, \;\; i, j = 0, 1, 2 \ldots n.$$

We can write the formula for the Newton iteration method [1]

$$\vec{x}_{m+1} = \vec{x}_m - \left[P'(\vec{x}_m)\right]^{-1}\left[P(\vec{x}_m)\right],$$

where $[P'(\vec{x})]$ is a linear map depending on the vector $\vec{x}$

$$[P'(\vec{x})]\,[\vec{h}] = A\vec{h} - 2\left\{\sum_{k=0}^{n}(S(k)\vec{x},\vec{h})\vec{e}_k + \sum_{k=0}^{n}(S^T(k)\vec{x},\vec{h})\vec{e}_k\right\}.$$

Calculations of shock profiles $K(x)$ for the Hopf equation in case $p = 8$, 9, 10, 11, give us the following pictures (Fig. 5, 6, 7, 8.) Here, we show only two different types of the shock type solutions of the Hopf equation. We can find more solutions if we take a different initial data for the Newton iteration method.

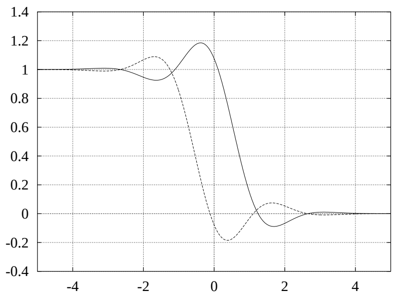

Figure 5. Shock profiles when $p = 8$.

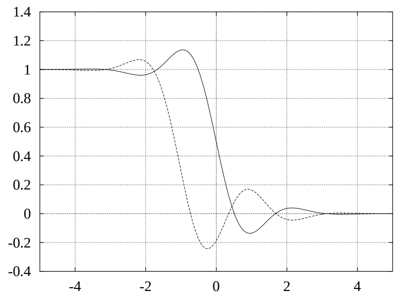

Figure 6. Shock profiles when $p = 9$.

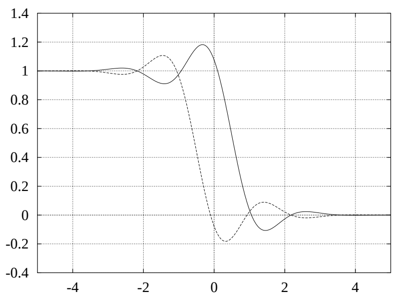

Figure 7. Shock profiles when $p = 10$.

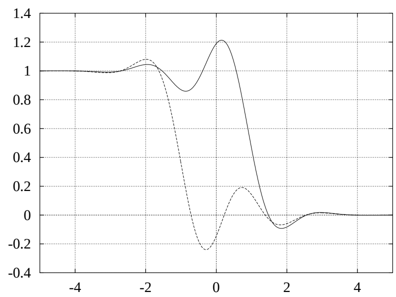

Figure 8. Shock profiles when $p = 11$.

Remark 2.1. It is not known if a function

$$\theta(x) = \sum_{n=1}^{\infty} a_n h_n(x), \quad \vec{a} = (a_0, a_1, \ldots, a_n, \ldots) \in l_2$$

exists such that

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty} x^k\theta(x)dx = \int\limits_{-\infty}^{+\infty} x^k\theta(x)\left(\int\limits_{-\infty}^{x}\theta(y)dy\right)dx,\ k \geq 0.$$

We think that such hypothesis is true.

3. CALCULATIONS OF THE MICROSCOPIC PROFILES OF THE SHOCK WAVE SOLUTIONS OF EQUATIONS OF ELASTICITY THEORY IN THE SENSE OF $\mathbf{R}\langle\varepsilon\rangle$–DISTRIBUTIONS

Let us consider the following system:

$$\begin{aligned} &u_t + (u^2)_x = \sigma_x \ \text{(the conservation law for momentum)},\\ &\sigma_t + u\sigma_x = k^2u_x \ \text{(the Hooke law)}, \end{aligned} \tag{3.1}$$

here u is the velocity of a medium and σ is the stress. We suppose that density of a medium is equal to 1 and k^2 is some constant.

We will seek a solution of this system in the following form

$$u(t,x,\varepsilon) = u_0 + \Delta u U\left(\frac{x-vt}{\varepsilon}\right), \tag{3.2}$$

where $u_0, \Delta u, v$ are real numbers, $\Delta u \neq 0$

$$U(x) = \int\limits_{-\infty}^{x}\widetilde{U}(y)dy, \quad \int\limits_{-\infty}^{+\infty}\widetilde{U}(y)dy = 1$$

and $\widetilde{U} \in \mathcal{S}(\mathbf{R})$. Similarly we represent

$$\sigma(t,x,\varepsilon) = \sigma_0 + \Delta\sigma\Sigma\left(\frac{x-vt}{\varepsilon}\right), \tag{3.3}$$

where $\sigma_0, \Delta\sigma, v$ are real numbers, $\Delta\sigma \neq 0$

$$\Sigma(x) = \int\limits_{-\infty}^{x}\widetilde{\Sigma}(y)dy, \quad \int\limits_{-\infty}^{+\infty}\widetilde{\Sigma}(y)dy = 1$$

and $\widetilde{\Sigma} \in \mathcal{S}(\mathbf{R})$. Note that v is a velocity of the shock waves.

On the other hand, we suppose

$$\widetilde{U}(x) = a_0h_0(x) + a_1h_1(x) + \ldots + a_nh_n(x),\ \ \vec{a} = (a_0, a_1, \ldots, a_n), \tag{3.4}$$

$$\widetilde{\Sigma}(x) = c_0 h_0(x) + c_1 h_1(x) + \ldots + c_n h_n(x), \;\; \vec{c} = (c_0, c_1, \ldots, c_n), \tag{3.5}$$

where $h_k(x)$ are Hermite functions.

DEFINITION 3.1. Functions $u \in J$ and $\sigma \in J$ are solutions of the system (3.1) up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any $t \in [0, T]$

$$\int_{-\infty}^{+\infty} \{u_t(t,x,\varepsilon) + 2u(t,x,\varepsilon)u_x(t,x,\varepsilon) - \sigma_x(t,x,\varepsilon)\} \, \psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle, \tag{3.6}$$

$$\int_{-\infty}^{+\infty} \left\{\sigma_t(t,x,\varepsilon) + u(t,x,\varepsilon)\sigma_x(t,x,\varepsilon) - k^2 u_x(t,x,\varepsilon)\right\} \psi(x)dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \tag{3.7}$$

are valid for every $\psi \in \mathcal{S}(\mathbf{R})$. When p is equal to $+\infty$ functions $u(t,x,\varepsilon)$ and $\sigma(t,x,\varepsilon))$ exactly satisfy the system (3.1) in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions. Substituting u and σ into (3.6), (3.7) we get the following relations for the moments

$$\{2u_0\Delta u - v\Delta u\} m_k(\widetilde{U}) + 2(\Delta u)^2 m_k(\widetilde{U}U) - \Delta\sigma m_k(\widetilde{\Sigma}) = 0, \;\; k = 0, 1, \ldots n, \tag{3.8}$$

$$\{u_0\Delta\sigma - v\Delta\sigma\} m_k(\widetilde{\Sigma}) + \Delta u \Delta\sigma m_k(\widetilde{\Sigma}U) - k^2 \Delta u m_k(\widetilde{U}) = 0, \;\; k = 0, 1, \ldots n. \tag{3.9}$$

We denote the moments as usual

$$m_k(\widetilde{U}) = \int_{-\infty}^{+\infty} x^k \widetilde{U}(x)dx,$$

$$m_k(\widetilde{U}U) = \int_{-\infty}^{+\infty} x^k \widetilde{U}(x) \left(\int_{-\infty}^{x} \widetilde{U}(y)dy \right) dx, \;\; k \geq 0,$$

$$m_k(\widetilde{\Sigma}U) = \int_{-\infty}^{+\infty} x^k \widetilde{\Sigma}(x) \left(\int_{-\infty}^{x} \widetilde{U}(y)dy \right) dx, \;\; k \geq 0.$$

It is easy to find v from (3.8) when k=0. Indeed,

$$\{2u_0\Delta u - v\Delta u\} + (\Delta u)^2 - \Delta\sigma = 0.$$

Therefore,

$$v = 2u_0 + \Delta u - \frac{\Delta\sigma}{\Delta u}. \tag{3.10}$$

Substituting v into the (3.8) we have

$$\{\Delta\sigma - (\Delta u)^2\} m_k(\widetilde{U}) + 2(\Delta u)^2 m_k(\widetilde{U}U) - \Delta\sigma m_k(\widetilde{\Sigma}) = 0, \quad k \geq 0. \qquad (3.11)$$

ΔU and $\Delta\sigma$ are some real numbers, therefore, all three vectors with coordinates $m_k(\widetilde{U})$, $m_k(\widetilde{U}U)$ and $m_k(\widetilde{\Sigma})$, $k = 0, 1, 2, \dots n$, respectively should be collinear. However, $m_0(\widetilde{U}) = m_0(\widetilde{\Sigma}) = 1$. Hence, $m_k(\widetilde{U}) = m_k(\widetilde{\Sigma})$, , $k = 0, 1, 2, \dots n$.

Thus, $\vec{a} = \vec{c}$ and it follows from (3.11) that $m_k(\widetilde{U}) = 2m_k(\widetilde{U}U)$, $k = 0, 1, 2, \dots n$. We have already shown how to solve this system by the Newton iteration method (see conditions (2.4) and solution in this case).

Substituting v into (3.9)and taking into account previous equalities we have

$$\left\{\frac{(\Delta\sigma)^2}{\Delta u} - u_0\Delta\sigma - \Delta u\Delta\sigma\right\} m_k(\widetilde{\Sigma}) + \Delta u\Delta\sigma m_k(\widetilde{\Sigma}\Sigma) - k^2\Delta u m_k(\widetilde{\Sigma}) = 0, \qquad (3.12)$$

where $k = 0, 1, 2, \dots n$. The last expression gives us a relation for constants $\Delta\sigma$, Δu, u_0, k^2 :

$$\left\{\frac{(\Delta\sigma)^2}{\Delta u} - u_0\Delta\sigma - \Delta u\Delta\sigma\right\} + \frac{1}{2}\Delta u\Delta\sigma - k^2\Delta u = 0$$

or

$$(\Delta\sigma)^2 - \left(u_0 + \frac{1}{2}\Delta u\right)\Delta u\Delta\sigma - k^2(\Delta u)^2 = 0.$$

If Δu, u_0, k^2 are known then from the last equation one can find $\Delta\sigma$

$$\Delta\sigma_{1,2} = \frac{1}{2}\left(u_0 + \frac{1}{2}\Delta u\right) \pm \frac{1}{2}|\Delta u|\sqrt{(u_0 + \frac{1}{2}\Delta u)^2 + 4k^2}.$$

In particular, if $\Delta u = -1$, $u_0 = 1$ then

$$\Delta\sigma_{1,2} = -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 + 16k^2}.$$

Shock profiles of the considered system (3.1) are given in Fig. 9, 10, 11, 12. We considered the case when $p = 13$ and $\Delta u = -1$, $u_0 = 1$, $k^2 = 0.1$. We can take any real σ_0, but here we used $\sigma_0 = 0.5$ and then calculated $\Delta\sigma_1$ (the velocity of shocks $v = 1.1531$ in Fig. 9, 11) and $\Delta\sigma_2$ (the velocity of shocks $v = 0.34689$ in Fig.10, 12). Two different types of shock profiles were considered. The first is given in the Fig. 9, 10. The second is given in the Fig. 11, 12.

Theorem 3.1. *For any integer p there exists a solution of the system of equations* (3.1) *in the sense of the definition 3.1. Moreover,*

$$v = 2u_0 + \Delta u - \frac{\Delta\sigma}{\Delta u}$$

and

$$(\Delta\sigma)^2 - \left(u_0 + \frac{1}{2}\Delta u\right)\Delta u \Delta\sigma - k^2(\Delta u)^2 = 0.$$

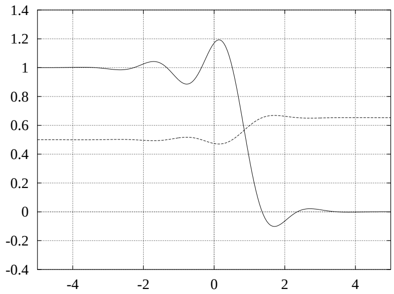

Figure 9. Shock profiles of velocity and stress for $v = 1.1531$.

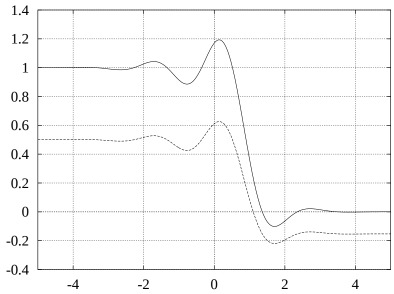

Figure 10. Shock profiles of velocity and stress for $v = 0.34689$.

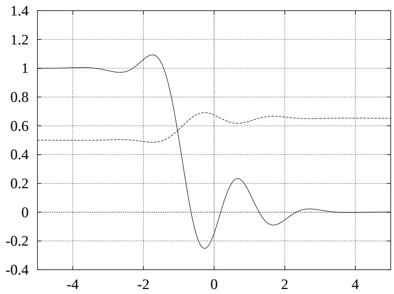

Figure 11. Shock profiles of velocity and stress for $v = 1.1531$.

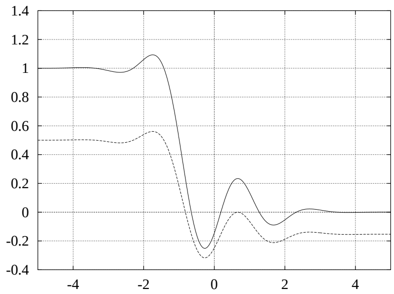

Figure 12. Shock profiles of velocity and stress for $v = 0.34689$.

Let us consider the following system

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0 \ \text{(the conservation law for mass)},\\
&(\rho u)_t + (\rho u^2)_x = \sigma_x \ \text{(the conservation law for momentum)},\\
&\sigma_t + u\sigma_x = k^2 u_x \ \text{(the Hooke law)}.
\end{aligned} \tag{3.13}$$

Here, u is the velocity of a medium and σ is the stress. We suppose that k^2 is a constant.

DEFINITION 3.2. Functions $u \in J$, $\rho \in J$ and $\sigma \in J$ are solutions of the system (3.13) up to e^{-p}, $p \in \mathbf{N}_0$ in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions if for any $t \in [0, T]$

$$\int_{-\infty}^{+\infty} \{\rho_t(t,x,\varepsilon) + \rho_x u + \rho u_x\}\, \psi(x)dx = \sum_{k=p}^{+\infty} \xi_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle,$$

$$\int_{-\infty}^{+\infty} \{\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x - \sigma_x\}\, \psi(x)dx = \sum_{k=p}^{+\infty} \zeta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle,$$

$$\int_{-\infty}^{+\infty} \{\sigma_t + u\sigma_x - k^2 u_x\}\, \psi(x)dx = \sum_{k=p}^{+\infty} \eta_k \varepsilon^k \in \mathbf{R}\langle\varepsilon\rangle \qquad (3.14)$$

are valid for every $\psi \in \mathcal{S}(\mathbf{R})$.

In case when p is equal to $+\infty$ functions $u(t,x,\varepsilon)$, $\rho(t,x,\varepsilon)$ and $\sigma(t,x,\varepsilon)$ exactly satisfy the system (3.13) in the sense of $\mathbf{R}\langle\varepsilon\rangle$–distributions.

We will seek a solution of this system in the form (3.2), (3.3), (3.4), (3.5),

$$\rho(t,x,\varepsilon) = \rho_0 + \Delta\rho R\left(\frac{x - vt}{\varepsilon}\right), \qquad (3.15)$$

$\rho_0, \Delta\rho, v$ are real numbers, $\Delta\rho \neq 0$ and $R(x) = \int_{-\infty}^{x} \widetilde{R}(y)dy$, $\int_{-\infty}^{+\infty} \widetilde{R}(y)dy = 1$ and $\widetilde{R} \in \mathcal{S}(\mathbf{R})$. We suppose

$$\widetilde{R}(x) = b_0 h_0(x) + b_1 h_1(x) + \ldots + b_n h_n(x), \quad \vec{b} = (b_0, b_1, \ldots, b_n). \qquad (3.16)$$

Substituting (3.2) and (3.15) into (3.14) we get the following relations for the moments

$$\{-v\Delta\rho + \Delta\rho u_0\} m_k(\widetilde{R}) + \Delta\rho\Delta u m_k(\widetilde{R}U) + \rho_0 \Delta u m_k(\widetilde{U}) + \Delta\rho\Delta u m_k(\widetilde{U}R) = 0, \qquad (3.17)$$

where $k = 0, 1, \ldots n$. Note that $m_0(\widetilde{R}U) = 1 - m_0(\widetilde{U}R)$. Let us suppose that $k = 0$, then we get

$$-v\Delta\rho + \Delta\rho u_0 + \Delta\rho\Delta u - \rho_0 \Delta u = 0$$

or

$$v = u_0 + \Delta u + \rho_0 \frac{\Delta u}{\Delta\rho}. \qquad (3.18)$$

The last expression gives us the following equation

$$-\Delta u(\rho_0+\Delta\rho)m_k(\widetilde{R})+\Delta\rho\Delta u m_k(\widetilde{R}U)+\rho_0\Delta u m_k(\widetilde{U})+\Delta\rho\Delta u m_k(\widetilde{U}R)=0 \tag{3.19}$$

where $k=0,1,\dots n$.

All four vectors with coordinates $m_k(\widetilde{R})$, $m_k(\widetilde{R}U)$, $m_k(\widetilde{U})$, $m_k(\widetilde{U}R)$ should be collinear. Let us consider $m_k(\widetilde{R})$ and $m_k(\widetilde{U})$. Because of $m_0(\widetilde{R})=m_0(\widetilde{U})=1$ then $m_k(\widetilde{R})=m_k(\widetilde{U})$ for k from 0 to n and therefore $\vec{a}=\vec{b}$. From the last equality we have

$$m_k(\widetilde{R}U)=m_k(\widetilde{U}R)=m_k(\widetilde{U}U),$$

where $k=0,1,\dots n$. Thus

$$-\Delta u(\rho_0+\Delta\rho)m_k(\widetilde{U})+2\Delta\rho\Delta u m_k(\widetilde{U}U)+\rho_0\Delta u m_k(\widetilde{U})=0,$$

where $k=0,1,\dots n$. It means

$$m_k(\widetilde{U})=2m_k(\widetilde{U}U),\ \ k=0,1,\dots n.$$

Substituting (3.2), (3.3) and (3.15) into (3.14) we get the following relations for the moments

$$\begin{aligned} &u_0\Delta\rho(u_0-v)m_k(\widetilde{R})+\rho_0\Delta u(2u_0-v)m_k(\widetilde{U})\\ &\quad+\Delta u\Delta\rho(2u_0-v)\{m_k(\widetilde{U}R)+m_k(\widetilde{R}U)\}\\ &\quad+2\rho_0(\Delta u)^2m_k(\widetilde{U}U)+\rho(\Delta u)^2m_k(\widetilde{R}U^2)\\ &\quad+2\Delta\rho(\Delta u)^2m_k(\widetilde{U}RU)-\Delta\sigma m_k(\Sigma)=0,\end{aligned} \tag{3.20}$$

where $k=0,1,\dots n$. Taking $k=0$ we get

$$\rho_0(\Delta u)^2\left(\frac{\rho_0}{\Delta\rho}+1\right)+\Delta\sigma=0. \tag{3.21}$$

Moreover $m_k(\widetilde{R})=m_k(\widetilde{\Sigma})$, $k=0,1,\dots n$ and then $\vec{b}=\vec{c}$. Finally,

$$\vec{a}=\vec{b}=\vec{c}\quad m_k(\widetilde{R})=3m_k(\widetilde{U}U^2),\ \ k=0,1,\dots n.$$

$$\{u_0\Delta\sigma-v\Delta\sigma\}m_k(\widetilde{\Sigma})+\Delta u\Delta\sigma m_k(\widetilde{\Sigma}U)-k^2\Delta u m_k(\widetilde{U})=0, \tag{3.22}$$

where $k=0,1,\dots n$. Supposing $k=0$ and using expression for the velocity (3.18) we get

$$\Delta\sigma\Delta u\left(\frac{\rho_0}{\Delta\rho}+\frac{1}{2}\right)+k^2\Delta u=0. \tag{3.23}$$

From the (3.22) we find the following equality for the velocity

$$v = u_0 + \Delta u + \frac{1}{2}u_0 - k^2\frac{\Delta u}{\Delta\sigma}. \tag{3.24}$$

It is the well known result in the elasticity theory. Thus, if ρ_0, Δu and k^2 are known then the rest constants can be obtained from the system

$$\begin{cases} \rho_0(\Delta u)^2\left(\rho_0 + \Delta\rho\right) + \Delta\sigma\Delta\rho = 0, \\ \Delta\sigma\left(\rho_0 + \frac{1}{2}\Delta\rho\right) + k^2\Delta\rho = 0. \end{cases} \tag{3.25}$$

Hence,

$$\Delta\sigma = -\frac{2k^2\Delta\rho}{2\rho_0 + \Delta\rho}$$

and

$$\Delta\rho_{1,2} = \frac{-\frac{3}{2}\rho_0^2(\Delta u)^2 \pm \sqrt{\frac{\rho_0^4(\Delta u)^4}{4} + 4k^2(\Delta u)^2\rho_0^3}}{\rho_0(\Delta u)^2 - 2k^2}.$$

Theorem 3.2. *For any integer p there exists a solution of the system of equations* (3.13) *in the sense of the definition 3.2. Moreover,*

$$\begin{cases} \rho_0(\Delta u)^2\left(\rho_0 + \Delta\rho\right) + \Delta\sigma\Delta\rho = 0, \\ \Delta\sigma\left(\rho_0 + \frac{1}{2}\Delta\rho\right) + k^2\Delta\rho = 0, \\ v = u_0 + \Delta u + \rho_0\dfrac{\Delta u}{\Delta\rho}. \end{cases}$$

Shock profiles of the considered system (3.13) are given in Fig. 13, 14, 15, 16. We considered the case when $p = 13$ and $\Delta u = -1$, $u_0 = 1$, $k^2 = 0.1$. We took $\rho_0 = 1.1$, $\sigma_0 = 0.5$ and then we calculated $\Delta\rho$ and $\Delta\sigma$. We consider two different types of shock profiles. The first type is presented in the Fig. 13, 14, the second one in the Fig. 15, 16.

4. CONCLUSIONS AND REMARKS

Our calculation method looks like the Fourier method for solving linear differential equations but it is applied to solve the nonlinear equations. Let us compare the method of mode superposition for a string and our method for the shock. Our method allows to obtain all known formulas for the shocks characteristics and, in addition, it finds a microscopic behaviour of shocks in the thin layer using an assumption that the profile of the shock can be approximated by the orthogonal system of functions. We can use Laguerre functions, harmonic functions or any other orthogonal system in calculations.

The proposed method can be applied to the problems of hydrodynamics, quantum mechanics and non-linear optics.

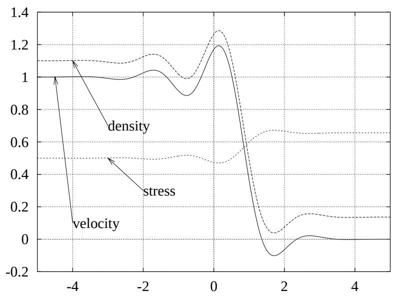

Figure 13. Shock profiles, $v = 1.1417$.

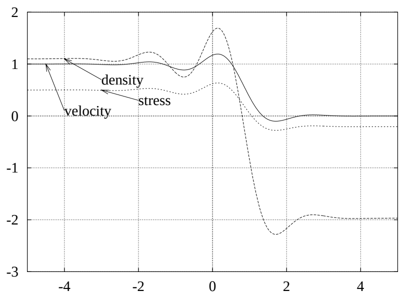

Figure 14. Shock profiles, $v = 0.35833$.

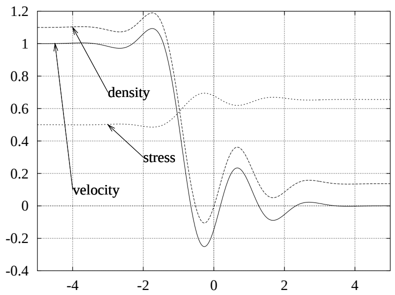

Figure 15. Shock profiles, $v = 1.1417$.

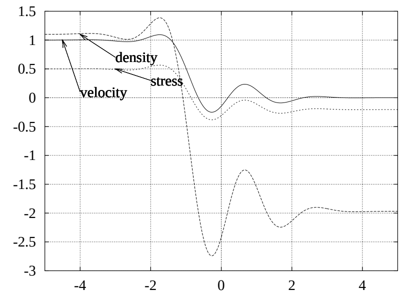

Figure 16. Shock profiles, $v = 0.35833$.

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Funkcionalai su reikšmėmis ne-Archmediniuose Laurent'o sekų laukuose ir jų taikymai elastiškumo teorijos lygtims. II

M. Radyna

Funkcionalai su reikšmėmis ne-archimediniuose Laurent'o sekų laukuose pritaikyti apibrėžti apibendrintąjį Hop'o lygties sprendinį solitono pavidalu. Pasiūlytas skaitinis algoritmas begalo siauro solitono profilio radimui. Taikant šį metodą, profilio radimas suvedamas į netiesinės algebrinių lygčių sistemos erdvėje $R^{n+1}, n > 1$, sprendimą. Parodyta, kad kai kuriuos sprendinius galima surasti naudojant Niutono iteracinį metodą. Pateikiami pavyzdžiai ir skaitiniai testai.