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# Types and Multiplicity of Solutions to Sturm–Liouville Boundary Value Problem*

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**Abstract.** We consider the second-order nonlinear boundary value problems (BVPs) with Sturm–Liouville boundary conditions. We define types of solutions and show that if there exist solutions of different types then there exist intermediate solutions also.

**Keywords:** nonlinear boundary value problem, multiplicity of solutions, Sturm–Liouville problem.

**AMS Subject Classification:** 34B15.

## 1 Introduction

We consider boundary value problem

$$x'' = f(t, x, x'), \quad f \in C^1\big([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big), \tag{1.1}$$

$$a_1 x(a) - a_2 x'(a) = A, \quad b_1 x(b) + b_2 x'(b) = B, \tag{1.2}$$

where $A, B \in \mathbb{R}$, $a_1, b_1 \in \mathbb{R}$, $a_2, b_2 \in \mathbb{R}^+ := (0, +\infty)$, $a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$. Our study continues a series of papers devoted to two-point boundary value problems for the second order nonlinear differential equations [2, 3, 4]. This research is motivated by the papers of L. Jackson [6], H. Knobloch [7] and L. Erbe [5], who studied BVPs for equation (1.1) provided that there exist the so called lower and upper functions. The method is very popular (consult the book C. de Coster, P. Habets [1] for more information) and a lot of papers were written devoted to various BVPs [8,9,10]. Generally the existence of lower and upper functions and some additional conditions, which take into account special features of the problem, imply the existence of a solution and provide information about location of it.

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It was H. Knobloch, who first observed that more can be said about a solution existing in presence of lower and upper functions. He showed for some BVP that there exists a solution with the property (B) which is described later. This interesting property implies that related equation of variations possesses the property that it is disconjugate in the interval $(a, b)$.

In articles [2,3,4] we investigated similar problems for the equation (1.1) in the cases of the Dirichlet and Neumann boundary conditions. This article is a continuation of research. Our study is based on the following works [5,7,12].

Assume $\alpha, \beta \in C[a, b]$ are lower and upper functions of problem (1.1), (1.2) such that $\alpha(t) \le \beta(t)$,

$$\begin{aligned} a_1\alpha(a) - a_2\alpha'(a) \le A, \qquad a_1\beta(a) - a_2\beta'(a) \ge A,\\ b_1\alpha(b) + b_2\alpha'(b) \le B, \qquad b_1\beta(b) + b_2\beta'(b) \ge B. \end{aligned} \tag{1.3}$$

Assume the Nagumo condition

$$\begin{aligned} &\forall t \in [a, b], \quad \alpha(t) \le x(t) \le \beta(t), \quad \forall x'(t) \in \mathbb{R}\\ &\text{there exists } \varphi(x) \in C(\mathbb{R}^+, \mathbb{R}^+) \text{ such that}\\ &\big|f(t, x, x')\big| \le \varphi\big(|x'|\big),\\ &\int_\lambda^N \frac{s\,ds}{\varphi(s)} > \max_{[a,b]} \beta(t) - \min_{[a,b]} \alpha(t) \end{aligned} \tag{1.4}$$

is satisfied, where

$$\lambda = \frac{2M}{b-a}, \quad M := \max_{[a,b]}\{|\beta(t)|, |\alpha(t)|\}.$$

The objective of this paper is to consider the Sturm–Liouville conditions for nonlinear second-order boundary value problem. We consider the case of multiple solutions. We have to distinguish between solutions of the BVP, therefore define an index for solutions and call it *the type of a solution*. The presence of the lower and upper functions $\alpha$ and $\beta$ guarantees the existence of solutions of the BVP. We show that there exists a solution with the specific property.

The paper is organized as follows. In Section 2 we give some preliminary facts on which to base our results. In Section 3 definitions are given. An example of the existence of solutions with several humps considered in Section 4. In Section 5 the main result is formulated and proved, in Section 6 the example of its application is shown.

## 2 Preliminary Results

It was H. Knobloch [7] who showed that the equation (1.1) in presence of regularly ordered ($\alpha \le \beta$) the upper $\beta$ and lower $\alpha$ functions, has a specific solution $x(t)$ located in-between $\alpha$ and $\beta$ which possesses the property (B). A solution $x(t)$ of (1.1) on a compact interval $I$ of the real line is said to have property (B) in case there exists a sequence of solutions $x_n$, of (1.1) such that

- $x_n \to x$ and $x'_n \to x'$ uniformly on $I$;

- $x - x_n \neq 0$, and has the same sign for all $n \geq 1$ and all $t \in I$;
- $|x'_n - x'| \leq c|x_n - x|$ for all $n \geq 1$ and all $t \in I$, where $c$ is a constant independent of $n$ and $t$.

Later L.H. Erbe [5] formulated property $(B)^*$: a solution $x$ of (1.1) is said to have property $(B)^*$ on $[a, b]$ in case there exists a sequence of solutions $x_n$ of (1.1), all having property $(B)$ on $[a, b]$, such that $x_n \to x$ and $x'_n \to x'$ uniformly on $[a, b]$.

*Remark 1.* [5, p. 459] Under certain continuity conditions on the partial derivatives $f_x$, and $f_{x'}$ the existence of a solution $x(t)$ of (1.1) with property $(B)$ implies that the corresponding equation of variation along $x(t)$

$$y'' = f_x\big(t, x(t), x'(t)\big)y + f_{x'}\big(t, x(t), x'(t)\big)y' \tag{2.1}$$

is disconjugate on $(a, b)$, that is, the only solution of (2.1) with more than one zero on $I$ is the identically zero solution. Using the limiting process, we can say that the same is true for solutions $x(t)$ which possess the property $(B)^*$.

Also interesting for us is the following theorem, which L. H. Erbe has proved in [5].

**Theorem 1.** [5, Th. 3.6, p. 465] *Let $\alpha(t)$ be a lower solution and $\beta(t)$ an upper solution of* (1.1) *with $\alpha(t) \leq \beta(t)$ on $[a, b]$ and $\alpha(a) < \beta(a)$, $\alpha(b) < \beta(b)$. Assume the Nagumo condition holds and let $g(x, x') \in G^1$ and $h(x, x') \in H$.*[2] *Then there is a solution $x_0(t)$ of the BVP*

$$x'' = f(t, x, x'), \quad g\big(x(a), x'(a)\big) = 0 = h\big(x(b), x'(b)\big),$$

*which satisfies $\alpha(t) < x_0(t) < \beta(t)$ on $[a, b]$.*

*Remark 2.* If we select the function $g(x(a), x'(a)) = -a_1x(a) + a_2x'(a) + A$ and $h(x(b), x'(b)) = b_1x(b) + b_2x'(b) - B$, then the inequalities (1.3) hold and the BVP (1.1), (1.2) has a solution on $[a, b]$.

**Theorem 2.** [5, Th. 4.5, p. 469] *Assume all hypotheses of Theorem* 1 *and, in addition, assume the Lipschitz condition*[3] *with respect to $x$ and $x'$ hold. Then there is a solution $x_0(t)$ of the BVP*

$$x'' = f(t, x, x'), \quad g\big(x(a), x'(a)\big) = 0 = h\big(x(b), x'(b)\big),$$

*which has property $(B)^*$ and satisfies $\alpha(t) \leq x_0(t) \leq \beta(t)$ on $[a, b]$.*

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[1] The class of all continuous functions $g(x, x')$ defined on $[\alpha(a), \beta(a)] \times R$ which are nondecreasing in $x'$ and satisfy $g(\alpha(a), \alpha'(a)) \geq 0$, $g(\beta(a), \beta'(a)) \leq 0$.

[2] The class of all continuous functions $h(x, x')$ defined on $[\alpha(b), \beta(b)] \times R$ which are nondecreasing in $x'$ and satisfy $h(\alpha(b), \alpha'(b)) \leq 0$, $h(\beta(b), \beta'(b)) \geq 0$.

[3] There exist two non-negative constants $K$ and $L$ such that whenever $(t, y, y')$ and $(t, x, x')$ are in the domain of $f$, the inequality $|f(t, y, y') - f(t, x, x')| \leq K|y - x| + L|y' - x'|$ holds.

## 3 Definitions

Provided that function $f(t, x, x')$ has continuous partial derivatives $f_x$ and $f_{x'}$ we construct the equation of variations for a particular solution $\xi(t)$

$$y'' = f_x\big(t, \xi(t), \xi'(t)\big)y + f_{x'}\big(t, \xi(t), \xi'(t)\big)y' \tag{3.1}$$

and consider this with the normalized initial conditions

$$a_1 y(a) - a_2 y'(a) = 0 \quad \text{and} \quad y^2(a) + y^2(a) = 1. \tag{3.2}$$

DEFINITION 1. Let $\xi(t)$ be a solution of problem (1.1), (1.2). We say that the type of a solution $\xi(t)$ is $\boldsymbol{i}$ $(i \neq 0)$, if the solution $y(t)$ of initial value problem (3.1), (3.2) has exactly $i$ zeros $\tau_i \in (a, b)$ and there is $\tau \in (\tau_{\mathrm{i}}, b)$ such that

$$b_1 y(\tau) + b_2 y'(\tau) = 0 \quad \text{and} \quad b_1 y(b) + b_2 y'(b) \neq 0. \tag{3.3}$$

Denote this *type*$(\xi) = i$. If moreover $b_1 y(b) + b_2 y'(b) = 0$, denote *type*$(\xi) = (i, i + 1)$.

*Remark 3.* A solution $x_0(t)$ of problem (1.1), (1.2) is of type zero if either the respective $y(t)$ does not vanish in $(a, b]$ or $y(t)$ has exactly one zero $\tau_1$ in $(a, b)$ but $b_1 y(t) + b_2 y'(t) \neq 0$ in $(\tau_1, b]$.

*Remark 4.* If $\xi(t)$ is an $i$-type solution of the problem (1.1), (1.2) according to Def. 1, then for sufficiently close to $\xi(t)$ solutions $x(t)$ of the problem (1.1), $a_1 x(a) - a_2 x'(a) = A$ the difference $u(t) = x(t) - \xi(t)$ has exactly $i$ zeros $\eta_i$ in the interval $(a, b)$ and in the interval $(\eta_i, b)$ there is a point $\zeta$ such that $u(\zeta) = 0$ and $u(b) \neq 0$. Solutions $x(t)$ will be called *neighboring solutions* to solution $\xi(t)$.

Similar definitions in other situation were introduced in [12].

## 4 Multibump Solutions

The type of a solution may be introduced also in this way (we do not use the below Definition 2 due to below argument).

DEFINITION 2. Let $\xi(t)$ be a solution of problem (1.1), (1.2). We say that the type of a solution $\xi(t)$ is $\boldsymbol{i}$ $(i \neq 0)$, if the solution $y(t)$ of initial value problem (3.1), (3.2) has exactly $i$ points $\tau \in (a, b)$ such that

$$b_1 y(\tau) + b_2 y'(\tau) = 0 \quad \text{and} \quad b_1 y(b) + b_2 y'(b) \neq 0. \tag{4.1}$$

We face then the following problem. There may be multiple points $\tau$ between two consecutive zeros of $y(t)$, where the conditions $b_1 y(\tau) + b_2 y'(\tau) = 0$ are fulfilled. To show that this is possible, let us consider the specific boundary conditions (the Neumann ones)

$$x'(a) = 0, \qquad x'(b) = 0.$$

We will construct the example of a linear equation of the form $y'' = -q(t)y$, which possesses the property that there are multiple points $\tau$ where $y'(t)$ vanishes.[4]

[4] Solutions of t his type in the theory of the fourth order ordinary differential equations are called in the literature "multibump" solutions [11]. This is why we use this term in our study.

**Proposition 1.** *Linear differential equations of second order with continuous coefficients may have multibump solutions.*

The proof follows from the following example.

*Example 1.* We consider the problem

$$y'' = -8\sin 10ty, \quad y(0) = 1, \quad y'(0) = 0. \tag{4.2}$$

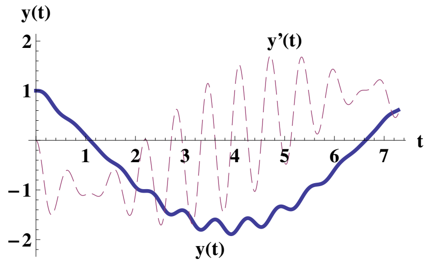


**Figure 1.** Graphs of $y(t)$ (solid line) and $y'(t)$.

As shown in Figure 1 there may be multiple zeros of $y'(t)$ between two consecutive zeros of $y(t)$.

## 5 Main Results

**Theorem 3.** *Suppose that solutions of $x'' = f(t, x, x')$ are extendable to the interval $[a, b]$. If there exist a solution $x_1$ of problem (1.1), (1.2) of type $i_1$ and a solution $x_2$ of type $i_2$, $|i_1 - i_2| \geq 2$, then there exist also all intermediate solutions.*

*Proof.* Consider boundary conditions (1.2) and denote the first one[5] (1.2$a$) and the second one[6] (1.2$b$). We wish to show that if there exist solutions $x_{i_1}$ and $x_{i_2}$ of the problem (1.1), (1.2) then there exist at least $|i_1 - i_2| - 1$ other solutions of the problem. We prove the result for the specific case $i_1 = 1$ and $i_2 = 3$. Our goal is to show that there exists at least one more solution of the problem. The proof for the general case can be conducted similarly.

Denote $x_1$ a solution of type 1 and, similarly, $x_3$ a solution of type 3. Both solutions satisfy the conditions (1.2$a$) and (1.2$b$). For simplicity of notation and for better understanding let us proceed with the Neumann boundary conditions

$$x'(a) = A, \quad x'(b) = B, \tag{5.1}$$

which are included in (1.2) ($a_1 = b_1 = 0$, $a_2 = -1$, $b_2 = 1$). Suppose for definiteness that $x_1(a) < x_3(a)$ and consider solutions $x(t; \gamma)$ of the Cauchy problems

$$x(a) = \gamma \in \big[x_1(a), x_3(a)\big], \quad x'(a) = A. \tag{5.2}$$

[5] $l_1(a) : a_1 x(a) - a_2 x'(a) = A$.
[6] $l_2(b) : b_1 x(b) + b_2 x'(b) = B$.

Introduce functions

$$u_1(t;\gamma) = x(t;\gamma) - x_1(t), \qquad u_3(t;\gamma) = x_3(t) - x(t;\gamma).$$

Both functions satisfy $u'(a;\gamma) = 0$ for all $\gamma \in [x_1(a), x_3(a)]$. If for some $\gamma_*$ one of the functions satisfies $u'(b;\gamma_*) = 0$ then a solution $x(t;\gamma_*)$ solves BVP (1.1), (5.1).

Consider $u_1(t;\gamma)$. For $\gamma > x_1(a)$ and close enough $u_1$ has a zero $t_1(\gamma)$ in $(a,b)$ and a point $\tau \in (t_1,b)$ such that $u'(\tau;\gamma) = 0$. This is true since $x_1$ by assumption is a solution of type 1. If for some $\gamma_1 \in (x_1(a), x_3(a))$ function $u_1(t;\gamma_1)$ has exactly two zeros $t_1$ and $t_2$ in $(a,b)$ and a point $\tau \in (t_1,b)$ such that $u'(\tau) < 0$, then, by continuity arguments, for some smaller $\gamma$ it had $u'(b;\gamma) = 0$. Therefore a solution $x(t;\gamma)$ solves BVP (1.1), (5.1) and is different of $x_1$ and $x_3$.

Suppose $u_1(t;\gamma)$ for all $\gamma \in [x_1(a), x_3(a)]$ has at most one zero in $(a,b)$. Consider function $u_3(t;\gamma)$. It has exactly three zeros $t_1$, $t_2$ and $t_3$ in $(a,b)$ for $\gamma < x_3(a)$ and close enough since $x_3$ is a solution of type 3. Notice that $u_3(t;x_1(a)) = x_3(t) - x_1(t) = u_1(t;x_3(a))$. By assumption $u_1(t;x_3(a))$ has at most 1 zero in $(a,b)$. Therefore $u_3(t;\gamma)$ for $\gamma$ close enough to $x_1(a)$ has one zero in $(a,b)$. Then there exist $\gamma_2 < \gamma_3$ in $(x_1(a), x_3(a))$ such that $u_3(b;\gamma_3) = 0$ (the third zero $t_3$ passes through $t = b$) and $u_3(b;\gamma_2) = 0$ (the second zero $t_2$ passes through $t = b$). Then there exists $\gamma^* \in (\gamma_2,\gamma_3)$ such that $u_3'(b;\gamma^*) = 0$. Therefore $x(t;\gamma^*)$ solves BVP (1.1), (5.1) and is different of $x_1$ and $x_3$.

It may happen that $u_1(t;\gamma)$ for some $\gamma \in [x_1(a), x_3(a)]$ has two zeros $t_1$ and $t_2$ but $u'(t) \neq 0$ in $(t_2,b)$ (then $u'(t) > 0$ for $t \in (t_2,b)$). The same type analysis can be made for $u_3$ showing the existence of a solution of BVP. □

**Theorem 4.** *Suppose there exist lower and upper functions $\alpha$ and $\beta$ in the problem* (1.1), (1.2) *and the Nagumo condition holds. Suppose also that there exists a solution $\xi(t)$ of the type $k$ $(k > 1)$, $\xi(t)$ is located between $\alpha(t)$ and $\beta(t)$. Then there exist at least $2k$ other solutions.*

*Proof.* Since $\xi(t)$ is located between $\alpha(t)$ and $\beta(t)$ it does not coincide neither with $\beta$ nor with $\alpha$. Consider the region $\omega(\xi,\beta)$ between $\xi$ and $\beta$. One may consider $\xi$ as a lower function for this region since all the conditions for lower functions are fulfilled. By Theorem 2 there exists a solution $x_0(t)$ of zero type in this region. Consider now solutions $\xi$ and $x_0$ in the region $\omega(\alpha,\beta)$. By Theorem 3, there exist more $k-1$ solutions of the problem. Totally there are $k$ solutions (with $x_0$) not counted a solution $\xi$.

The same analysis, made for the region $\omega(\alpha,\xi)$, shows that there are at least $k$ solutions in this region. Then the total number of solutions of the BVP in the region $\omega(\alpha,\beta)$ is at least $2k$, a solution $\xi$ not counted. □

## 6 Example

*Example 2.* We consider

$$x'' = x^3 - \left(\frac{5\pi}{2}\right)^2 x,$$
$$x(0) - x'(0) = 0, \quad x(1) + x'(1) = 0. \tag{6.1}$$

The upper and lower functions are $\beta(t) \equiv 8$ and $\alpha(t) \equiv -8$. According to Theorem 2 this problem has solution $\xi^*$ (and $\xi_*$) of zero type.

Let us construct the equation of variations for $\xi_2(t) \equiv 0$ and consider with the initial conditions:

$$y''(t) = -\left(\frac{5\pi}{2}\right)^2 y, \tag{6.2}$$
$$y(0) - y'(0) = 0, \quad y^2(0) + y'^2(0) = 1.$$

The solution of this problem has two zeros in $(0, 1)$, hence, *type*$(\xi_2 \equiv 0) = 2$. According to Theorem 3 this problem has solutions of type one: there is a solution $\xi_1(t)$ and the symmetric of $\xi_1(t)$.

Figure 2 shows all types of this problem. Figure 3 shows the phase plane respectively.

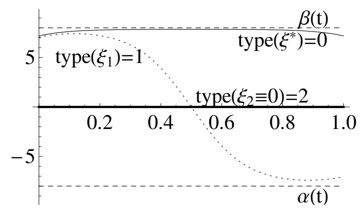


**Figure 2.** All types of the problem (6.1)

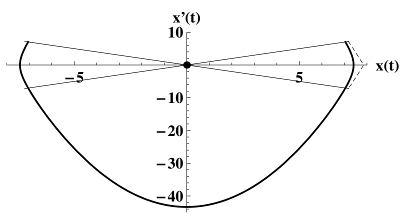


**Figure 3.** Phase plane (the trivial solution $\xi_2$ in the phase plane is a point at the origin).

## 7 Conclusions

We can introduce types of solutions of Sturm–Liouville BVP. The BVP can have solutions of different types. For instance, if upper and lower functions exist, then there exists a solution of zero type. As a consequence, if there exists a solution of type $k$, $k > 0$, then there exist more solutions of BVP. If there exist solutions of different types $k_1$ and $k_2$, $|k_1 - k_2| \geq 2$, then there exist at least $|k_1 - k_2| - 1$ solutions of BVP.

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