Approximate Transmission Conditions for a Poisson Problem at Mid-Diffusion

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Abstract. This work consists in the asymptotic analysis of the solution of Poisson equation in a bounded domain of $\mathbb{R}^P$ ($P = 2, 3$) with a thin layer. We use a method based on hierarchical variational equations to derive an explicitly asymptotic expansion of the solution with respect to the thickness of the thin layer. We determine the first two terms of the expansion and prove the error estimate made by truncating the expansion after a finite number of terms. Next, using the first two terms of the asymptotic expansion, we show that we can model the effect of the thin layer by a problem with transmission conditions of order two.

Keywords: asymptotic analysis, asymptotic expansion, approximate transmission conditions, thin layer, Poisson equation.

AMS Subject Classification: 35C20; 35B40; 78M35; 41A60; 35J25; 78M30.

1 Introduction

This paper deals with the study of the asymptotic behavior of the solution of the Poisson equation in a bounded domain Ω of $\mathbb{R}^P$ ($P = 2, 3$) consisting of two sub-domains separated by a thin layer of thickness δ (destined to tend to 0). The mesh of these thin geometries leads to systems with a condition number that increases significantly (cf. [27]). To overcome this difficulty, we adopt asymptotic methods to model the effect of the thin layer by problems with appropriate transmission conditions. These methods are widely used in different frameworks (cf., e.g., [1, 2, 7, 9, 10, 12, 17, 18]). Especially, effective boundary conditions have been obtained for the Helmholtz and Maxwell equations in [3, 4, 15, 16] in smooth geometries and in [11] for Poisson problem in a polygonal domain. The case of approximate transmission conditions has been studied in [6, 13, 21, 25, 26] for thin layer and in [13, 22, 23] for rough and periodic thin layer. In [24, Part II] Schmidt derived an approximation of the solution inside the thin layer based on families of functions called optimal basis functions.

Our motivation comes from [20, 25], in which the authors have worked on problems of electromagnetic and biological origins. Our Problem is inspired from the thesis of Poignard [20, Chapter 2]. He considered a cell immersed in an ambient medium and studied the electric field in the transverse magnetic (TM) mode at mid-frequency.

Let us give now precise notations. The domain Ω is composed of smooth sub-domains: an open bounded subset $\Omega_{\text{int},\delta}$ with regular boundary $\Gamma_{\delta,1}$, an exterior domain $\Omega_{\text{ext},\delta}$ with disjoint regular boundaries $\Gamma_{\delta,2}$ and $\partial\Omega$, and a membrane Ω_δ (thin layer) of thickness δ separating $\Omega_{\text{int},\delta}$ from $\Omega_{\text{ext},\delta}$ (see Fig. 1).
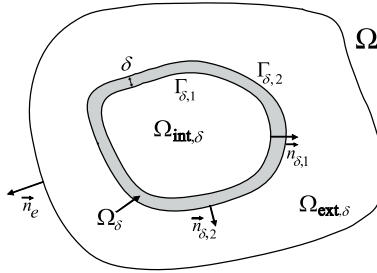

Figure 1. Geometric data.

Define the piecewise regular function α by

$$\alpha(x) = \begin{cases} \alpha_e & \text{if } x \in \Omega_{\text{ext},\delta}, \\ \alpha_\delta & \text{if } x \in \Omega_\delta, \\ \alpha_i & \text{if } x \in \Omega_{\text{int},\delta}, \end{cases}$$

where α_e, α_δ and α_i are strictly positive constants satisfying $\alpha_i < \alpha_\delta < \alpha_e$ or $\alpha_e < \alpha_\delta < \alpha_i$ which correspond to the case of mid-diffusion. For a given f in $\mathcal{C}^\infty(\overline{\Omega})$, we are interested in the solution u_δ of the following diffusion problem

$$\begin{cases} -\text{div}\,(\alpha\nabla u_\delta) = f & \text{in } \Omega, \\ u_{\delta|\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The main result of this paper consists of an asymptotic expansion of the solution u_δ of Problem (1.1) in presence of a thin layer in terms of its thickness δ and the derivation of the associated approximated model of order 1. The position of the limit interface Γ (see Fig. 2 and Fig. 3) for δ tends to 0 is varied and a particular choice of Γ leads to a simple problem involving the Poisson equation with transmission conditions of order two on the interface Γ which lies between $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$.

We use an approach based on variational equations (cf. [3, 6]) to derive transmission conditions with tangential derivatives of order two on Γ, modeling the effect of the thin layer. However, it seems that the existence and uniqueness of the solution of this problem is not obvious. Therefore, we rewrite the problem as a pseudodifferential equation (cf. [5]) and show that in the case of mid-diffusion, we can find the appropriate position of the surface Γ
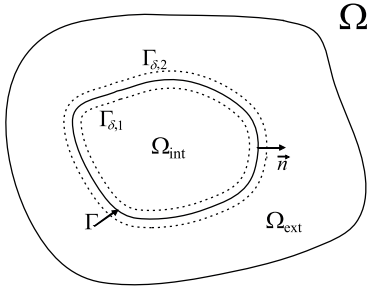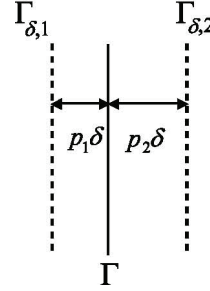
Figure 2. Geometry of the studied problem.

Figure 3. The thin layer Ω_δ.

to solve this equation. The problems are similar in 2D and 3D. We treat the three-dimensional case and we give a remark to the two-dimensional case.

The present paper is organized as follows. In Section 2, we give the statement of the model problem considered. In Section 3, we collect basic results of differential geometry of surfaces. Sections 4 and 5 are devoted to the asymptotic analysis of our problem. We present, in Section 4, hierarchical variational equations suited to the construction of a formal asymptotic expansion up to any order, while Section 5 focuses on the convergence of this ansatz. With the help of the asymptotic expansion of the solution u_δ, we model, in the last section, the effect of the thin layer by a problem with appropriate transmission conditions.

2 Problem Setting

We consider a parallel surface Γ to $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$ dividing Ω_δ into two thin layers $\Omega_{\delta,1}$ and $\Omega_{\delta,2}$ of thickness respectively $p_1\delta$ and $p_2\delta$, where p_1 and p_2 are nonnegative real numbers satisfying $p_1 + p_2 = 1$ and such that p_1 and p_2 belong to a small neighborhood of $1/2$ (see Fig. 2 and Fig. 3). The term *small neighborhood* means that the constants p_1 and p_2 are not too close to 1 or 0, in order to avoid having a layer too thin compared to the other because the following analysis does not lend itself to this case. Let us denote by $u_{\text{ext},\delta}$, $u_{d_2,\delta}$, $u_{d_1,\delta}$ and $u_{\text{int},\delta}$ the restrictions of u_δ respectively to the domains $\Omega_{\text{ext},\delta}$, $\Omega_{\delta,2}$, $\Omega_{\delta,1}$ and $\Omega_{\text{int},\delta}$. Under the aforementioned assumptions, we investigate in $H^1(\Omega)$ the solution u_δ of the following problem

$$
\begin{cases}
-\operatorname{div}\left(\alpha_e \nabla u_{\text{ext},\delta}\right) = f_{|\Omega_{\text{ext},\delta}} & \text{in } \Omega_{\text{ext},\delta}, & (2.1a) \\
-\operatorname{div}\left(\alpha_\delta \nabla u_{d_2,\delta}\right) = f_{|\Omega_{\delta,2}} & \text{in } \Omega_{\delta,2}, & (2.1b) \\
-\operatorname{div}\left(\alpha_\delta \nabla u_{d_1,\delta}\right) = f_{|\Omega_{\delta,1}} & \text{in } \Omega_{\delta,1}, & (2.1c) \\
-\operatorname{div}\left(\alpha_i \nabla u_{\text{int},\delta}\right) = f_{|\Omega_{\text{int},\delta}} & \text{in } \Omega_{\text{int},\delta}, & (2.1d) \\
u_{\text{ext},\delta|\partial\Omega} = 0 & \text{on } \partial\Omega & (2.1e)
\end{cases}
$$

with transmission conditions

$$
\begin{cases}
u_{d_2,\delta|\Gamma_{\delta,2}} = u_{\text{ext},\delta|\Gamma_{\delta,2}} & \text{on } \Gamma_{\delta,2}, & (2.1f) \\
\alpha_\delta \partial_{\mathbf{n}_{\delta,2}} u_{d_2,\delta|\Gamma_{\delta,2}} = \alpha_e \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} & \text{on } \Gamma_{\delta,2}, & (2.1g)
\end{cases}
$$

$$\begin{cases} u_{d_1,\delta|\Gamma} = u_{d_2,\delta|\Gamma} & \text{on } \Gamma, & (2.1\text{h}) \\ \alpha_\delta \partial_{\mathbf{n}} u_{d_1,\delta|\Gamma} = \alpha_\delta \partial_{\mathbf{n}} u_{d_2,\delta|\Gamma} & \text{on } \Gamma, & (2.1\text{i}) \\ u_{\text{int},\delta|\Gamma_{\delta,1}} = u_{d_1,\delta|\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1}, & (2.1\text{j}) \\ \alpha_i \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}} = \alpha_\delta \partial_{\mathbf{n}_{\delta,1}} u_{d_1,\delta|\Gamma_{\delta,1}} & \text{on } \Gamma_{\delta,1}, & (2.1\text{k}) \end{cases}$$

where $\partial_{\mathbf{n}_{\delta,1}}$, $\partial_{\mathbf{n}}$, $\partial_{\mathbf{n}_{\delta,2}}$ and $\partial_{\mathbf{n}_e}$ denote the derivatives in the direction of the unit normal vectors $\mathbf{n}, \mathbf{n}_{\delta,1}, \mathbf{n}_{\delta,2}$ and $\mathbf{n}_e$ to $\Gamma_{\delta,1}, \Gamma, \Gamma_{\delta,2}$ and $\partial\Omega$ respectively (see Fig. 2). The following theorem gives the existence and uniqueness of the solution u_δ to Problem (2.1).

Theorem 1. *Problem (2.1) admits a unique solution u_δ in $H^1(\Omega)$. Furthermore, there exists a constant c independent of δ such that*

$$\|u_\delta\|_{H^1(\Omega)} \le c \, \|f\|_{L^2(\Omega)} \, .$$

Proof.   Existence and uniqueness of u_δ follow from Lax–Milgram lemma [8]. $\square$

3   Recalls of Differential Geometry

The goal of this section is to define and to collect the main features of differential geometry [14] (see also [19]) in order to formulate our problem in a fixed domain (independent of δ) which is a key tool to determine the asymptotic expansion of the solution u_δ. In the sequel, the greek indice β takes the values 1 and 2. Let $I_{\delta,1} = (-\delta, 0)$ and $I_{\delta,2} = (0, \delta)$. We parameterize the thin shell $\Omega_{\delta,\beta}$ by the manifold $\Gamma \times I_{\delta,\beta}$ through the mapping ψ_β defined by

$$\begin{cases} \Gamma \times I_{\delta,\beta} & \overset{\psi_\beta}{\to} & \Omega_{\delta,\beta} \\ (m, \eta_\beta) & \to & x := m + p_\beta \eta_\beta \mathbf{n}(m). \end{cases}$$

As well-known [14], if the thickness of $\Omega_{\delta,\beta}$ is small enough, ψ_β is a C^∞-diffeomorphism of manifolds and it is also known [21, Remark 2.1] that the normal vector $\mathbf{n}_{\delta,\beta}$ to $\Gamma_{\delta,\beta}$ can be identified to $\mathbf{n}$. To each function v_β defined on $\Omega_{\delta,\beta}$, we associate the function $\widetilde{v}_\beta$ defined on $\Gamma \times I_{\delta,\beta}$ by

$$\begin{cases} \widetilde{v}_\beta(m, \eta_\beta) := v_\beta(x), \\ \qquad\quad x = \psi_\beta(m, \eta_\beta), \end{cases}$$

then, we have

$$\nabla v_\beta = (I + p_\beta \eta_\beta \mathcal{R})^{-1} \nabla_\Gamma \widetilde{v}_\beta + p_\beta^{-1} \frac{\partial \widetilde{v}_\beta}{\partial \eta_\beta} \mathbf{n},$$

where $\nabla_\Gamma \widetilde{v}_\beta(m)$ and $\mathcal{R}$ are respectively the surface gradient of $\widetilde{v}$ at $m \in \Gamma$ and the curvature operator $\mathcal{R}$ of Γ at point m. The volume element on the thin shell $\Omega_{\delta,\beta}$ is given by

$$d\Omega_{\delta,\beta} = p_\beta \det (I + p_\beta \eta_\beta \mathcal{R}) \, d\Gamma d\eta_\beta.$$

Now, we introduce the scaling $s_\beta = \eta_\beta / \delta$, and the intervals $I_1 = (-1, 0)$ and $I_2 = (0, 1)$ such that the C^∞-diffeomorphism Φ_β, defined by

$$\begin{cases} \Omega^\beta := \Gamma \times I_\beta & \overset{\Phi_\beta}{\to} \quad \Omega_{\delta,\beta} \\ \quad\quad (m, s_\beta) & \to \quad x := m + \delta p_\beta s_\beta \mathbf{n}(m) \end{cases}$$

parameterizes the thin shell $\Omega_{\delta,\beta}$. To any function v_β defined on $\Omega_{\delta,\beta}$, we associate the function $v^{[\beta]}$ defined on Ω^β through

$$\begin{cases} v^{[\beta]}(m, s_\beta) := v_\beta(x), \\ \quad\quad x = \Phi_\beta(m, s_\beta), \end{cases}$$

then the gradient takes the form

$$\nabla v_\beta = (I + \delta p_\beta s_\beta \mathcal{R})^{-1} \nabla_\Gamma v^{[\beta]} + p_\beta^{-1} \delta^{-1} \frac{\partial v^{[\beta]}}{\partial s_\beta} \mathbf{n}. \tag{3.1}$$

The volume element on the thin shell $\Omega_{\delta,\beta}$ becomes

$$d\Omega_{\delta,\beta} = p_\beta \delta \det J_{\delta,\beta} \, d\Gamma ds_\beta, \tag{3.2}$$

where $J_{\delta,\beta} := I + p_\beta \delta s_\beta \mathcal{R}$. Let u_β and v_β be two regular functions defined on $\Omega_{\delta,\beta}$. From (3.1) and (3.2), we get the change of variables formula

$$\int_{\Omega_{\delta,\beta}} \nabla u_\beta . \nabla v_\beta \, d\Omega_{\delta,\beta} = p_\beta \delta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta$$

$$+ p_\beta^{-1} \delta^{-1} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta. \tag{3.3}$$

Remark 1. For any function u defined in a neighborhood of Γ, we denote, for convenience, by $u_{|\Gamma}$ the trace of u on Γ indifferently in local coordinates or in Cartesian coordinates.

4 The Asymptotic Analysis

This section is devoted to the asymptotic analysis of the solution of Problem (2.1). We show that the latter is equivalent to a variational equation from which we derive the asymptotic expansion of u_δ. We give a hierarchy of variational equations needed to determine the terms of the expansion and we calculate the first two terms of the expansion.

Let v_d be in $H^1(\Omega_\delta)$. We denote by v_{d_β} its restriction to $\Omega_{\delta,\beta}$. Multiplying Equation

$$-\text{div}\,(\alpha_\delta \nabla u_{d,\delta}) = f_{|\Omega_\delta} \quad \text{in } \Omega_\delta$$

by test functions v_d, using (2.1g), (2.1i), (2.1k) and Green's formula, we get

$$\langle \alpha_i \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}}, v_{d_1|\Gamma_{\delta,1}} \rangle_{H^{-1/2}(\Gamma_{\delta,1}) \times H^{1/2}(\Gamma_{\delta,1})} + \alpha_\delta \int_{\Omega_\delta} \nabla u_{d,\delta} . \nabla v_d \, d\Omega_\delta$$

$$- \langle \alpha_e \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}}, v_{d_2|\Gamma_{\delta,2}} \rangle_{H^{-1/2}(\Gamma_{\delta,2}) \times H^{1/2}(\Gamma_{\delta,2})} = \int_{\Omega_\delta} f_{|\Omega_\delta} v_d \, d\Omega_\delta,$$

in which $\langle .,. \rangle_{H^{-1/2}(\Gamma_{\delta,\beta}) \times H^{1/2}(\Gamma_{\delta,\beta})}$ denotes the duality pairing between $H^{-1/2}(\Gamma_{\delta,\beta})$ and $H^{1/2}(\Gamma_{\delta,\beta})$. We use the dilation in the thin layer and Formula (3.3), to obtain

$$
\left\langle \alpha_i \partial_{\mathbf{n}_{\delta,1}} u_{\text{int},\delta|\Gamma_{\delta,1}} \circ \Phi_1(m,-1), v_d^{[1]}(m,-1) \right\rangle_{H^{-1/2}(\Gamma \times \{-1\}) \times H^{1/2}(\Gamma \times \{-1\})}
$$
$$
- \left\langle \alpha_e \partial_{\mathbf{n}_{\delta,2}} u_{\text{ext},\delta|\Gamma_{\delta,2}} \circ \Phi_2(m,1), v_d^{[2]}(m,1) \right\rangle_{H^{-1/2}(\Gamma \times \{1\}) \times H^{1/2}(\Gamma \times \{1\})}
$$
$$
+ \sum_{\beta=1}^{2} \left[\alpha_\delta \delta a_\delta^{[\beta]} \left(u_{d,\delta}^{[\beta]}, v_d^{[\beta]} \right) \right] = \int_{\Omega_\delta} f_{|\Omega_\delta} v_d \, d\Omega_\delta, \tag{4.1}
$$

which is the starting point for the asymptotic analysis, where the bilinear form $a_\delta^{[\beta]}(.,.)$ is defined by

$$
a_\delta^{[\beta]} \left(u^{[\beta]}, v^{[\beta]} \right) := p_\beta \int_{\Omega^\beta} J_{\delta,\beta}^{-2} \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta
$$
$$
+ p_\beta^{-1} \delta^{-2} \int_{\Omega^\beta} \partial_{s_\beta} u^{[\beta]} \partial_{s_\beta} v^{[\beta]} \det J_{\delta,\beta} \, d\Gamma ds_\beta \tag{4.2}
$$

for every $u^{[\beta]}$ and $v^{[\beta]}$ in $H^1(\Omega^\beta)$.

4.1   Hierarchy of the variational equations

In the spirit of [21, 25], we will consider two asymptotic expansions. Exterior expansions corresponding to the asymptotic expansion of u_δ restricted to $\Omega_{\text{ext},\delta}$ and to $\Omega_{\text{int},\delta}$ characterized by the ansatz

$$
u_{\text{ext},\delta} = u_{\text{ext},0} + \delta u_{\text{ext},1} + \cdots , \tag{4.3}
$$
$$
u_{\text{int},\delta} = u_{\text{int},0} + \delta u_{\text{int},1} + \cdots , \tag{4.4}
$$

in which the terms $u_{\text{int},n}$ and $u_{\text{ext},n}$ ($n \in \mathbb{N}$) are independent of δ and defined on Ω_{int}, where Ω_{int} is a bounded domain of $\mathbb{R}^P$ ($P = 2,3$) with boundary Γ (see Fig. 2) and on $\Omega_{\text{ext}} := \Omega \backslash \overline{\Omega}_{\text{int}}$. They are respectively the limits of $\Omega_{\text{int},\delta}$ and $\Omega_{\text{ext},\delta}$ for $\delta \to 0$. They fulfill

$$
\begin{cases}
-\text{div}\,(\alpha_i \nabla u_{\text{int},n}) = \delta_{0,n} f_{|\Omega_{\text{int}}} & \text{in } \Omega_{\text{int}}, \\
-\text{div}\,(\alpha_e \nabla u_{\text{ext},n}) = \delta_{0,n} f_{|\Omega_{\text{ext}}} & \text{in } \Omega_{\text{ext}}, \\
u_{\text{ext},n|\partial\Omega} = 0 & \text{on } \partial\Omega,
\end{cases} \tag{4.5}
$$

where $\delta_{0,n}$ indicates the Kronecker symbol. And an interior expansion corresponding to the asymptotic expansion of $u_{d_\beta,\delta}$ written in a fixed domain and defined by the ansatz

$$
u_{d,\delta}^{[\beta]} = u_0^{[\beta]} + \delta u_1^{[\beta]} + \cdots , \quad \text{in } \Omega^\beta, \tag{4.6}
$$

where the terms $u_n^{[\beta]}$, $n \in \mathbb{N}$, are independent of δ. Using a Taylor expansion in the normal variable, we infer formally

$$\left(\sum_{n \geq 0} \delta^n u_{\text{int},n} \right) \circ \Phi_1(m, s_1) = u_{\text{int},0|\Gamma} + \delta(u_{\text{int},1|\Gamma} + s_1 p_1 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}) + \cdots$$

$$:= U_{\text{int},0} + \delta U_{\text{int},1} + \delta^2 U_{\text{int},2} + \cdots, \qquad (4.7)$$

$$\left(\sum_{n \geq 0} \delta^n u_{\text{ext},n} \right) \circ \Phi_2(m, s_2) = u_{\text{ext},0|\Gamma} + \delta(u_{\text{ext},1|\Gamma} + s_2 p_2 \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}) + \cdots$$

$$:= U_{\text{ext},0} + \delta U_{\text{ext},1} + \delta^2 U_{\text{ext},2} + \cdots. \qquad (4.8)$$

Transmission Conditions (2.1f), (2.1h) and (2.1j) become

$$u_{\text{ext},0|\Gamma} + \delta(u_{\text{ext},1|\Gamma} + p_2 \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma}) + \cdots = u_{0|s_2=1}^{[2]} + \delta u_{1|s_2=1}^{[2]} + \cdots, \qquad (4.9)$$

$$u_{0|s_1=0}^{[1]} + \delta u_{1|s_1=0}^{[1]} + \cdots = u_{0|s_2=0}^{[2]} + \delta u_{1|s_2=0}^{[2]} + \cdots, \qquad (4.10)$$

$$u_{\text{int},0|\Gamma} + \delta(u_{\text{int},1|\Gamma} - p_1 \partial_{\mathbf{n}} u_{\text{int},0|\Gamma}) + \cdots = u_{0|s_1=-1}^{[1]} + \delta u_{1|s_1=-1}^{[1]} + \cdots. \qquad (4.11)$$

Inserting (4.3) in the second equation of (4.5) and using Green's formula in $\Omega_{\delta,2}$, we get

$$\left\langle \alpha_e \partial_{\mathbf{n}} \left(\sum_{n \geq 0} \delta^n u_{\text{ext},n|\Gamma} \right), v_{d_2|\Gamma} \right\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

$$- \left\langle \alpha_e \partial_{\mathbf{n}_{\delta,2}} \left(\sum_{n \geq 0} \delta^n u_{\text{ext},n|\Gamma_{\delta,2}} \right), v_{d_2|\Gamma_{\delta,2}} \right\rangle_{H^{-1/2}(\Gamma_{\delta,1}) \times H^{1/2}(\Gamma_{\delta,1})}$$

$$+ \alpha_e \int_{\Omega_{\delta,2}} \nabla \left(\sum_{n \geq 0} \delta^n u_{\text{ext},n} \right) \cdot \nabla v_{d_2} \, d\Omega_{\delta,2} = \int_{\Omega_{\delta,2}} f_{|\Omega_{\delta,2}} v_{d_2} \, d\Omega_{\delta,2}.$$

Using the scaling $s_2 = \eta_2/\delta$, we obtain

$$\int_{\Gamma} \alpha_e \partial_{\mathbf{n}} \left(\sum_{n \geq 0} \delta^n u_{\text{ext},n|\Gamma} \right) v_d^{[2]}(m, 0) \, d\Gamma + \alpha_e \delta a_\delta^{[2]} \left(\sum_{n \geq 0} \delta^n U_{\text{ext},n}, v_d^{[2]} \right)$$

$$- \left\langle \alpha_e \partial_{\mathbf{n}_{\delta,2}} \left(\sum_{n \geq 0} \delta^n u_{\text{ext},n|\Gamma_{\delta,2}} \right) \circ \Phi_2(m, 1), v_d^{[2]}(m, 1) \right\rangle_{H^{-1/2}(\Gamma \times \{1\}) \times H^{1/2}(\Gamma \times \{1\})}$$

$$= \int_{\Omega_{\delta,2}} f_{|\Omega_{\delta,2}} v_{d_2} \, d\Omega_{\delta,2}. \qquad (4.12)$$

In the same way, we obtain the equation for $\alpha_i \partial_{\mathbf{n}_{\delta,1}} (\sum_{n \geq 0} \delta^n u_{\text{int},n|\Gamma_{\delta,1}}) \circ \Phi_1(m, -1)$

$$\alpha_i \delta a_\delta^{[1]} \left(\sum_{n \geq 0} \delta^n U_{\text{int},n}, v_d^{[1]} \right) - \int_{\Gamma} \alpha_i \partial_{\mathbf{n}} \left(\sum_{n \geq 0} \delta^n u_{\text{int},n|\Gamma} \right) v_d^{[1]}(m, 0) \, d\Gamma$$

$$+ \left\langle \alpha_i \partial_{\mathbf{n}_{\delta,1}} \left(\sum_{n \geq 0} \delta^n u_{\text{int},n|\Gamma_{\delta,1}} \right) \circ \Phi_1(m, -1), \right.$$

$$\left. v_d^{[1]}(m,-1)\right\rangle_{H^{-1/2}(\Gamma\times\{-1\})\times H^{1/2}(\Gamma\times\{-1\})} = \int_{\Omega_{\delta,1}} f_{|\Omega_{\delta,1}} v_{d_1}\,d\Omega_{\delta,1}. \quad (4.13)$$

Inserting expansions (4.3), (4.4) and (4.6) in (4.1), using (4.7)–(4.8) and (4.12)-(4.13), we get

$$\int_\Gamma \alpha_i \partial_{\mathbf{n}}\left(\sum_{n\geq 0}\delta^n u_{\mathrm{int},n|\Gamma}\right)v_d^{[1]}(m,0)\,d\Gamma - \alpha_i\delta a_\delta^{[1]}\left(\sum_{n\geq 0}\delta^n U_{\mathrm{int},n}, v_d^{[1]}\right)$$

$$+ \sum_{\beta=1}^2\left[\alpha_\delta \delta a_\delta^{[\beta]}\left(\sum_{n\geq 0}\delta^n u_n^{[\beta]}, v_d^{[\beta]}\right)\right] - \alpha_e\delta a_\delta^{[2]}\left(\sum_{n\geq 0}\delta^n U_{\mathrm{ext},n}, v_d^{[2]}\right)$$

$$- \int_\Gamma \alpha_e \partial_{\mathbf{n}}\left(\sum_{n\geq 0}\delta^n u_{\mathrm{ext},n|\Gamma}\right)v_d^{[2]}(m,0)\,d\Gamma = 0. \quad (4.14)$$

Now, we use the identity (see [3, p. 1680])

$$J_{\delta,\beta}^{-2} := I - 2s_\beta p_\beta \delta\mathcal{R} + 3\left(p_\beta s_\beta \delta\mathcal{R}\right)^2 + \cdots + n\left(-p_\beta s_\beta \delta\mathcal{R}\right)^{n-1}$$
$$+ \left(-s_\beta p_\beta \delta\mathcal{R}\right)^n\left[nJ_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}\right].$$

Since

$$\det J_{\delta,\beta} = 1 + 2p_\beta s_\beta \delta\mathcal{H} + \left(p_\beta s_\beta \delta\right)^2\mathcal{K},$$

where $2\mathcal{H} := tr\mathcal{R}$ and $\mathcal{K} := \det\mathcal{R}$ are respectively the mean and the Gaussian curvatures of the surface Γ, the bilinear form $a_\delta^{[\beta]}(.,.)$ admits the expansion

$$a_\delta^{[\beta]}(.,.) = \delta^{-2}a_{0,2}^{[\beta]} + \delta^{-1}a_{1,2}^{[\beta]} + \left(a_{2,2}^{[\beta]} + a_{0,1}^{[\beta]}\right) + \delta a_{1,1}^{[\beta]} + \cdots$$
$$+ \delta^{n-1}a_{n-1,1}^{[\beta]} + \delta^n r_n^{[\beta]}(\delta;.,.), \quad (4.15)$$

where the forms $a_{k,l}^{[\beta]}$ are independent of δ and are given by

$$a_{0,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} p_\beta^{-1}\partial_{s_\beta}u^{[\beta]}\partial_{s_\beta}v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{1,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} 2\mathcal{H}s_\beta\partial_{s_\beta}u^{[\beta]}\partial_{s_\beta}v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{2,2}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} p_\beta\mathcal{K}s_\beta^2\partial_{s_\beta}u^{[\beta]}\partial_{s_\beta}v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{0,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} p_\beta\nabla_\Gamma u^{[\beta]}.\nabla_\Gamma v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{1,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} 2p_\beta^2 s_\beta\left(\mathcal{H}I - \mathcal{R}\right)\nabla_\Gamma u^{[\beta]}.\nabla_\Gamma v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{2,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} p_\beta^3\left(\mathcal{K}I - 4\mathcal{H}\mathcal{R} + 3\mathcal{R}^2\right)s_\beta^2\nabla_\Gamma u^{[\beta]}.\nabla_\Gamma v^{[\beta]}\,d\Gamma ds_\beta,$$

$$a_{n-1,1}^{[\beta]}\left(u^{[\beta]}, v^{[\beta]}\right) := \int_{\Omega^\beta} p_\beta^n\left[(n-2)\mathcal{K}\mathcal{R}^{n-3} - (n-1)2\mathcal{H}\mathcal{R}^{n-2}\right.$$
$$\left. + n\mathcal{R}^{n-1}\right](-s_\beta)^{n-1}\nabla_\Gamma u^{[\beta]}.\nabla_\Gamma v^{[\beta]}\,d\Gamma ds_\beta, \quad n > 3.$$

The form $r_n^{[\beta]}(\delta;.,.)$ is the remainder of Expansion (4.15) and is expressed by

$$r_n^{[\beta]}\big(\delta; u^{[\beta]}, v^{[\beta]}\big) := \int_{\Omega^\beta} \big(B_{n,\delta} + 2\mathcal{H}B_{n-1,\delta} + \mathcal{K}B_{n-2,\delta}\big)\, s_\beta^n \nabla_\Gamma u^{[\beta]} . \nabla_\Gamma v^{[\beta]} d\Gamma ds_\beta$$

with

$$B_{n,\delta} := \begin{cases} (-\mathcal{R})^n \left(n J_{\delta,\beta}^{-1} + J_{\delta,\beta}^{-2}\right) & \text{if } n \geq 0, \\ J_{\delta,\beta}^{-2} & \text{otherwise.} \end{cases}$$

Inserting Expansion (4.15) in (4.14) and matching the same powers of δ, we obtain the following variational equations

$$a_{0,2}^{[1]}\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\text{int},0}, v^{[1]}\big) + a_{0,2}^{[2]}\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\text{ext},0}, v^{[2]}\big) = 0, \qquad (4.16)$$

$$\begin{aligned}
a_{1,2}^{[1]}\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\text{int},0}, v^{[1]}\big) &+ a_{0,2}^{[1]}\big(\alpha_\delta u_1^{[1]} - \alpha_i U_{\text{int},1}, v^{[1]}\big) \\
&+ a_{1,2}^{[2]}\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\text{ext},0}, v^{[2]}\big) + a_{0,2}^{[2]}\big(\alpha_\delta u_1^{[2]} - \alpha_e U_{\text{ext},1}, v^{[2]}\big) \\
&= \alpha_e \int_\Gamma \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} v^{[2]}(m,0)\, d\Gamma - \alpha_i \int_\Gamma \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} v^{[1]}(m,0)\, d\Gamma \qquad (4.17)
\end{aligned}$$

for all $v^{[\beta]}$ in $H^1\big(I_\beta; L^2(\Gamma)\big)$ such that $v^{[1]}(.,0) = v^{[2]}(.,0)$,

$$\begin{aligned}
a_{0,2}^{[1]}\big(\alpha_\delta u_2^{[1]} - \alpha_i U_{\text{int},2}, v^{[1]}\big) &+ a_{1,2}^{[1]}\big(\alpha_\delta u_1^{[1]} - \alpha_i U_{\text{int},1}, v^{[1]}\big) \\
&+ \big(a_{2,2}^{[1]} + a_{0,1}^{[1]}\big)\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\text{int},0}, v^{[1]}\big) \\
&+ a_{0,2}^{[2]}\big(\alpha_\delta u_2^{[2]} - \alpha_e U_{\text{ext},2}, v^{[2]}\big) + a_{1,2}^{[2]}\big(\alpha_\delta u_1^{[2]} - \alpha_e U_{\text{ext},1}, v^{[2]}\big) \\
&+ \big(a_{2,2}^{[2]} + a_{0,1}^{[2]}\big)\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\text{ext},0}, v^{[2]}\big) \\
&= \alpha_e \int_\Gamma \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} v^{[2]}(m,0)\, d\Gamma - \alpha_i \int_\Gamma \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} v^{[1]}(m,0)\, d\Gamma, \qquad (4.18)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
a_{0,2}^{[1]}\big(\alpha_\delta u_{n+1}^{[1]} - \alpha_i U_{\text{int},n+1}, v^{[1]}\big) &+ a_{1,2}^{[1]}\big(\alpha_\delta u_n^{[1]} - \alpha_i U_{\text{int},n}, v^{[1]}\big) \\
&+ a_{2,2}^{[1]}\big(\alpha_\delta u_{n-1}^{[1]} - \alpha_i U_{\text{int},n-1}, v^{[1]}\big) + a_{0,1}^{[1]}\big(\alpha_\delta u_{n-1}^{[1]} - \alpha_i U_{\text{int},n-1}, v^{[1]}\big) \\
&+ a_{1,1}^{[1]}\big(\alpha_\delta u_{n-2}^{[1]} - \alpha_i U_{\text{int},n-2}, v^{[1]}\big) + a_{2,1}^{[1]}\big(\alpha_\delta u_{n-3}^{[1]} - \alpha_i U_{\text{int},n-3}, v^{[1]}\big) \\
&+ \sum_{l=4}^n a_{l-1,1}^{[1]}\big(\alpha_\delta u_{n-l}^{[1]} - \alpha_i U_{\text{int},n-l}, v^{[1]}\big) + a_{0,2}^{[2]}\big(\alpha_\delta u_{n+1}^{[2]} - \alpha_e U_{\text{ext},n+1}, v^{[2]}\big) \\
&+ a_{1,2}^{[2]}\big(\alpha_\delta u_n^{[2]} - \alpha_e U_{\text{ext},n}, v^{[2]}\big) + a_{2,2}^{[2]}\big(\alpha_\delta u_{n-1}^{[2]} - \alpha_e U_{\text{ext},n-1}, v^{[2]}\big) \\
&+ a_{0,1}^{[2]}\big(\alpha_\delta u_{n-1}^{[2]} - \alpha_e U_{\text{ext},n-1}, v^{[2]}\big) + a_{1,1}^{[2]}\big(\alpha_\delta u_{n-2}^{[2]} - \alpha_e U_{\text{ext},n-2}, v^{[2]}\big) \\
&+ a_{2,1}^{[2]}\big(\alpha_\delta u_{n-3}^{[2]} - \alpha_e U_{\text{ext},n-3}, v^{[2]}\big) + \sum_{l=4}^n a_{l-1,1}^{[2]}\big(\alpha_\delta u_{n-l}^{[2]} - \alpha_e U_{\text{ext},n-l}, v^{[2]}\big) \\
&= \alpha_e \int_\Gamma \partial_{\mathbf{n}} u_{\text{ext},n|\Gamma} v^{[2]}(m,0)\, d\Gamma - \alpha_i \int_\Gamma \partial_{\mathbf{n}} u_{\text{int},n|\Gamma} v^{[1]}(m,0)\, d\Gamma, \quad n \geq 4
\end{aligned}$$
$$(4.19)$$

for all $v^{[\beta]}$ in $H^1\left(\Gamma \times I_\beta\right)$ such that $v^{[1]}(.,0) = v^{[2]}(.,0)$.

4.2   Calculation of the first terms

In this paragraph, we first recall some theoretical results needed for our calculation. After this, we calculate explicitly the first two terms of Expansions (4.3)–(4.4) and (4.6) in order to present a recursive method to define successively the terms of these expansions. We need the following theorem.

Theorem 2. *Let $F_{int} \in L^2(\Omega_{int})$, $F_{ext} \in L^2(\Omega_{ext})$, $h \in H^{1/2}(\Gamma)$ and $\zeta \in H^{-1/2}(\Gamma)$. Then the following problem*

$$
\begin{cases}
-div\left(\alpha_i \nabla U_{int}\right) = F_{int} & in\ \Omega_{int}, \\
-div\left(\alpha_e \nabla U_{ext}\right) = F_{ext} & in\ \Omega_{ext}, \\
U_{int|\Gamma} - U_{ext|\Gamma} = h & on\ \Gamma, \\
\alpha_i \partial_{\mathbf{n}} U_{int|\Gamma} - \alpha_e \partial_{\mathbf{n}} U_{ext|\Gamma} = \zeta & on\ \Gamma, \\
U_{ext|\partial\Omega} = 0 & on\ \partial\Omega
\end{cases}
\tag{4.20}
$$

admits a unique solution (U_{int}, U_{ext}) in $H^1(\Omega_{int}) \times H^1(\Omega_{ext})$. Moreover, for $k_0 \in \mathbb{N}$, $F_{int} \in H^{k_0-2}(\Omega_{int})$, $F_{ext} \in H^{k_0-2}(\Omega_{ext})$, $h \in H^{k_0-1/2}(\Gamma)$, $\zeta \in H^{k_0-3/2}(\Gamma)$ and $\Gamma \cup \partial\Omega\ \mathcal{C}^{k_0}$-continuous, let $(U_{int}, U_{ext}) \in H^1(\Omega_{int}) \times H^1(\Omega_{ext})$ be the solution of (4.20). For any positive integer $k \leq k_0$, there exists a constant c_k such that

$$
\|U_{int}\|_{H^k(\Omega_{int})} + \|U_{ext}\|_{H^k(\Omega_{ext})} \leq c_k \big(\|F_{int}\|_{H^{k-2}(\Omega_{int})} + \|F_{ext}\|_{H^{k-2}(\Omega_{ext})}
$$
$$
+ \|h\|_{H^{k-1/2}(\Gamma)} + \|\zeta\|_{H^{k-3/2}(\Gamma)}\big).
$$

Proof.   Applying the techniques of Lemma 2.6 and Proposition 2.8 in [24, p. 37] we get the statement of theorem.   □

We also need the following technical lemma. This construction was motivated by [3] and its proof is not difficult.

Lemma 1. *For $\beta = 1, 2$, let $k^{[\beta]}$ be a vectorial function in $L^2(\Omega^\beta, \mathbb{C}^P)$ such that the partial application $s_\beta \to k^{[\beta]}(.\,, s_\beta)$ is valued in the space of vectorial fields tangent to Γ and also $div_\Gamma k^{[\beta]} \in L^2(\Omega^\beta)$. Then the solution $h^{[\beta]}$ of the variational equation*

$$
\mathcal{L}^{[\beta]} v^{[\beta]} := \int_{\Omega^\beta} h^{[\beta]} \partial_{s_\beta} v^{[\beta]}\, d\Gamma ds_\beta + \int_{\Omega^\beta} k^{[\beta]}.\nabla_\Gamma v^{[\beta]}\, d\Gamma ds_\beta = 0
$$
$$
\forall v^{[\beta]} \in H^1(\Omega^\beta), \quad v^{[\beta]}(.\,, 0) = 0
$$

is explicitly given by

$$
h^{[\beta]}(m, s_\beta) = \int_{s_\beta}^{(-1)^\beta} div_\Gamma k^{[\beta]}(m, \lambda)\, d\lambda.
$$

Moreover, if $v^{[\beta]}(.\,, 0) \neq 0$, we have

$$
\mathcal{L}^{[\beta]} v^{[\beta]} = (-1)^{\beta+1} \int_\Gamma h^{[\beta]}(m, 0) v^{[\beta]}(m, 0)\, d\Gamma
$$
$$
= \int_\Gamma \left[(-1)^{\beta+1} \int_0^{(-1)^\beta} div_\Gamma k^{[\beta]}(m, s_\beta)\, ds_\beta \right] v^{[\beta]}(m, 0)\, d\Gamma.
$$

4.2.1 Term of order 0

The choice of $v^{[2]} = 0$ in (4.16) gives

$$a_{0,2}^{[1]}\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\mathrm{int},0}, v^{[1]}\big) = 0.$$

Applying Lemma 1 with

$$h^{[1]} = p_1^{-1}\alpha_\delta \partial_{s_1} u_0^{[1]} - p_1^{-1}\alpha_i \partial_{s_1} U_{\mathrm{int},0} = p_1^{-1}\alpha_\delta\ \partial_{s_1} u_0^{[1]}, \quad k^{[1]} = 0,$$

we obtain $\partial_{s_1} u_0^{[1]} = 0$. Similarly, the choice of $v^{[1]} = 0$ in (4.16) gives

$$a_{0,2}^{[2]}\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\mathrm{ext},0}, v^{[2]}\big) = 0.$$

We apply Lemma 1 with

$$h^{[2]} = p_2^{-1}\alpha_\delta \partial_{s_2} u_0^{[2]} - p_2^{-1}\alpha_e \partial_{s_2} U_{\mathrm{ext},0} = p_2^{-1}\alpha_\delta\ \partial_{s_2} u_0^{[2]}, \quad k^{[2]} = 0$$

to obtain $\partial_{s_2} u_0^{[2]} = 0$. Using (4.9), (4.10) and (4.11), we get

$$u_{\mathrm{int},0|\Gamma} = u_0^{[1]}(m, s_1) = u_0^{[2]}(m, s_2) = u_{\mathrm{ext},0|\Gamma}, \quad m \in \Gamma, \ \forall (m, s_\beta) \in \Omega^\beta. \quad (4.21)$$

In the same manner as above, using Lemma 1 and (4.17), we find

$$p_1^{-1}\alpha_\delta \partial_{s_1} u_1^{[1]} - \alpha_i \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} = 0, \qquad (4.22)$$

$$p_2^{-1}\alpha_\delta \partial_{s_2} u_1^{[2]} - \alpha_e \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} = 0. \qquad (4.23)$$

Moreover, for all $v^{[\beta]}$, we have

$$\alpha_i \int_\Gamma \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} v^{[1]}(m, 0) d\Gamma = \alpha_e \int_\Gamma \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma} v^{[2]}(m, 0) d\Gamma.$$

So

$$\alpha_i \partial_{\mathbf{n}} u_{\mathrm{int},0|\Gamma} = \alpha_e \partial_{\mathbf{n}} u_{\mathrm{ext},0|\Gamma}. \qquad (4.24)$$

Let us define α_0 and u_n by

$$\alpha_0(x) := \begin{cases} \alpha_e & \text{if } x \in \Omega_{\mathrm{ext}}, \\ \alpha_i & \text{if } x \in \Omega_{\mathrm{int}}, \end{cases} \quad \text{and} \quad u_n := \begin{cases} u_{\mathrm{ext},n} & \text{in } \Omega_{\mathrm{ext}}, \\ u_{\mathrm{int},n} & \text{in } \Omega_{\mathrm{int}}. \end{cases}$$

Therefore, with (4.5), (4.21), (4.24) and Theorem 2, u_0 is the unique solution of the following problem

$$\begin{cases} -\mathrm{div}\,(\alpha_0 \nabla u_0) = f & \text{in } \Omega, \\ u_{0|\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that u_0 is nothing but the solution of the initial problem without the thin layer.

4.2.2   Term of order 1

Integrating Relations (4.22) and (4.23) in s_β and identifying terms of order 1 in (4.9) and (4.11), yields

$$u_1^{[1]}(m, s_1) = u_1^{[1]}(m, -1) + (s_1 + 1)p_1\alpha_i\alpha_\delta^{-1}\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}$$
$$= u_{\text{int},1|\Gamma} + p_1\big[(s_1 + 1)\alpha_i\alpha_\delta^{-1} - 1\big]\partial_{\mathbf{n}}u_{\text{int},0|\Gamma}, \quad \forall(m, s_1) \in \Omega^1$$

and

$$u_1^{[2]}(m, s_2) = u_1^{[2]}(m, 1) + (s_2 - 1)p_2\alpha_e\alpha_\delta^{-1}\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}$$
$$= u_{\text{ext},1|\Gamma} + p_2\big[(s_2 - 1)\alpha_e\alpha_\delta^{-1} + 1\big]\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}, \quad \forall(m, s_2) \in \Omega^2.$$

The identification of first order terms of (4.10) gives the first transmission condition on Γ

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1\big(1 - \alpha_i\alpha_\delta^{-1}\big)\partial_{\mathbf{n}}u_{\text{int},0|\Gamma} + p_2\big(1 - \alpha_e\alpha_\delta^{-1}\big)\partial_{\mathbf{n}}u_{\text{ext},0|\Gamma}. \quad (4.25)$$

The second one follows the same lines as for order 0. The choice of $v^{[2]} = 0$ in (4.18) gives

$$a_{0,2}^{[1]}\big(\alpha_\delta u_2^{[1]} - \alpha_i U_{\text{int},2}, v^{[1]}\big) + a_{0,1}^{[1]}\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\text{int},0}, v^{[1]}\big) = 0.$$

We apply Lemma 1 with

$$h^{[1]} = \frac{\alpha_\delta}{p_1}\partial_{s_1}u_2^{[1]} - \frac{\alpha_i}{p_1}\partial_{s_1}U_{\text{int},2} = \frac{\alpha_\delta}{p_1}\partial_{s_1}u_2^{[1]} - \alpha_i\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - s_1p_1\alpha_i\partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma},$$
$$k^{[1]} = p_1\nabla_\Gamma\big(\alpha_\delta u_0^{[1]} - \alpha_i U_{\text{int},0}\big) = p_1(\alpha_\delta - \alpha_i)\nabla_\Gamma u_{\text{int},0|\Gamma}$$

to find

$$\frac{\alpha_\delta}{p_1}\partial_{s_1}u_2^{[1]}(m, s_1) - \alpha_i\partial_{\mathbf{n}}u_{\text{int},1|\Gamma} - s_1p_1\alpha_i\partial_{\mathbf{n}}^2 u_{\text{int},0|\Gamma}$$
$$= -(s_1 + 1)p_1(\alpha_\delta - \alpha_i)\Delta_\Gamma u_{\text{int},0|\Gamma}.$$

Moreover, for all $v^{[1]}$ we obtain

$$\mathcal{L}^{[1]}v^{[1]} = -\int_\Gamma p_1(\alpha_\delta - \alpha_1)\Delta_\Gamma u_{\text{int},0|\Gamma}\ v^{[1]}(m, 0)\,d\Gamma.$$

Similarly, the choice of $v^{[1]} = 0$ in (4.18) gives

$$a_{0,2}^{[2]}\big(\alpha_\delta u_2^{[2]} - \alpha_e U_{\text{ext},2}, v^{[2]}\big) + a_{0,1}^{[2]}\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\text{ext},0}, v^{[2]}\big) = 0.$$

We apply Lemma 1 with

$$h^{[2]} = \frac{\alpha_\delta}{p_2}\partial_{s_2}u_2^{[2]} - \frac{\alpha_e}{p_2}\partial_{s_2}U_{\text{ext},2} = \frac{\alpha_\delta}{p_2}\partial_{s_2}u_2^{[2]} - \alpha_e\partial_{\mathbf{n}}u_{\text{ext},1|\Gamma} - s_2p_2\alpha_e\partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma},$$
$$k^{[2]} = p_2\nabla_\Gamma\big(\alpha_\delta u_0^{[2]} - \alpha_e U_{\text{ext},0}\big) = p_2(\alpha_\delta - \alpha_e)\nabla_\Gamma u_{\text{ext},0|\Gamma}$$

to find

$$\frac{\alpha_\delta}{p_2} \partial_{s_2} u_2^{[2]}(m, s_2) - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} - s_2 p_2 \alpha_e \partial_{\mathbf{n}}^2 u_{\text{ext},0|\Gamma}$$
$$= (1 - s_2) p_2 (\alpha_\delta - \alpha_e) \Delta_\Gamma u_{\text{ext},0|\Gamma}.$$

Furthermore, for all $v^{[2]}$ we obtain

$$\mathcal{L}^{[2]} v^{[2]} = -\int_\Gamma p_2 (\alpha_\delta - \alpha_e) \Delta_\Gamma u_{\text{ext},0|\Gamma} \; v^{[2]}(m, 0) \, d\Gamma.$$

As a consequence,

$$\alpha_i \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} = p_1(\alpha_\delta - \alpha_i) \Delta_\Gamma u_{\text{int},0|\Gamma} + p_2(\alpha_\delta - \alpha_e) \Delta_\Gamma u_{\text{ext},0|\Gamma}.$$
(4.26)

It follows from (4.5), (4.25), (4.26) and Theorem 2 that u_1 is the unique solution of the following problem

$$\begin{cases} -\text{div} (\alpha_i \nabla u_{\text{int},1}) = 0 & \text{in } \Omega_{\text{int}}, \\ -\text{div} (\alpha_e \nabla u_{\text{ext},1}) = 0 & \text{in } \Omega_{\text{ext}}, \\ u_{\text{ext},1|\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

with transmission conditions on Γ

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = p_1 \left(1 - \frac{\alpha_i}{\alpha_\delta} \right) \partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + p_2 \left(1 - \frac{\alpha_e}{\alpha_\delta} \right) \partial_{\mathbf{n}} u_{\text{ext},0|\Gamma},$$

$$\alpha_i \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} = p_1(\alpha_\delta - \alpha_i) \Delta_\Gamma u_{\text{int},0|\Gamma} + p_2(\alpha_\delta - \alpha_e) \Delta_\Gamma u_{\text{ext},0|\Gamma}$$

or

$$u_{\text{int},1|\Gamma} - u_{\text{ext},1|\Gamma} = \left[p_1 \left(1 - \frac{\alpha_i}{\alpha_\delta} \right) + p_2 \left(\frac{\alpha_i}{\alpha_e} - \frac{\alpha_i}{\alpha_\delta} \right) \right] \partial_{\mathbf{n}} u_{\text{int},0|\Gamma},$$

$$\alpha_i \partial_{\mathbf{n}} u_{\text{int},1|\Gamma} - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},1|\Gamma} = \left[p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e) \right] \Delta_\Gamma u_{\text{int},0|\Gamma}.$$

Remark 2. The determination of the first two terms of the asymptotic expansion in the two-dimensional case does not differ from the case $N = 3$.

4.3 Existence and uniqueness of $(u_{\text{int},j})$, $(u_{\text{ext},j})$ and $(u_j^{[\beta]})$

Theorem 3. *The sequences $(u_{int,j})$, $(u_{ext,j})$ and $(u_j^{[\beta]})$ exist and are unique. Furthermore, for any $k \in \mathbb{N}$ and $j \in \mathbb{N}$, we have $u_{int,j} \in H^k(\Omega_{int})$, $u_{ext,j} \in H^k(\Omega_{ext})$ and $u_j^{[\beta]} \in H^k(\Omega^\beta)$.*

Proof. In the previous paragraphs, we have determined the first two terms of Expansions (4.3), (4.4), (4.6), and have shown, provided that f is $\mathcal{C}^\infty$, that for $j = 0, 1$ and for any $k \in \mathbb{N}$, there exists a constant c independent of δ such that $\|u_{\text{int},j}\|_{H^k(\Omega_{\text{int}})} + \|u_{\text{ext},j}\|_{H^k(\Omega_{\text{ext}})} \leq c$. Also, we have shown that $u_j^{[\beta]}$ is a polynomial of degree j in s_β and using the regularity of $(u_{\text{int},j})_{0 \leq j \leq 1}$ and $(u_{\text{ext},j})_{0 \leq j \leq 1}$, we deduce that $\|u_j^{[\beta]}\|_{H^k(\Omega^\beta)} \leq c$. Let us now present the general

construction : Let us assume that the terms $u_{\text{int},j}$, $u_{\text{ext},j}$ and $u_j^{[\beta]}$ were built for $j < n$. The construction of $u_{\text{int},j}$, $u_{\text{ext},j}$ and $u_j^{[\beta]}$ consists of four steps.

Step 1. We define $\partial_{s_\beta} u_n^{[\beta]}$ from Lemma 1 and Problem (4.19) at rank $(n-1)$. Thus the jump condition $u_{\text{int},n|\Gamma} - u_{\text{ext},n|\Gamma}$ can be determined by the relation

$$u_{\text{int},n|\Gamma} - u_{\text{ext},n|\Gamma} := \sum_{j=1}^{n} \frac{1}{j!} p_2^j \partial_{\mathbf{n}}^j u_{\text{ext},n-j|\Gamma} - \sum_{j=1}^{n} \frac{(-1)^j}{j!} p_1^j \partial_{\mathbf{n}}^j u_{\text{int},n-j|\Gamma}$$
$$- \int_{-1}^{0} \partial_{s_1} u_n^{[1]}(m, s_1)\, ds_1 - \int_{0}^{1} \partial_{s_2} u_n^{[1]}(m, s_2)\, ds_2.$$

Step 2. Lemma 1 and Problem (4.19) at rank n determine the transmission condition $\alpha_i \partial_{\mathbf{n}} u_{\text{int},n|\Gamma} - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},n|\Gamma}$ in terms of the surfacic divergence of $\partial_{s_\beta} u_j^{[\beta]}$ $(j < n)$.

Step 3. Step 1, Step 2, (4.5) and Theorem 2 guarantee the existence, uniqueness and regularity of $u_{\text{int},n}$ and $u_{\text{ext},n}$.

Step 4. The determination of $u_n^{[\beta]}$ can be achieved by integrating $\partial_{s_\beta} u_n^{[\beta]}$, defined in Step 1, in s_β and using transmission conditions (4.9) and (4.11). The smoothness of $u_n^{[\beta]}$ follows from the regularity of $(u_{\text{int},j})_{j \leq n}$ and $(u_{\text{ext},j})_{j \leq n}$. $\qquad\square$

5 Convergence Theorem

The process described in the previous section can be continued up to any order provided that the data are sufficiently regular. We can also estimate the error made by truncating the series after a finite number of terms. Let n be in $\mathbb{N}$, we set

$$u_{\text{int},\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{\text{int},j}, \qquad u_{\text{ext},\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{\text{ext},j},$$

$$u_{d,\delta}^{(n)} := \begin{cases} u_{d_1,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_1,j} & \text{in } \Omega_{\delta,1}, \\ u_{d_2,\delta}^{(n)} := \sum_{j=0}^{n} \delta^j u_{d_2,j} & \text{in } \Omega_{\delta,2}, \end{cases}$$

where $u_{d_\beta,j}(x) := \widetilde{u}_{d_\beta,j}(m, \delta s_\beta) := u_j^{[\beta]}(m, s_\beta); \ \forall x = \Phi_\beta(m, s_\beta) \in \Omega_{\delta,\beta}$.

Theorem 4 [Convergence Theorem]. *For all integers n, there exists a constant c independent of δ such as*

$$\left\| u_{int,\delta} - u_{int,\delta}^{(n)} \right\|_{H^1(\Omega_{int,\delta})} + \delta^{1/2} \left\| u_{d,\delta} - u_{d,\delta}^{(n)} \right\|_{H^1(\Omega_\delta)}$$
$$+ \left\| u_{ext,\delta} - u_{ext,\delta}^{(n)} \right\|_{H^1(\Omega_{ext,\delta})} \leq c\delta^{n+1}.$$

Proof. Since f is $\mathcal{C}^\infty$, all terms in Expansions (4.3), (4.4) and (4.6) up to order $n+1$ may be obtained from Equations (4.16)–(4.19). Let us define the remainders $R_{D_1,n}$, $R_{D_2,n}$, $R_{1,n}$ and $R_{2,n}$ of Taylor expansions in the normal

variable with respect to δ up to order n of $u^{(n)}_{\text{int},\delta|\Gamma_{\delta,1}}$, $u^{(n)}_{\text{ext},\delta|\Gamma_{\delta,2}}$, $u^{(n)}_{\text{int},\delta}$ and $u^{(n)}_{\text{ext},\delta}$ respectively by

$$R_{D_1,n} := u^{(n)}_{\text{int},\delta|\Gamma_{\delta,1}} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(-1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^l u_{\text{int},j|\Gamma}, \tag{5.1}$$

$$R_{D_2,n} := u^{(n)}_{\text{ext},\delta|\Gamma_{\delta,2}} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{\delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^l u_{\text{ext},j|\Gamma}, \tag{5.2}$$

$$R_{1,n}(x) := R^{[1]}_{1,n}(m,s_1) := \left(u^{(n)}_{\text{int},\delta}\right)^{[1]} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(s_1)^l \delta^{j+l}}{l!} p_1^l \partial_{\mathbf{n}}^l u_{\text{int},j|\Gamma}$$

$$:= \left(u^{(n)}_{\text{int},\delta}\right)^{[1]} - \sum_{j=0}^{n} \delta^j U_{\text{int},j}, \quad \forall x = \Phi_1(m,s_1) \in \Omega_{\delta,1}, \tag{5.3}$$

$$R_{2,n}(x) := R^{[2]}_{2,n}(m,s_2) := \left(u^{(n)}_{\text{ext},\delta}\right)^{[2]} - \sum_{j=0}^{n}\sum_{l=0}^{n-j} \frac{(s_2)^l \delta^{j+l}}{l!} p_2^l \partial_{\mathbf{n}}^l u_{\text{ext},j|\Gamma}$$

$$:= \left(u^{(n)}_{\text{ext},\delta}\right)^{[2]} - \sum_{j=0}^{n} \delta^j U_{\text{ext},j}, \quad \forall x = \Phi_2(m,s_2) \in \Omega_{\delta,2}, \tag{5.4}$$

where $s_\beta \in I_\beta$. We shall rely on the following proposition to show the estimates of the remainders $R_{D_\beta,n}$ and $R_{\beta,n}$. The steps of the proof are very similar to those given in [25, Section 5]. We refer the reader to this paper. □

Proposition 1. *There exists a constant $c > 0$, independent of δ, such as*

$$\left\|\nabla R_{\beta,n}\right\|_{L^2(\Omega_{\delta,\beta})} \le c\delta^{n+1/2}, \quad \left\|\nabla_\Gamma^{(j)} R_{D_\beta,n}\right\|_{L^2(\Gamma)} \le c\delta^{n+1/2}, \quad \text{for } j = 0,1.$$

Moreover, there exists an extension $\mathcal{P}R$ of $R_{D_\beta,n}$ into Ω_δ with

$$\left\|\mathcal{P}R\right\|_{H^1(\Omega_\delta)} \le c\delta^n.$$

Continuation of the proof of Theorem 4. Let $r^n_{\text{int},\delta}$, $r^n_{d,\delta}$ and $r^n_{\text{ext},\delta}$ be the remainders made by truncating Series (4.3), (4.4) and (4.6)

$$r^n_{\text{int},\delta} := u_{\text{int},\delta} - u^{(n)}_{\text{int},\delta}, \qquad r^n_{\text{ext},\delta} := u_{\text{ext},\delta} - u^{(n)}_{\text{ext},\delta}, \qquad r^n_{d,\delta} := u_{d,\delta} - u^{(n)}_{d,\delta}$$

and $\mathcal{L}_\delta$ be the linear form defined on $H^1_0(\Omega)$

$$\mathcal{L}_\delta v := \alpha_i \int_{\Omega_{\text{int},\delta}} \nabla r^n_{\text{int},\delta} . \nabla v_{\text{int},\delta} \, d\Omega_{\text{int},\delta} + \alpha_\delta \int_{\Omega_\delta} \nabla(r^n_{d,\delta} - \mathcal{P}R) . \nabla v_d \, d\Omega_\delta$$

$$+ \alpha_e \int_{\Omega_{\text{ext},\delta}} \nabla r^n_{\text{ext},\delta} . \nabla v_{\text{ext},\delta} \, d\Omega_{\text{ext},\delta}, \tag{5.5}$$

in which $\mathcal{P}R$ is the extension function of $R_{D_\beta,n}$ into Ω_δ and $v_{\text{int},\delta}$, v_d, and $v_{\text{ext},\delta}$ are the restrictions of v respectively to the domains $\Omega_{\text{int},\delta}$, Ω_δ and $\Omega_{\text{ext},\delta}$.

Using Green's formula in Ω_{int} and in Ω_{ext} with the help of (4.5), we obtain

$$\mathcal{L}_\delta v = \int_\Omega fv\, d\Omega - \alpha_i \int_{\Omega_{\text{int}}} \nabla u_{\text{int},\delta}^{(n)}.\nabla v_{|\Omega_{\text{int}}}\, d\Omega_{\text{int}} + \alpha_i \int_{\Omega_{\delta,1}} \nabla u_{\text{int},\delta}^{(n)}.\nabla v_{d,1}\, d\Omega_{\delta,1}$$

$$- \alpha_\delta \int_{\Omega_\delta} \nabla u_{d,\delta}^{(n)}.\nabla v_d\, d\Omega_\delta - \alpha_e \int_{\Omega_{\text{ext}}} \nabla u_{\text{ext},\delta}^{(n)}.\nabla v_{|\Omega_{\text{ext}}}\, d\Omega_{\text{ext}}$$

$$+ \alpha_e \int_{\Omega_{\delta,1}} \nabla u_{\text{ext},\delta}^{(n)}.\nabla v_{d,1}\, d\Omega_{\delta,1} - \alpha_\delta \int_{\Omega_\delta} \nabla \mathcal{P}R.\nabla v_d\, d\Omega_\delta$$

$$= -\alpha_i \int_\Gamma (\partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + \cdots + \delta^n \partial_{\mathbf{n}} u_{\text{int},n|\Gamma}) v_{d,1|\Gamma}\, d\Gamma$$

$$+ \alpha_i \int_{\Omega_{\delta,1}} \nabla u_{\text{int},\delta}^{(n)}.\nabla v_{d,1}\, d\Omega_{\delta,1} - \alpha_\delta \int_{\Omega_\delta} \nabla u_{d,\delta}^{(n)}.\nabla v_d\, d\Omega_\delta$$

$$+ \alpha_e \int_{\Omega_{\delta,2}} \nabla u_{\text{ext},\delta}^{(n)}.\nabla v_{d,2}\, d\Omega_{\delta,2} - \alpha_\delta \int_{\Omega_\delta} \nabla \mathcal{P}R.\nabla v_d\, d\Omega_\delta$$

$$+ \alpha_e \int_\Gamma (\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} + \cdots + \delta^n \partial_{\mathbf{n}} u_{\text{ext},n|\Gamma}) v_{d,2|\Gamma}\, d\Gamma.$$

It follows, from (5.1)–(5.4), that

$$\mathcal{L}_\delta v = -\alpha_i \int_\Gamma \left(\partial_{\mathbf{n}} u_{\text{int},0|\Gamma} + \cdots + \delta^n \partial_{\mathbf{n}} u_{\text{int},n|\Gamma}\right) v_{d,1|\Gamma}\, d\Gamma$$

$$+ \alpha_i \delta a_\delta^{[1]}\left(U_{\text{int},0} + \cdots + \delta^n U_{\text{int},n}, v_d^{[1]}\right) + \alpha_i \int_{\Omega_{\delta,1}} \nabla R_{1,n}.\nabla v_{d,1}\, d\Omega_{\delta,1}$$

$$- \sum_{\beta=1}^2 \alpha_\delta \delta a_\delta^{[\beta]}\left(u_0^{[\beta]} + \cdots + \delta^n u_n^{[\beta]}, v_d^{[\beta]}\right) + \alpha_e \int_{\Omega_{\delta,2}} \nabla R_{2,n}.\nabla v_{d,2}\, d\Omega_{\delta,2}$$

$$+ \alpha_e \delta a_\delta^{[2]}\left(U_{\text{ext},0} + \cdots + \delta^n U_{\text{ext},n}, v_d^{[2]}\right) - \alpha_\delta \int_{\Omega_\delta} \nabla \mathcal{P}R.\nabla v_d\, d\Omega_\delta$$

$$+ \alpha_e \int_\Gamma \left(\partial_{\mathbf{n}} u_{\text{ext},0|\Gamma} + \cdots + \delta^n \partial_{\mathbf{n}} u_{\text{ext},n|\Gamma}\right) v_{d,2|\Gamma}\, d\Gamma.$$

Now, we use the fact that $u_0^{[\beta]}, \ldots, u_{n+1}^{[\beta]}$ ($\beta = 1, 2$) are solutions of Equations (4.16)–(4.19), and obtain

$$\mathcal{L}_\delta v = \delta^{n+1}\Big\{\delta^{-1} a_{0,2}^{[1]}\left(\alpha_\delta u_{n+1}^{[1]} - \alpha_i U_{\text{int},n+1}, v_d^{[1]}\right)$$

$$- \left(a_{2,2}^{[1]} + a_{0,1}^{[1]}\right)\left(\alpha_\delta u_n^{[1]} - \alpha_i U_{\text{int},n}, v_d^{[1]}\right)$$

$$- a_{1,1}^{[1]}\left(\alpha_\delta u_{n-1}^{[1]} - \alpha_i U_{\text{int},n-1} + \delta\alpha_\delta u_n^{[1]} - \delta\alpha_i U_{\text{int},n}, v_d^{[1]}\right)$$

$$- a_{2,1}^{[1]}\big(\alpha_\delta u_{n-2}^{[1]} - \alpha_i U_{\text{int},n-2} + \delta\alpha_\delta u_{n-1}^{[1]} - \delta\alpha_i U_{\text{int},n-1}$$

$$+ \delta^2 \alpha_\delta u_n^{[1]} - \delta^2 \alpha_i U_{\text{int},n}, v_d^{[1]}\big) - \cdots - a_{n-1,1}^{[1]}\left(\alpha_\delta u_1^{[1]} - \alpha_i U_{\text{int},1}\right.$$

$$+ \cdots + \delta^{n-1} \alpha_\delta u_{n-1}^{[1]} - \delta^{n-1} \alpha_i U_{\text{int},n-1}, v_d^{[1]}\big)$$

$$- r_n^{[1]}\left(\delta; \alpha_\delta u_1^{[1]} - \alpha_i U_{\text{int},1} + \cdots + \delta^n \alpha_\delta u_n^{[1]} - \delta^n \alpha_i U_{\text{int},n}, v_d^{[1]}\right)$$

$$+ \delta^{-1} a_{0,2}^{[2]}\left(\alpha_\delta u_{n+1}^{[2]} - \alpha_e U_{\text{ext},n+1}, v_d^{[2]}\right)$$

$$- \left(a_{2,2}^{[2]} + a_{0,1}^{[2]}\right)\left(\alpha_\delta u_n^{[2]} - \alpha_e U_{\text{ext},n}, v_d^{[2]}\right)$$

$$- a_{1,1}^{[2]}\left(\alpha_\delta u_{n-1}^{[2]} - \alpha_e U_{\text{ext},n-1} + \delta\alpha_\delta u_n^{[2]} - \delta\alpha_e U_{\text{ext},n}, v_d^{[2]}\right)$$

$$- a_{2,1}^{[2]}\left(\alpha_\delta u_{n-2}^{[2]} - \alpha_e U_{\text{ext},n-2} + \delta\alpha_\delta u_{n-1}^{[2]} - \delta\alpha_e U_{\text{ext},n-1}\right.$$

$$\left. + \delta^2 \alpha_\delta u_n^{[2]} - \delta^2 \alpha_e U_{\text{ext},n}, v_d^{[2]}\right) - \cdots - a_{n-1,1}^{[2]}\left(\alpha_\delta u_1^{[2]} - \alpha_e U_{\text{ext},1}\right.$$

$$\left. + \cdots + \delta^{n-1}\alpha_\delta u_{n-1}^{[2]} - \delta^{n-1}\alpha_e U_{\text{ext},n-1}, v_d^{[2]}\right)$$

$$- r_n^{[2]}\left(\delta; \alpha_\delta u_1^{[2]} - \alpha_e U_{\text{ext},1} + \cdots + \delta^n \alpha_\delta u_n^{[2]} - \delta^n \alpha_e U_{\text{ext},n}, v_d^{[2]}\right)\Big\}$$

$$+ \alpha_i \int_{\Omega_{\delta,1}} \nabla R_{1,n}.\nabla v_{d,1}\, d\Omega_{\delta,1} + \alpha_e \int_{\Omega_{\delta,2}} \nabla R_{2,n}.\nabla v_{d,2}\, d\Omega_{\delta,2}$$

$$- \alpha_\delta \int_{\Omega_\delta} \nabla \mathcal{P}R.\nabla v_d\, d\Omega_\delta.$$

By the estimates based on the explicit expressions of the bilinear form $a_{k,l}^{[\beta]}(.\,,.)$ and those of Propositions 1, we have

$$|\mathcal{L}_\delta v| \le c\delta^{n+1} \sum_{\beta=1}^{2}\left(\left\|\nabla_\Gamma v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)} + \delta^{-1}\left\|\partial_{s_\beta} v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)} + \left\|v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)}\right)$$

$$+ c\delta^n \left\|v_d\right\|_{H^1(\Omega_{\delta,\beta})}.$$

This implies

$$|\mathcal{L}_\delta v| \le c\delta^{n+\frac{1}{2}} \sum_{\beta=1}^{2}\left(\delta^{\frac{1}{2}}\left\|\nabla_\Gamma v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)} + \delta^{-\frac{1}{2}}\left\|\partial_{s_\beta} v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)}\right.$$

$$\left. + \delta^{\frac{1}{2}}\left\|v_d^{[\beta]}\right\|_{L^2(\Omega^\beta)}\right) + c\delta^n \left\|v_d\right\|_{H^1(\Omega_{\delta,\beta})}.$$

Then

$$|\mathcal{L}_\delta v| \le c\delta^n \left\|v\right\|_{H^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{5.6}$$

We set in (5.5) $v_{\text{int},\delta} = r_{\text{int},\delta}^n$, $v_d = r_{d,\delta}^n - \mathcal{P}R$ and $v_{\text{ext},\delta} = r_{\text{ext},\delta}^n$, we obtain

$$\left\|r_{\text{int},\delta}^n\right\|_{H^1(\Omega_{\text{int},\delta})} + \left\|r_{d,\delta}^n - \mathcal{P}R\right\|_{H^1(\Omega_\delta)} + \left\|r_{\text{ext},\delta}^n\right\|_{H^1(\Omega_{\text{ext},\delta})} \overset{(5.6)}{\le} c\delta^n.$$

Thanks to Proposition 1, we find

$$\left\|r_{\text{int},\delta}^n\right\|_{H^1(\Omega_{\text{int},\delta})} + \left\|r_{d,\delta}^n\right\|_{H^1(\Omega_\delta)} + \left\|r_{\text{ext},\delta}^n\right\|_{H^1(\Omega_{\text{ext},\delta})} \le c\delta^n. \tag{5.7}$$

Moreover, since f is $\mathcal{C}^\infty$ and for every integer j, we have $\|u_{\text{ext},j}\|_{H^1(\Omega_{\text{ext},\delta})} = O(1)$, $\|u_{\text{int},j}\|_{H^1(\Omega_{\text{int},\delta})} = O(1)$ and $\|u_{d_\beta,j}\|_{H^1(\Omega_{\delta,\beta})} = O(\delta^{-1/2})$, therefore (see, e.g., [25])

$$\left\|r_{\text{int},\delta}^n\right\|_{H^1(\Omega_{\text{int},\delta})} = \left\|\delta^{n+1}u_{\text{int},n+1} + r_{\text{int},\delta}^{n+1}\right\|_{H^1(\Omega_{\text{int},\delta})}$$

$$\overset{(5.7)}{\le} c\delta^{n+1} + c\delta^{n+1} \le c\delta^{n+1},$$

$$\left\|r_{\text{ext},\delta}^{n}\right\|_{H^1(\Omega_{\text{ext},\delta})} = \left\|\delta^{n+1}u_{\text{ext},n+1} + r_{\text{ext},\delta}^{n+1}\right\|_{H^1(\Omega_{\text{ext},\delta})}$$

$$\overset{(5.7)}{\leq} c\delta^{n+1} + c\delta^{n+1} \leq c\delta^{n+1},$$

$$\left\|r_{d,\delta}^{n}\right\|_{H^1(\Omega_\delta)} = \left\|r_{d,\delta}^{n+1} + \delta^{n+1}u_{d,n+1}\right\|_{H^1(\Omega_\delta)} \overset{(5.7)}{\leq} c\delta^{n+1} + c\delta^{n+1/2} \leq c\delta^{n+1/2}.$$

This completes the proof.   □

6   Approximate Transmission Conditions

This section is devoted to the approximation of u_δ by a solution of a problem modelling the effect of the thin layer with a precision of order two in δ. We truncate the series defining the asymptotic expansions, keeping only the first two terms

$$u_{\text{int},\delta} \simeq u_{\text{int},\delta}^{(1)} = u_{\text{int},0} + \delta u_{\text{int},1} \ \ \text{in } \Omega_{\text{int}},$$

$$u_{\text{ext},\delta} \simeq u_{\text{ext},\delta}^{(1)} = u_{\text{ext},0} + \delta u_{\text{ext},1} \ \ \text{in } \Omega_{\text{ext}},$$

$$u_{d_1,\delta}(x) \simeq u_{d_1,\delta}^{(1)}(m,s_1) = u_0^{[1]}(m,s_1) + \delta u_1^{[1]}(m,s_1) \quad \forall x = \Phi_1(m,s_1) \in \Omega_{\delta,1},$$

$$u_{d_2,\delta}(x) \simeq u_{d_2,\delta}^{(1)}(m,s_2) := u_0^{[2]}(m,s_2) + \delta u_1^{[2]}(m,s_2) \quad \forall x = \Phi_2(m,s_2) \in \Omega_{\delta,2},$$

where

$$U_\delta^{(1)} := \begin{cases} u_{\text{ext},\delta}^{(1)} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta}^{(1)} & \text{in } \Omega_{\text{int}}, \end{cases}$$

is the solution of

$$\begin{cases} -\text{div}\big(\alpha_i \nabla u_{\text{int},\delta}^{(1)}\big) = f_{|\Omega_{\text{int}}} & \text{in } \Omega_{\text{int}}, \\ -\text{div}\big(\alpha_e \nabla u_{\text{ext},\delta}^{(1)}\big) = f_{|\Omega_{\text{ext}}} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta|\Gamma}^{(1)} - u_{\text{ext},\delta|\Gamma}^{(1)} = \delta\mathcal{A}\big(u_{\text{int},\delta}^{(1)}\big) - \delta^2\xi_\delta & \text{on } \Gamma, \\ \alpha_i \partial_{\mathbf{n}} u_{\text{int},\delta|\Gamma}^{(1)} - \alpha_e \partial_{\mathbf{n}} u_{\text{ext},\delta|\Gamma}^{(1)} = \delta\mathcal{B}\big(u_{\text{int},\delta}^{(1)}\big) - \delta^2\rho_\delta & \text{on } \Gamma, \\ u_{\text{ext},\delta|\partial\Omega}^{(1)} = 0 & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

with

$$\mathcal{A}(u) := \big[p_1\big(1 - \alpha_i\alpha_\delta^{-1}\big) + p_2\big(\alpha_i\alpha_e^{-1} - \alpha_i\alpha_\delta^{-1}\big)\big]\big(\partial_{\mathbf{n}} u_{|\Gamma}\big),$$

$$\mathcal{B}(u) := \big[p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e)\big]\Delta_\Gamma u_{|\Gamma},$$

$$\xi_\delta := \big[p_1\big(1 - \alpha_i\alpha_\delta^{-1}\big) + p_2\big(\alpha_i\alpha_e^{-1} - \alpha_i\alpha_\delta^{-1}\big)\big]\partial_{\mathbf{n}} u_{\text{int},1|\Gamma},$$

$$\rho_\delta := \big[p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e)\big]\Delta_\Gamma u_{\text{int},1|\Gamma}.$$

Let

$$U_\delta^{ap} := \begin{cases} u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext}}, \\ u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int}} \end{cases}$$

be the solution of (6.1) with $\rho_\delta = 0$ and $\xi_\delta = 0$. We obtain a problem $(\mathcal{P}_\delta^{ap})$ with transmission conditions of order equal to that of the differential operator. The new transmission conditions on Γ are defined by

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = \delta\big[p_1\big(1 - \alpha_i\alpha_\delta^{-1}\big) + p_2\big(\alpha_i\alpha_e^{-1} - \alpha_i\alpha_\delta^{-1}\big)\big]\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap}, \\ \alpha_i\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap} - \alpha_e\partial_{\mathbf{n}}u_{\text{ext},\delta|\Gamma}^{ap} = \delta\big[p_1(\alpha_\delta - \alpha_i) + p_2(\alpha_\delta - \alpha_e)\big]\Delta_\Gamma u_{\text{int},\delta|\Gamma}^{ap}. \end{cases}$$
$$(6.2)$$

However, it is not obvious if the bilinear form associated to Problem $(\mathcal{P}_\delta^{ap})$ is positive or negative. Then the existence and uniqueness of the solution are not ensured by the Lax–Milgram lemma. Therefore, we reformulate Problem $(\mathcal{P}_\delta^{ap})$ into a nonlocal equation on the interface Γ (cf. [5]). A direct use of transmission conditions (6.2) leads to an operator which is not self-adjoint. So, we choose the position of Γ in such a way that the jump of the trace of the solution on Γ is null. We put

$$p_1\big(1 - \alpha_i\alpha_\delta^{-1}\big) + p_2\big(\alpha_i\alpha_e^{-1} - \alpha_i\alpha_\delta^{-1}\big) = 0$$

and obtain

$$p_1 = \frac{\alpha_i\,(\alpha_e - \alpha_\delta)}{\alpha_\delta\,(\alpha_e - \alpha_i)} \quad \text{and} \quad p_2 = \frac{\alpha_e\,(\alpha_\delta - \alpha_i)}{\alpha_\delta\,(\alpha_e - \alpha_i)},$$

which is valid only when $\alpha_i < \alpha_\delta < \alpha_e$ or $\alpha_e < \alpha_\delta < \alpha_i$. This corresponds to the case of mid-diffusion. Transmission conditions (6.2) become

$$\begin{cases} u_{\text{int},\delta|\Gamma}^{ap} - u_{\text{ext},\delta|\Gamma}^{ap} = 0, \\ \alpha_i\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap} - \alpha_e\partial_{\mathbf{n}}u_{\text{ext},\delta|\Gamma}^{ap} = \delta\frac{(\alpha_e - \alpha_\delta)\,(\alpha_i - \alpha_\delta)}{\alpha_\delta}\Delta_\Gamma u_{\text{int},\delta}^{ap}. \end{cases}$$

After, we remove the right-hand side of Problem $(\mathcal{P}_\delta^{ap})$ by a standard lift: let G be in $H_0^1(\Omega)$ such that $-div\,(\alpha_0\nabla G) = f$. Then the function $\Psi = U_\delta^{ap} - G$ solves the following problem

$$\begin{cases} -div\,(\alpha_i\nabla\Psi_{\text{int}}) = 0 & \text{in } \Omega_{\text{int}}, \\ -div\,(\alpha_e\nabla\Psi_{\text{ext}}) = 0 & \text{in } \Omega_{\text{ext}}, \\ \Psi_{\text{int}|\Gamma} - \Psi_{\text{ext}|\Gamma} = 0 & \text{on } \Gamma, \\ \alpha_i\partial_{\mathbf{n}}\Psi_{\text{int}|\Gamma} - \alpha_e\partial_{\mathbf{n}}\Psi_{\text{ext}|\Gamma} - \delta\frac{(\alpha_e - \alpha_\delta)\,(\alpha_i - \alpha_\delta)}{\alpha_\delta}\Delta_\Gamma\Psi_{\text{int}|\Gamma} = g & \text{on } \Gamma, \\ \Psi_{\text{ext}|\partial\Omega} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g = (\alpha_e - \alpha_i)\,\partial_{\mathbf{n}}G_{|\Gamma} + \delta\frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta}\Delta_\Gamma G_{|\Gamma}$, $\Psi_{\text{int}} := \Psi_{|\Omega_{\text{int}}}$ and $\Psi_{\text{ext}} := \Psi_{|\Omega_{\text{ext}}}$.

We introduce the Steklov-Poicaré operators S_{int} and S_{ext} (called also Dirichlet-to-Neumann operators) defined from $H^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ by $S_{\text{int}}\varphi := \partial_{\mathbf{n}}u_{\text{int}|\Gamma}$, where u_{int} is the solution of the boundary value problem

$$\begin{cases} -\Delta u_{\text{int}} = 0 & \text{in } \Omega_{\text{int}}, \\ u_{\text{int}|\Gamma} = \varphi & \text{on } \Gamma, \end{cases}$$

and by $S_{\text{ext}}\psi := \partial_{-\mathbf{n}}u_{\text{ext}|\Gamma}$, where u_{ext} is the solution of the boundary value problem

$$\begin{cases} -\Delta u_{\text{ext}} = 0 & \text{in } \Omega_{\text{ext}}, \\ u_{\text{ext}|\Gamma} = \psi & \text{on } \Gamma, \\ u_{\text{ext}|\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $(\mathcal{P}_\delta^{ap})$ is equivalent to the boundary equation

$$\alpha_i S_{\text{int}}\omega + \alpha_e S_{\text{ext}}\omega - \delta\frac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta}\Delta_\Gamma\omega = g \text{ on } \Gamma,$$

where ω is the trace of Ψ on the surface Γ. We are now in position to state the existence and uniqueness theorem, whose proof is similar to that of Theorem 2.5 in [5].

Theorem 5. *The operator* $P_\delta := \delta\frac{(\alpha_e-\alpha_\delta)(\alpha_i-\alpha_\delta)}{\alpha_\delta}\Delta_\Gamma - \alpha_i S_{int} - \alpha_e S_{ext}$ *is an elliptic self-adjoint semi-bounded from below pseudodifferential operator of order 2. Moreover, there exists series* $(\lambda_n)_{n\in\mathbb{N}}$ *growing to infinity such that for any* $g \in H^s(\Gamma)$ *with* $s \in \mathbb{R}$, *we have the following:*

1. *If* $0 \notin (\lambda_n)_{n\in\mathbb{N}}$, *then equation* $-P_\delta\omega = g$ *admits a unique solution in* $\mathcal{S}'(\Gamma)$ *which, in addition, belongs to* $H^{s+2}(\Gamma)$;

2. *If* $0 \in (\lambda_n)_{n\in\mathbb{N}}$, *then there is either no solution or a complete affine finite dimensional space of* $H^{s+2}(\Gamma)$ *solutions.*

Finally, we give an error estimate between the solution u_δ of (1.1) and the approximate solution u_δ^{ap} defined on Ω by

$$u_\delta^{ap} := \begin{cases} u_{\text{int},\delta}^{ap} & \text{in } \Omega_{\text{int},\delta}, \\ u_{d,\delta}^{ap} & \text{in } \Omega_\delta, \\ u_{\text{ext},\delta}^{ap} & \text{in } \Omega_{\text{ext},\delta}, \end{cases}$$

where

$$u_{d,\delta|\Omega_{\delta,1}}^{ap}(x) := u_{d_1,\delta}^{ap}(x) := u_{d_1,\delta}^{[1],ap}(m, s_1)$$

$$:= u_{\text{int},\delta|\Gamma}^{ap} + \delta\frac{\alpha_i(\alpha_e - \alpha_\delta)}{\alpha_\delta(\alpha_e - \alpha_i)}\left[(s_1 + 1)\frac{\alpha_i}{\alpha_\delta} - 1\right]\partial_{\mathbf{n}}u_{\text{int},\delta|\Gamma}^{ap},$$

$$u_{d,\delta|\Omega_{\delta,2}}^{ap}(x) := u_{d_2,\delta}^{ap}(x) := u_{d_2,\delta}^{[2],ap}(m, s_2)$$

$$:= u_{\text{ext},\delta|\Gamma}^{ap} + \delta\frac{\alpha_e(\alpha_\delta - \alpha_i)}{\alpha_\delta(\alpha_e - \alpha_i)}\left[(s_2 - 1)\frac{\alpha_e}{\alpha_\delta} + 1\right]\partial_{\mathbf{n}}u_{\text{ext},\delta|\Gamma}^{ap}.$$

We can now formulate our main result.

Theorem 6. *There exists a constant c independent of δ such as*

$$\left\|u_{int,\delta} - u_{int,\delta}^{ap}\right\|_{H^1(\Omega_{int,\delta})} + \delta^{\frac{1}{2}}\left\|u_{d,\delta} - u_{d,\delta}^{ap}\right\|_{H^1(\Omega_\delta)}$$
$$+ \left\|u_{ext,\delta} - u_{ext,\delta}^{ap}\right\|_{H^1(\Omega_{ext,\delta})} \leq c\delta^2.$$

Proof. Since

$$\left\|u^{[1],ap}_{d_1,\delta} - u^{(1)}_{d_1,\delta}\right\|_{H^1(\Omega^1)} \le c\Big(\left\|u^{ap}_{\text{int},\delta} - u^{(1)}_{\text{int},\delta}\right\|_{H^1(\Omega_{\text{int}})} + \delta^2 \left\|u_{\text{int},1}\right\|_{H^1(\Omega_{\text{int}})}\Big),$$

$$\left\|u^{[2],ap}_{d_2,\delta} - u^{(1)}_{d_2,\delta}\right\|_{H^1(\Omega^2)} \le c\Big(\left\|u^{ap}_{\text{ext},\delta} - u^{(1)}_{\text{ext},\delta}\right\|_{H^1(\Omega_{\text{ext}})} + \delta^2 \left\|u_{\text{ext},1}\right\|_{H^1(\Omega_{\text{ext}})}\Big),$$

where c is a constant independent of δ and according to the Convergence Theorem, it is sufficient to estimate the error $U^{ap}_\delta - U^{(1)}_\delta$. Therefore, as in [28, p. 36], we perform an asymptotic expansion for U^{ap}_δ. The ansatz

$$U^{ap}_\delta = \sum_{j \ge 0} \delta^j w_j, \tag{6.3}$$

where $w_{j|\Omega_{\text{ext}}} := w_{\text{ext},j}$ and $w_{j|\Omega_{\text{int}}} := w_{\text{int},j}$, gives the recurrence relations

$$\begin{cases} -div\left(\alpha_i \nabla w_{\text{int},j}\right) = \delta_{j,0} f_{|\Omega_{\text{int}}} & \text{in } \Omega_{\text{int}}, \\ -div\left(\alpha_e \nabla w_{\text{ext},j}\right) = \delta_{j,0} f_{|\Omega_{\text{ext}}} & \text{in } \Omega_{\text{ext}}, \\ w_{\text{int},j|\Gamma} - w_{\text{ext},j|\Gamma} = 0 & \text{on } \Gamma, \\ \alpha_i \partial_\mathbf{n} w_{\text{int},j|\Gamma} - \alpha_e \partial_\mathbf{n} w_{\text{ext},j|\Gamma} = \dfrac{(\alpha_e - \alpha_\delta)(\alpha_i - \alpha_\delta)}{\alpha_\delta} \Delta_\Gamma w_{\text{int},j-1|\Gamma} & \text{on } \Gamma, \\ w_{\text{ext},j|\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

with the convention that $w_{-1} = 0$. A simple calculation shows that the first two terms w_0 and w_1 coincide with the first two terms u_0 and u_1 of (4.3) and (4.4). Furthermore, since f is $\mathcal{C}^\infty$, each term of (6.3) is bounded in $H^1(\Omega)$. Then, by setting $\mathcal{R}_w := U^{ap}_\delta - w_0 - \delta w_1 - \delta^2 w_2$, there exists $c > 0$, such as $\|\mathcal{R}_w\|_{H^1(\Omega)} \le c\delta^2$, which gives the desired result.　□

7　Conclusion

We derived and justified an asymptotic expansion of the exact solution of Problem (1.1). We successfully modeled the effect of the thin layer by a problem with transmission condition with accuracy up to $O(\delta^2)$.

An interesting perspective is to investigate the cases of high-diffusion i.e. $\alpha_e < \alpha_i < \alpha_\delta$ or $\alpha_i < \alpha_e < \alpha_\delta$, low-diffusion i.e. $\alpha_\delta < \alpha_i < \alpha_e$ or $\alpha_\delta < \alpha_e < \alpha_i$, and the case where diffusion constants depend on δ. Lastly, the cases of Helmholtz and Maxwell equations would be interesting.

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