

Convergence of Comonotone Histopolating Splines

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Abstract. The convergence rate of histopolation on an interval with combined splines of class C^1 having linear/linear rational or quadratic polynomial pieces is studied. The function to histopolate may have finite number of derivative zeros and established convergence rate depends mainly on the behaviour of the derivative near its zeros. Given numerical results are completely consistent with theoretical ones.

Keywords: histopolation, rational spline, quadratic spline, convergence rate, comonotone shape-preserving.

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1 Introduction

It is known that the histopolation problem could be reduced to the solution of a corresponding interpolation problem, see, e.g., [14]. Then the derivative of the interpolant serves as histopolant. This idea works in the case of polynomial spline interpolant whose derivative is also a polynomial spline. The convergence rates of polynomial spline interpolants and their derivatives are well known. On the other hand, polynomial spline interpolants and histopolants do not preserve geometric properties like positivity, monotonicity, convexity of the function to approximate. An appropriate tool here is the use of rational or combined splines. But, e.g., the derivative of the quadratic/linear function as a piece of convexity preserving rational spline is not linear/linear function which is a natural piece of monotonicity preserving rational spline. Therefore, the study of histopolation with splines using linear/linear rational pieces is an independent problem. In [3] we established the convergence rate of linear/linear rational spline histopolants in case of strict (uniform) monotonicity. The main purpose of this paper is to find out the convergence rate of histopolating combined splines consisting of linear/linear rational or quadratic polynomial pieces when the function to histopolate is not necessarily monotone.

While interpolating polynomial splines is a classical research area [1, 15, 16], there is also extensive literature about shape-preserving spline interpolation, see [6]. Interpolating rational splines of low degree with free parameters as a shape-preserving tool appeared later [7, 8, 10]. An example of the use of interpolating rational splines at the solution of boundary value problems is given in [5]. Histopotation problems with splines are also treated in many works, let us mention of them [16] with several references therein, and [9, 11, 12, 13]. In our paper we will follow the main ideas, representation and basic equations from [2, 4] for histopolating linear/linear rational or combined splines. The most complicated problem in our research occurs to be the analysis of basic equations determining spline parameters.

2 Histopotation Problem

In this section we introduce basic notions and recall some results about the histopotation problem from [4].

Let it be given a mesh $a = x_0 < x_1 < \dots < x_n = b$ and real numbers $z_i, i = 1, \dots, n$, corresponding to subintervals $[x_{i-1}, x_i]$, i.e. we have a given histogram. We look for a function $S \in C^1[a, b]$ which is on each subinterval either a quadratic polynomial or a linear/linear rational function of the form

$$S(x) = \frac{a_i + b_i(x - x_{i-1})}{1 + d_i(x - x_{i-1})} \tag{2.1}$$

with $1 + d_i(x - x_{i-1}) > 0$ for all $x \in [x_{i-1}, x_i]$ and satisfies the histopotation conditions

$$\int_{x_{i-1}}^{x_i} S(x)dx = z_i(x_i - x_{i-1}), \quad i = 1, \dots, n. \tag{2.2}$$

In addition, we impose the boundary conditions

$$S'(x_0) = \alpha, \quad S'(x_n) = \beta \tag{2.3}$$

or

$$S(x_0) = \alpha, \quad S(x_n) = \beta \tag{2.4}$$

for given α and β . It is known that there are no such a two different combined splines satisfying the same histopotation conditions (2.2) and boundary conditions (2.3) or (2.4), see [4, 8].

To determine the kind of subintervals to be rational or quadratic we need the following numbers each of them corresponding to one mesh point

$$\begin{aligned} \delta_i &= z_{i+1} - z_i, \quad i = 1, \dots, n - 1, \\ \delta_0 &= \alpha, \quad \delta_n = \beta \quad \text{for boundary conditions (2.3),} \\ \delta_0 &= z_1 - \alpha, \quad \delta_n = \beta - z_n \quad \text{for boundary conditions (2.4).} \end{aligned}$$

In general, we choose $[x_{i-1}, x_i]$ to be rational if $\delta_{i-1}\delta_i > 0$ and quadratic otherwise. We call this choice the comonotone shape-preserving strategy.

We say that a quadratic section, i.e., a maximal sequence of adjacent quadratic intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$, has a weak alternation of data if the interval $[x_{i-1}, x_i]$ being rational, intervals $[x_i, x_{i+1}], \dots, [x_{i+k-1}, x_{i+k}]$ quadratic and $[x_{i-k}, x_{i+k+1}]$ again rational then,

$$\delta_i > 0, \delta_{i+1} \leq 0, \dots, (-1)^k \delta_{i+k-1} \leq 0, (-1)^k \delta_{i+k} > 0$$

or

$$\delta_i < 0, \delta_{i+1} \geq 0, \dots, (-1)^k \delta_{i+k-1} \geq 0, (-1)^k \delta_{i+k} < 0.$$

In the case of weak alternation of data a solution of the histopolation problem exists (and is unique as was already mentioned). The histopolation problem with the choice of subintervals according to the comonotone shape-preserving strategy and without the weak alternation of data may not have a solution. Because of that we consider the modified comonotone shape-preserving strategy where in absence of the weak alternation of data one of the intervals $[x_i, x_{i+1}]$ or $[x_{i+k-1}, x_{i+k}]$ is taken to be rational. Then the histopolation problem has always a solution.

3 Representation and Basic Equations

In the study of spline algorithms the representation plays always an important role, especially, in nonlinear case. We chose the representation via the derivative values $m_i = S'(x_i)$, $i = 0, \dots, n$, and histogram heights z_i , $i = 1, \dots, n$. Then, with the help of (2.2), (2.1) on $[x_{i-1}, x_i]$ transforms to

$$S(x) = z_i + h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2} \log\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - h_i \frac{m_{i-1}}{\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)(1 + t\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right))} \tag{3.1}$$

with $x = x_{i-1} + th_i$, $h_i = x_i - x_{i-1}$, and on a quadratic interval $[x_{i-1}, x_i]$ we have

$$S(x) = z_i + \frac{h_i}{6} \left((-2 + 6t - 3t^2)m_{i-1} + (-1 + 3t^2)m_i \right). \tag{3.2}$$

We will use the function

$$\varphi(x) = \begin{cases} \frac{x^2(\log x - 1) + x}{(x-1)^2} & \text{for } x > 0, x \neq 0, \\ \frac{1}{2} & \text{for } x = 1. \end{cases}$$

From the properties of φ we need the following

$$\begin{aligned} \varphi(x) > 0, \quad \varphi'(x) > 0, \quad \varphi''(x) < 0 \quad \text{for } x > 0, \\ \lim_{x \rightarrow 0^+} \frac{\varphi(x)}{x} = 1, \quad \lim_{x \rightarrow \infty} \frac{\varphi(x)}{\log x} = 1, \quad \varphi'(1) = \frac{1}{3}, \quad \varphi''(1) = -\frac{1}{6}. \end{aligned}$$

The continuity of S at x_i between rational intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ gives the equation

$$m_i \left(h_i \varphi \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} \right) + h_{i+1} \varphi \left(\left(\frac{m_{i+1}}{m_i} \right)^{1/2} \right) \right) = \delta_i, \tag{3.3}$$

between quadratic intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$

$$\frac{h_i}{6} m_{i-1} + \frac{h_i + h_{i+1}}{3} m_i + \frac{h_{i+1}}{6} m_{i+1} = \delta_i, \tag{3.4}$$

between rational $[x_{i-1}, x_i]$ and quadratic $[x_i, x_{i+1}]$

$$m_i \left(h_i \varphi \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} \right) + \frac{h_{i+1}}{3} \right) + \frac{h_{i+1}}{6} m_{i+1} = \delta_i, \tag{3.5}$$

and symmetrical counterpart of equation (3.5) for quadratic $[x_{i-1}, x_i]$ and rational $[x_i, x_{i+1}]$. The boundary conditions (2.3) fix the values $m_0 = \alpha$ and $m_n = \beta$, but (2.4) gives according to (3.1) or (3.2) two additional equations containing m_0, m_1 or m_{n-1}, m_n , respectively.

In the study of convergence we consider a given sufficiently smooth function $f : [a, b] \rightarrow \mathbb{R}$ and calculate

$$z_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x) dx, \quad i = 1, \dots, n.$$

The conditions (2.3) and (2.4) are posed in the form

$$S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n) \tag{3.6}$$

and

$$S(x_0) = f(x_0), \quad S(x_n) = f(x_n). \tag{3.7}$$

We are interested in the convergence rate of $S - f$ and $S' - f'$ in uniform norm on $[a, b]$ as $h = \max_{1 \leq i \leq n} h_i \rightarrow 0$.

4 Analysis of Basic Equations

Suppose that the given function f has a finite number of points c_i in $[a, b]$ such that $f'(c_i) = 0$. Then the technique of [3] cannot be applied directly. At each point c_i we may allow the Taylor expansion

$$f(x) = f(c_i) + f'(c_i)(x - c_i) + \dots + \frac{f^{(k_i-1)}(c_i)}{(k_i - 1)!} (x - c_i)^{k_i-1} + \frac{f^{(k_i)}(c_i)}{k_i!} (x - c_i)^{k_i} + o((x - c_i)^{k_i})$$

with $f'(c_i) = 0, \dots, f^{(k_i-1)}(c_i) = 0, f^{(k_i)}(c_i) \neq 0$. Generalizing the situation, we carry out our analysis for the differentiable functions f such that

$\lim_{x \rightarrow c_i, x > c_i} f'(x)/|x - c_i|^{\alpha_1} = \gamma_1 \neq 0$ and $\lim_{x \rightarrow c_i, x < c_i} f'(x)/|x - c_i|^{\alpha_2} = \gamma_2 \neq 0$ with some constants α_1, α_2 being positive and γ_1, γ_2 . Our reasoning is adequate in the small neighbourhood of points c_i . For the regions far enough from c_i the analysis of [3] could be applied.

Let us indicate some general observations.

Supposing that $f' \in \text{Lip } \alpha, \quad 0 < \alpha \leq 1$, i.e. $|f'(x) - f'(y)| \leq L|x - y|^\alpha$ for some L , we have in the Taylor expansion $f(x) = f(a) + f'(a)(x - a) + R$ that $|R| \leq (L/(1 + \alpha))|x - a|^{1+\alpha}$. Then

$$\delta_i = \frac{1}{h_{i+1}} \int_{x_i}^{x_{i+1}} f(x)dx - \frac{1}{h_i} \int_{x_{i-1}}^{x_i} f(x)dx = \frac{1}{2}(h_i + h_{i+1})f'(x_i) + R_i$$

with $|R_i| \leq (L/((1 + \alpha)(2 + \alpha)))(h_i^{1+\alpha} + h_{i+1}^{1+\alpha})$.

At histoplation always the estimate $\|S' - f'\|_\infty = O(h^\alpha)$ yields $\|S - f\|_\infty = O(h^{1+\alpha})$, see [3]. We also discuss in Section 5 how the estimate $|m_i - f'_i| = O(h^\alpha)$ gives $\|S' - f'\|_\infty = O(h^\alpha)$.

We will perform the reasoning in particular cases and see later the ways of generalization. For simplicity of presentation, take the uniform partition of $[a, b]$, i.e., $x_i = a + ih, \quad i = 0, \dots, n, \quad h = (b - a)/n$. In the case of sufficiently smooth function f we have

$$\frac{1}{h}\delta_i = f'_i + \frac{h^2}{12}f''_i + O(h^3),$$

here and in the sequel we mean $f'_i = f'(x_i)$ with similar significance for other functions. Observe that outside of certain (small) neighbourhood of each c_i the reasoning of [3] is applicable and the estimate $|m_i - f'_i| = O(h^2)$ for smooth functions holds (or $|m_i - f'_i| = O(h^\alpha), \quad 0 < \alpha < 2$, in the case of lower smoothness, see [3]). Thus, the study only in the neighbourhood of c_i is needed. Nevertheless, we give a complete analysis independent of [3].

Let us start with the function $f(x) = x^2 \text{sgn } x, \quad x \in [-1, 1]$.

For n even, we have $x_i = 0, i = n/2$, and elementary calculations give

$$\delta_i = \frac{2}{3}h^2, \quad \delta_{i+k} = \delta_{i-k} = 2kh^2, \quad k = 1, \dots, \frac{n}{2} - 1.$$

As boundary conditions (3.6) are consistent we choose all subintervals to be rational. For n odd, the point 0 is the midpoint of the interval $[x_i, x_{i+1}]$ with $i = (n - 1)/2$. Then

$$\delta_i = \delta_{i+1} = \frac{13}{12}h^2, \quad \delta_{i+k} = (2k - 1)h^2, \quad k = 2, \dots, (n - 1)/2,$$

and

$$\delta_{i-1} = \delta_{i+k+1} = (2k + 1)h^2, \quad k = 1, \dots, (n - 3)/2.$$

Here also all subintervals are rational. We will focus our attention mainly to the case of n even.

The equations of the type (3.3) corresponding to the points x_j are now

$$m_j \left(\varphi \left(\left(\frac{m_{j-1}}{m_j} \right)^{1/2} \right) + \varphi \left(\left(\frac{m_{j+1}}{m_j} \right)^{1/2} \right) \right) = \frac{1}{h} \delta_j = f'_j \quad (4.1)$$

and thus

$$m_j - f'_j = f'_j \frac{1 - \varphi\left(\left(\frac{m_{j-1}}{m_j}\right)^{1/2}\right) - \varphi\left(\left(\frac{m_{j+1}}{m_j}\right)^{1/2}\right)}{\varphi\left(\left(\frac{m_{j-1}}{m_j}\right)^{1/2}\right) + \varphi\left(\left(\frac{m_{j+1}}{m_j}\right)^{1/2}\right)}. \tag{4.2}$$

Denote $\mu_j = m_{j-1}/m_j$, $j = 1, \dots, i$, and $\mu_j = m_j/m_{j-1}$, $j = i + 1, \dots, n$. Our aim is to estimate the values μ_j and then use them in (4.2). Let us consider the situation at right hand side from $x_i = 0$ (by symmetry, $\mu_{i+k} = \mu_{i-k+1}$ for all k), then the equations (4.1) for $j = i + k$ and $j + 1$ give

$$\mu_{j+1} = \frac{\delta_{j+1} \varphi(\mu_j^{-1/2}) + \varphi(\mu_{j+1}^{1/2})}{\delta_j \varphi(\mu_{j+1}^{-1/2}) + \varphi(\mu_{j+2}^{1/2})} \tag{4.3}$$

with $\delta_{j+1}/\delta_j = 1 + 1/k$. Denote $\mu = (\mu_1, \dots, \mu_n)$ and introduce the functions

$$\Phi_j(\mu) = \frac{\varphi(\mu_{j-1}^{-1/2}) + \varphi(\mu_j^{1/2})}{\varphi(\mu_j^{-1/2}) + \varphi(\mu_{j+1}^{1/2})}, \quad \Psi_j(\mu) = \frac{\delta_j}{\delta_{j-1}} \Phi_j(\mu).$$

Then the equations (4.3) are $\mu_j = \Psi_j(\mu)$. We need the following technical

Lemma. *Suppose $\mu_{j+l} \in [1 + 1/(k+l-1) - \delta, 1 + 1/(k+l-1) + \delta]$, $l = 0, 1, 2$, with sufficiently small $\delta > 0$. Then for some positive constants c_1, c_2 it holds*

$$\Phi_{j+1}(\mu) \in \left[1 - \frac{c_1}{k^3} - \frac{2}{3}\delta - c_2\delta^2, 1 + \frac{c_1}{k^3} + \frac{2}{3}\delta + c_2\delta^2 \right].$$

Proof. Take $\bar{\mu}_{j+l} = 1 + 1/(k+l-1)$, $l = 0, 1, 2$, and estimate then $\Phi_{j+1}(\bar{\mu})$. The use of the Taylor expansion

$$\begin{aligned} \varphi(\bar{\mu}_{j+1}^{1/2}) &= \varphi\left(\left(1 + \frac{1}{k}\right)^{1/2}\right) = \varphi(1) + \varphi'(1)\left(\left(1 + \frac{1}{k}\right)^{1/2} - 1\right) \\ &\quad + \frac{\varphi''(1)}{2}\left(\left(1 + \frac{1}{k}\right)^{1/2} - 1\right)^2 + \frac{\varphi'''(\xi)}{6}\left(\left(1 + \frac{1}{k}\right)^{1/2} - 1\right)^3 \end{aligned}$$

and inside that

$$x^{1/2} - 1 = \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + O((x - 1)^3) \tag{4.4}$$

gives

$$\varphi(\bar{\mu}_{j+1}^{1/2}) = \frac{1}{2} + \frac{1}{6k} - \frac{1}{16k^2} + O\left(\frac{1}{k^3}\right).$$

We get also

$$\varphi(\bar{\mu}_{j+1}^{-1/2}) = \varphi\left(\left(1 - \frac{1}{k+1}\right)^{1/2}\right) = \frac{1}{2} - \frac{1}{6(k+1)} - \frac{1}{16(k+1)^2} + O\left(\frac{1}{k^3}\right).$$

Then

$$\varphi(\bar{\mu}_j^{-1/2}) + \varphi(\bar{\mu}_{j+1}^{1/2}) = 1 - \frac{1}{8k^2} + O\left(\frac{1}{k^3}\right)$$

and

$$\varphi(\bar{\mu}_{j+1}^{-1/2}) + \varphi(\bar{\mu}_{j+2}^{1/2}) = 1 - \frac{1}{8(k+1)^2} + O\left(\frac{1}{k^3}\right).$$

Now

$$\Phi_{j+1}(\bar{\mu}) = \left(1 - \frac{1}{8k^2} + O\left(\frac{1}{k^3}\right)\right) \left(1 + \frac{1}{8(k+1)^2} + O\left(\frac{1}{k^3}\right)\right) = 1 + O\left(\frac{1}{k^3}\right).$$

If the components of μ are such that $\mu_{j+l} \in [\bar{\mu}_{j+l} - \delta, \bar{\mu}_{j+l} + \delta]$, $l = 0, 1, 2$, we take into account $\varphi'(1) = 1/3$, the boundedness of φ'' and the Taylor expansion (4.4) to arrive at the inclusion of $\Phi_{j+1}(\mu)$ stated in the assertion. \square

As a consequence, in the assumption of Lemma, we get the inclusion

$$\Psi_{j+1}(\mu) \in \left[1 + \frac{1}{k} - \frac{c_1}{k^3} - \frac{2}{3}\delta - c_2\delta^2, 1 + \frac{1}{k} + \frac{c_1}{k^3} + \frac{2}{3}\delta + c_2\delta^2\right], \quad (4.5)$$

where the constants c_1, c_2 may be different compared to those of Lemma.

Take in Lemma $\delta = c_0/k$, $c_0 > 0$, then $\Psi_{j+1}(\mu) \in [1 + 1/k - \delta, 1 + 1/k + \delta]$ if $c_1/k^2 + c_2c_0^2/k \leq c_0/3$ for c_1, c_2 in (4.5). This holds for some $c_0 > 0$ and for $k \geq k_0$ with some fixed k_0 taken after the choice of c_0 . Basing on the proof of Lemma we get for the solution of $\mu_{j+1} = \Psi_{j+1}(\mu)$ the estimate

$$\frac{1 - \varphi(\mu_j^{-1/2}) - \varphi(\mu_{j+1}^{1/2})}{\varphi(\mu_j^{-1/2}) + \varphi(\mu_{j+1}^{1/2})} = O\left(\frac{1}{k^2}\right) + O(\delta)$$

and, taking into account $f'_j = 2kh$,

$$m_j - f'_j = 2kh \left(O\left(\frac{1}{k^2}\right) + O(\delta) \right) = O(h). \quad (4.6)$$

It remains to study the behaviour of $m_j - f'_j$ for $j = i, \dots, i + k_0 - 1$. Because of $f'_j = 2kh$, $j = i + k$, it is sufficient to establish the boundedness of μ_j from below and from above by positive constants (not depending on h) for finite number of indices k . The symmetry considerations allow to assert that $m_{i-1} = m_{i+1}$ (still for n even and $i = n/2$) and then $\mu_i = \mu_{i+1}$. The equation (4.3) is in this case

$$\mu_i = 3 \frac{\varphi(\mu_i^{-1/2}) + \varphi(\mu_i^{1/2})}{\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})}. \quad (4.7)$$

First, assume that $\mu_i \rightarrow \infty$. Then $\varphi(\mu_i^{1/2}) \sim (\log \mu_i)/2$, $\mu_i^{-1} \rightarrow 0$ and $\varphi(\mu_i^{-1/2}) \sim \mu_i^{-1/2} \rightarrow 0$ (here and in the sequel \sim means that the quotient of these terms converges to some positive constant, mainly to 1). In the equation (4.7) in the form

$$\mu_i (\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})) = 3(\varphi(\mu_i^{-1/2}) + \varphi(\mu_i^{1/2})) \quad (4.8)$$

the right hand side of (4.8) behaves as $3(\log \mu_i)/2$ but the left hand side increases at least as $\mu_i^{1/2}$ which gives a contradiction. Secondly, let $\mu_i \rightarrow 0$. Then $\varphi(\mu_i^{1/2}) \rightarrow 0$, $\mu_i^{-1/2} \rightarrow \infty$ and the right hand side of (4.8) is of order $3(\log \mu_i^{-1})/2$ while in the left hand side this order has to have the term $\mu_i \varphi(\mu_{i+2}^{1/2})$. This yields that $\mu_{i+2} \rightarrow \infty$ and $\varphi(\mu_i^{-1/2})/\varphi(\mu_{i+2}^{1/2}) \sim 2\mu_i/3 \rightarrow 0$. The particular case of (4.3) for $j = i + 1$ gives

$$\mu_{i+2}(\varphi(\mu_{i+2}^{-1/2}) + \varphi(\mu_{i+3}^{1/2})) = 2(\varphi(\mu_i^{-1/2}) + \varphi(\mu_{i+2}^{1/2})),$$

where at right the main term is of order $\log \mu_{i+2}$ but at left we have at least $\mu_{i+2}^{1/2}$ which is impossible.

This completes the proof of boundedness of μ_i (and μ_{i+1}). It remains to carry out the induction step which differs from the just presented reasoning only by details and we omit it.

We have proved the estimate $m_i - f'_i = O(h)$ for the investigated function $f(x) = x^2 \operatorname{sgn} x$, $x \in [-1, 1]$.

Let us consider now briefly some other particular cases.

The function $f(x) = x^3$, $x \in [-1, 1]$, generates $\delta_j > 0$ for all j and any partition. On uniform partition, for n even and $i = n/2$ we get $\delta_i/h = h^2/2$ and $\delta_{i+k}/h = h^2(6k^2 + 1)/2$, $k \geq 1$, and then

$$\frac{\delta_{i+k}}{\delta_{i+k-1}} = 1 + \frac{2}{k} + O\left(\frac{1}{k^2}\right). \tag{4.9}$$

For n odd and $i = (n - 1)/2$ we have $x_i = -h/2$, $x_{i+1} = h/2$, $\delta_i/h = \delta_{i+1}/h = 5h^2/4$, $\delta_{i+k+1}/h = (3(2k + 1)^2/4 + 1/2)h^2$ and still (4.9) holds. The assertion of Lemma about the inclusion of the value $\Phi_{j+1}(\mu)$ takes places if we choose $\mu_{j+l} \in [1 + 2/(k + l - 1) - \delta, 1 + 2/(k + l - 1) + \delta]$, $l = 0, 1, 2$, we only have to take $\bar{\mu}_{j+l} = 1 + 2/(k + l - 1)$ in the proof. Then

$$\varphi(\bar{\mu}_j^{-1/2}) + \varphi(\bar{\mu}_{j+1}^{1/2}) = 1 - \frac{1}{6k^2} + O\left(\frac{1}{k^2}\right)$$

and other changes in the proof of Lemma are obvious. Taking into account (4.9) we establish the inclusion $\Psi_{j+1}(\mu) \in [1 + 2/k - \delta, 1 + 2/k + \delta]$ if the inequality

$$\left(1 + \frac{c_1}{k^3} + \frac{2}{3}\delta + c_2\delta^2\right) \left(1 + \frac{2}{k} + \frac{c_3}{k^2}\right) \leq 1 + \frac{2}{k} + \delta$$

holds (c_3 reflects (4.9)). This is achieved with $\delta = c_0/k^2$ where c_0 is sufficiently great and $k \geq k_0$ for some fixed k_0 . Now $\delta_j/h = 3k^2h^2 + h^2f_j'''/12$ and instead of (4.2) we have

$$m_j - f'_j = f'_j \frac{1 - \varphi\left(\left(\frac{m_{j-1}}{m_j}\right)^{1/2}\right) - \varphi\left(\left(\frac{m_{j+1}}{m_j}\right)^{1/2}\right)}{\varphi\left(\left(\frac{m_{j-1}}{m_j}\right)^{1/2}\right) + \varphi\left(\left(\frac{m_{j+1}}{m_j}\right)^{1/2}\right)} + \frac{\frac{h^2}{12}f_j'''}{\varphi\left(\left(\frac{m_{j-1}}{m_j}\right)^{1/2}\right) + \varphi\left(\left(\frac{m_{j+1}}{m_j}\right)^{1/2}\right)},$$

thus (4.6) is replaced by

$$m_j - f'_j = 3k^2 h^2 \left(O\left(\frac{1}{k^2}\right) + O(\delta) \right) + O(h^2) = O(h^2). \quad (4.10)$$

The boundedness of μ_j for finite number of j could be obtained exactly as presented above and, consequently, we have $m_j - f'_j = O(h^2)$ for the function $f(x) = x^3$, $x \in [-1, 1]$.

Next, consider the function $f(x) = x^2$, $x \in [-1, 1]$, and uniform partition on $[-1, 1]$. For n even, we have $x_i = 0$, $i = n/2$, and $z_i = z_{i+1}$, thus $\delta_i = 0$. We have also $\delta_j < 0$, $j = 0, \dots, i-1$, and $\delta_j > 0$, $j = i+1, \dots, n$. By the comonotone strategy it should be chosen $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ quadratic, all other particular intervals rational. But there is no weak alternation of data in this section of quadratic intervals and, by the modified strategy, we choose $[x_{i-1}, x_i]$ to be rational, too. For n odd, the point 0 is the midpoint of the interval $[x_i, x_{i+1}]$ with $i = (n-1)/2$. Then $\delta_j < 0$, $j = 0, \dots, i$, and $\delta_j > 0$, $j = i+1, \dots, n$. The comonotone strategy makes the subinterval $[x_i, x_{i+1}]$ to be quadratic and all others rational. We have here the weak alternation of data. It is clear that only a finite number of basic equations corresponding to the neighbourhood of $[x_i, x_{i+1}]$ need the study. The equations of the types (3.3)–(3.5) corresponding to the points x_{i-1} , x_i , x_{i+1} are now

$$m_{i-1} \left(\varphi \left(\left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \right) + \varphi \left(\left(\frac{m_i}{m_{i-1}} \right)^{1/2} \right) \right) = \frac{1}{h} \delta_{i-1} = f'_{i-1}, \quad (4.11)$$

$$m_i \varphi \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} \right) + \frac{1}{3} m_i + \frac{1}{6} m_{i+1} = \frac{1}{h} \delta_i = f'_i, \quad (4.12)$$

$$\frac{1}{6} m_i + \frac{1}{3} m_{i+1} + m_{i+1} \varphi \left(\left(\frac{m_{i+2}}{m_{i+1}} \right)^{1/2} \right) = \frac{1}{h} \delta_{i+1} = f'_{i+1}. \quad (4.13)$$

Actually, we have to estimate $m_j - f'_j$ for $j = i-1, i, i+1$, for $j < i-1$ and $j > i+1$ the reasoning about the function $f(x) = x^2 \operatorname{sgn} x$ is valid provided we prove the boundedness of $\mu_i = m_{i-1}/m_i$ and $\mu_{i+2} = m_{i+2}/m_{i+1}$. We see that, for n even, in (4.11) $f'_{i-1} = -2h$, in (4.12) $f'_i = 0$, in (4.13) $f'_{i+1} = 2h$, for n odd, $f'_{i-1} = -3h$, $f_i = -h$, $f_{i+1} = h$.

Suppose $m_{i-1}/m_i \rightarrow \infty$. First, consider the case $c_1 \leq m_i/h \leq c_2$ with some $c_1, c_2 < 0$. Dividing (4.12) by h , we see that $m_{i+1}/h \rightarrow \infty$. Then (4.13) divided by h gives a contradiction. Secondly, let $m_i/h \rightarrow -\infty$. Dividing (4.12) by m_i we conclude that $m_{i+1}/m_i \rightarrow -\infty$. Then (4.13) divided by m_i leads to a contradiction. Thirdly, let $m_i/h \rightarrow 0$, however $m_i < 0$. Then $h/m_i \rightarrow -\infty$ and $|h/m_i| \rightarrow \infty$. Consider the case n even (then $f'_i = 0$). Dividing (4.12) by m_i gives $m_{i+1}/m_i \rightarrow -\infty$, we have also $\varphi((m_{i-1}/m_i)^{1/2}) \sim |m_{i+1}/m_i|/6$ or

$$\frac{m_{i-1}}{m_i} \sim e^{\frac{1}{3} \left| \frac{m_{i+1}}{m_i} \right|}. \quad (4.14)$$

Dividing (4.11) by m_i we get

$$\left(\frac{m_{i-1}}{m_i} \right)^{1/2} + \frac{m_{i-1}}{m_i} \varphi \left(\left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \right) \sim 2 \left| \frac{h}{m_i} \right|$$

and according to (4.14)

$$e^{\frac{1}{6} \left| \frac{m_{i+1}}{m_i} \right|} + e^{\frac{1}{3} \left| \frac{m_{i+1}}{m_i} \right|} \varphi \left(\left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \right) \sim 2 \left| \frac{h}{m_i} \right|. \tag{4.15}$$

Dividing (4.13) by m_i gives

$$\frac{1}{3} \left| \frac{m_{i+1}}{m_i} \right| + \left| \frac{m_{i+1}}{m_i} \right| \varphi \left(\left(\frac{m_{i+2}}{m_{i+1}} \right)^{1/2} \right) \sim 2 \left| \frac{h}{m_i} \right|.$$

The boundedness of m_{i+2}/m_{i+1} means that $|m_{i+1}/m_i|$ and $2|h/m_i|$ have the same order and the left hand side of (4.15) has also the order of $|m_{i+1}/m_i|$ which is impossible. It remains to consider the case $m_{i+2}/m_{i+1} \rightarrow \infty$. Dividing the counterpart of (4.11) corresponding to x_{i+2} by m_{i+1} gives

$$\frac{m_{i+2}}{m_{i+1}} \left(\varphi \left(\left(\frac{m_{i+1}}{m_{i+2}} \right)^{1/2} \right) + \varphi \left(\left(\frac{m_{i+3}}{m_{i+2}} \right)^{1/2} \right) \right) \sim \frac{h}{m_{i+1}}, \tag{4.16}$$

which means also that $h/m_{i+1} \rightarrow \infty$ because $m_{i+1}/m_{i+2} \rightarrow 0$. Then (4.13) divided by m_{i+1} implies that h/m_{i+1} has the order of $\log(m_{i+2}/m_{i+1})$ which is in contradiction with (4.16). Consider now the case of n odd. Dividing (4.12) by m_i we get

$$\varphi \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} \right) + \frac{1}{3} = \frac{1}{6} \left| \frac{m_{i+1}}{m_i} \right| + \left| \frac{h}{m_i} \right|.$$

The possibility $\varphi((m_{i-1}/m_i)^{1/2}) \sim |m_{i+1}/m_i|$ (the quotient is bounded from above and below by positive constants) could be treated as in the case of n even. If $\varphi((m_{i-1}/m_i)^{1/2}) \sim |h/m_i|$ then dividing (4.11) by m_i we get

$$\frac{m_{i-1}}{m_i} \varphi \left(\left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \right) + \frac{m_{i-1}}{m_i} \varphi \left(\left(\frac{m_i}{m_{i-1}} \right)^{1/2} \right) = 3 \left| \frac{h}{m_i} \right|.$$

Here the right hand side is of order $\log(m_{i-1}/m_i)$ while the left hand side is of order at least $(m_{i-1}/m_i)^{1/2}$ which is a contradiction. This completes the proof of the boundedness of m_{i-1}/m_i from above.

Suppose next that $m_{i-1}/m_i \rightarrow 0$, then $m_i/m_{i-1} \rightarrow \infty$. Dividing (4.11) by m_{i-1} we get

$$\varphi \left(\left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \right) + \varphi \left(\left(\frac{m_i}{m_{i-1}} \right)^{1/2} \right) = l \left| \frac{h}{m_{i-1}} \right|$$

with $l = 2$ or $l = 3$ depending on n to be even or odd. This means that $|h/m_{i-1}| \rightarrow \infty$. If $\varphi((m_{i-2}/m_{i-1})^{1/2})$ is of order $|h/m_{i-1}|$ then

$$\frac{m_i}{m_{i-1}} \sim e^{k \left| \frac{h}{m_{i-1}} \right|} \tag{4.17}$$

with some constant k (here also, \sim means that the quotient is two-sided bounded). Dividing (4.12) by m_{i-1} we see that

$$\frac{m_{i+1}}{m_{i-1}} / \frac{m_i}{m_{i-1}} \rightarrow -2. \tag{4.18}$$

Dividing (4.13) by m_{i-1} gives

$$\frac{1}{6} \frac{m_i}{m_{i-1}} + \left(\frac{1}{3} + \varphi \left(\left(\frac{m_{i+2}}{m_{i+1}} \right)^{1/2} \right) \right) \frac{m_{i+2}}{m_{i-1}} = -l \left| \frac{h}{m_{i-1}} \right|$$

with $l = 1$ or $l = 2$ and this is a contradiction due to (4.18) and (4.17). If $\varphi((m_{i-2}/m_{i-1})^{1/2})$ is of order $|h/m_{i-1}|$ then

$$\frac{m_{i-2}}{m_{i-1}} \sim e^{k|h/m_{i-1}|}. \tag{4.19}$$

Now the counterpart of (4.11) corresponding to the point x_{i-2} divided by m_{i-1} gives

$$\frac{m_{i-2}}{m_{i-1}} \varphi \left(\left(\frac{m_{i-3}}{m_{i-2}} \right)^{1/2} \right) + \left(\frac{m_{i-2}}{m_{i-1}} \right)^{1/2} \sim \left| \frac{h}{m_{i-1}} \right|,$$

which is a contradiction due to (4.19) and this ends the proof of boundedness of m_{i-1}/m_i from below by a positive constant.

As any point x_j is an endpoint of a rational interval and the quotient of two values m_j in any rational interval is bounded from above and below by positive constants, we obtain the estimate $m_j - f'_j = O(h)$ for all j like in the case of function $f(x) = x^2 \operatorname{sgn} x$.

The function $f(x) = |x|^3$, $x \in [-1, 1]$, could be treated joining the arguments from treatments of two previous functions. Here we have the estimate $m_j - f'_j = O(h^2)$.

Ending this section let us indicate the framework at establishing the estimates. First, we prove the boundedness of μ_j for finite number of indices $j = i, \dots, i + k_0$ (or for $j < i$), with suitable fixed k_0 and then use (4.6) or (4.10). For $j > i + k_0$ (or $j < i - k_0$) we use Bohl–Brouwer fixed point principle as it was done by us several times [2, 3, 4] basing on inclusions (4.5) or its analogue in the analysis of function $f(x) = x^3$.

However, we need a boundary value $\mu_{j+1} = \mu_{i+k_0}$ but, e.g., we take it from the estimates $m_j = f'_j + \alpha_1$, $m_{j+1} - f'_{j+1} = \alpha_2$, $|\alpha_1| \leq ch$, $|\alpha_2| \leq ch$, $c = \text{const}$, then $\mu_{j+1} = m_{j+1}/m_j = 1 + 1/k_0 + \delta$, $|\delta| \leq c_0/k_0$. Such an estimate is valid due to (4.6) but works as well if we use (4.10). At the other end we choose as boundary condition $\mu_{n+1} = 1 + 2/n$ or $\mu_{n+1} = 1 + 1/n$ for the cases $f(x) = x^2$ or $f(x) = x^3$, respectively.

5 Convergence Results

In Section 4 we established the estimates $m_i - f'_i = O(h^\alpha)$, $0 < \alpha \leq 2$, depending on the function of f to histopolate. In the beginning we show how this implies the estimate $\|S' - f'\|_\infty = O(h^\alpha)$. We mentioned already that this yields $\|S - f\|_\infty = O(h^{1+\alpha})$.

The representation (3.1) gives on a rational interval $[x_{i-1}, x_i]$

$$S'(x) = \frac{m_{i-1}}{\left(1 + \frac{x-x_{i-1}}{h} \left(\left(\frac{m_{i-1}}{m_1} \right)^{1/2} - 1 \right) \right)^2} \tag{5.1}$$

and on a quadratic interval $[x_{i-1}, x_i]$ from (3.2) follows

$$S'(x) = (1 - t)m_{i-1} + tm_i, \tag{5.2}$$

actually, as the linear interpolation representation for derivative S' . On rational intervals we have using (5.1) and (4.1)

$$\begin{aligned} S'(x) - f'(x) &= S'(x) - f'_{j-1} + f'_{j-1} - f'(x) \\ &= \frac{\frac{1}{h}\delta_{i-1}}{\left(1 + \frac{x-x_{i-1}}{h} \left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^2 \left(\varphi\left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2}\right) + \varphi\left(\left(\frac{m_i}{m_{i-1}}\right)^{1/2}\right)\right)\right)} \\ &\quad - f'_{i-1} + O(h). \end{aligned} \tag{5.3}$$

In the case of function $f(x) = x^2$ and $f(x) = x^2 \operatorname{sgn} x$ it holds $\delta_{i-1}/h = f'_{i-1}$ and (5.3) gives $S'(x) - f'(x) = O(h)$ because $f'_{j+1} = 2kh$, $j = n/2 + k$ (or $j + 1 = (n - 1)/2 + k$), $j \geq n/2$ and $\mu_{j+1} = 1 + O(1/k)$. Near the central knot we only use the boundedness of μ_j . The same reasoning works as well in the case of functions $f(x) = x^3$ and $f(x) = |x|^3$ for $k \leq k_0$, k_0 fixed, because of $f'_{j+1} = 3k^2h^2$ and $f'_{i-1} - f'(x) = O(h^2)$ due to $f''(x) = O(h)$, $0 \leq x \leq x_{n/2+k_0}$. Then we obtain $S'(x) - f'(x) = O(h^2)$. For $k > k_0$, use the expansion

$$\begin{aligned} S'(x) - f'(x) &= m_{i-1} - f'_{i-1} + (x - x_{i-1})(S''_{i-1} - f''_{i-1}) \\ &\quad + \frac{(x - x_{i-1})^2}{2}(S''' - f''')(\xi). \end{aligned} \tag{5.4}$$

Let us show how to establish $S''_{i-1} - f''_{i-1} = O(h)$. From (5.1) follows

$$S''_{i-1} = -\frac{2m_{i-1}}{h} \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} - 1 \right).$$

It was shown that ($i \geq n/2 + k_0$) $\mu_i = m_i/m_{i-1} = 1 + 2/k + O(1/k^2)$, then $(m_{i-1}/m_i)^{1/2} - 1 = -1/k + O(1/k^2)$ (see, e.g., [3]), $f'_{i-1} = 3k^2h^2$ and we get due to $m_{i-1} = f'_{i-1} + O(h^2)$

$$S''_{i-1} = \frac{-2(3k^2h^2 + O(h^2))}{h} \left(-\frac{1}{k} + O\left(\frac{1}{k^2}\right) \right) = 6kh + O(h).$$

Now, as $f''_{i-1} = 6kh$, we get $S''_{i-1} - f''_{i-1} = O(h)$. In addition, (5.1) gives

$$S'''(x) = \frac{6m_{i-1} \frac{1}{h^2} \left(\left(\frac{m_{i-1}}{m_i} \right)^{1/2} - 1 \right)^2}{\left(1 + \frac{x-x_{i-1}}{h} \left(\left(\frac{m_{i-1}}{m_i}\right)^{1/2} - 1\right)^4\right)}$$

and using again $(m_{i-1}/m_i)^{1/2} - 1 = -1/k + O(1/k^2)$, $m_{i-1} = 3k^2h^2 + O(h^2)$, we obtain the uniform boundedness of S''' and from (5.4) $S'(x) - f'(x) = O(h^2)$ for $k > k_0$, too. On the quadratic interval (5.2) gives

$$S'(x) - f'(x) = (1 - t)(m_{i-1} - f'_{i-1}) + t(m_i - f'_i) + (1 - t)f'_{i-1} + tf'_i - f'(x)$$

and from this the required order follows if we take into account the order of linear interpolation on dependence on the smoothness of f .

It is quite clear how all presented reasonings could be generalized to the case considered at the beginning of Section 4. Let us indicate some important moments. If in the point c_i the function f' do not change the sign, we should argue as in the case of $f(x) = x^2 \operatorname{sgn} x$ or $f(x) = x^3$. Then in (4.3) $\delta_{j+1}/\delta_j = 1 + l/k + O(1/k^2)$ with some constant l which can be fractional. Such a fractionality of constants appears also in other formulae but all presented steps of the proofs are valid. If f' changes the sign in the point c_i , we follow the proof of the cases of $f(x) = x^2$ or $f(x) = |x|^3$ and again the fractionality phenomena should be taken into account. We have proved the following

Theorem. *Suppose that a function f has a finite number of points c_i in $[a, b]$ such that $f'(c_i) = 0$ and*

$$\lim_{\substack{x \rightarrow c_i \\ x < c_i}} \frac{f'(x)}{|x - c_i|^{\alpha_{i1}}} = \gamma_{i1} \neq 0, \quad \lim_{\substack{x \rightarrow c_i \\ x > c_i}} \frac{f'(x)}{|x - c_i|^{\alpha_{i2}}} = \gamma_{i2} \neq 0.$$

Let $\alpha = \min_i \min\{\alpha_{i1}, \alpha_{i2}, 2\}$. We also assume that $f' \in \operatorname{Lip} \alpha$ if $0 < \alpha \leq 1$ or $f'' \in \operatorname{Lip}(\alpha - 1)$ if $1 < \alpha \leq 2$. Then the combined histopolating spline S which is constructed by comonotone or modified comonotone strategy, has in the uniform norm on $[a, b]$ the convergence rate $\|S - f\|_\infty = O(h^{1+\alpha})$ together with $\|S' - f'\|_\infty = O(h^\alpha)$.

Remark. We formulated and proved the results in the case of uniform mesh. It is quite evident that our arguments work also in the case of a mesh with $0 < q_1 \leq h_j/h_i \leq q_2$ for $|i - j| = 1$ where q_1, q_2 are constants. The reason here is that far enough from the points c_i the results hold by [3], but in finite number of intervals around the points c_i such nonuniformity can change only the constants, not the order of all given estimates. However, around the points c_i where the theoretical rate of convergence is lower than in regions with strict uniform monotonicity, we can use the mesh with smaller step as the idea of adaptive meshes. This compensates such a lower rate.

6 Numerical Tests

We histopolated the functions $f(x) = x^2 \operatorname{sgn} x$, $f(x) = x^3$, $f(x) = x^2$ and $f(x) = |x|^3$ on the interval $[-1, 1]$. For the first and third functions we obtain convergence rate $O(h^2)$ and for others the rate $O(h^3)$. In our tests we used the uniform mesh, for the third function we used also a nonuniform mesh, where central knots were calculated as $x_{(n+1)/2} = h/2$, $x_{(n+1)/2-1} = -h/2$, $x_{(n+1)/2-2} = -3h/2$, $x_{(n+1)/2+1} = 3h/5$. Other knots were spaced uniformly on rest parts of the interval, i.e., $x_i = x_0 + i(x_{(n+1)/2-2} - x_0)/((n+1)/2 - 2)$, $i = 1, \dots, (n+1)/2 - 3$ and $x_{(n+1)/2+1+i} = x_n - (n-1)(x_n - x_{(n+1)/2+1})/((n+1)/2 - 2)$, $i = 1, \dots, n - 1$. For the function $f(x) = x^2$ we made test for both, n odd and n even. Other functions were tested only for n odd or only for n even and on uniform mesh. Selection of the subinterval was made by comonotone shape-preserving strategy in case of n odd and by modified comonotone shape-preserving strategy in case of n even.

Table 1. Numerical results for $f(x) = x^2$, $x \in [-1, 1]$, uniform mesh, n odd.

n	5	15	45	135	405
ε_n	7.34×10^{-3}	8.39×10^{-4}	9.33×10^{-5}	1.04×10^{-5}	1.15×10^{-6}
$\varepsilon_n/\varepsilon_{n/3}$		8.741958	9.000082	9.000000	9.000000

Table 2. Numerical results for $f(x) = x^2$, $x \in [-1, 1]$, uniform mesh, n even.

n	8	16	32	64	128
ε_n	7.88×10^{-3}	1.97×10^{-3}	4.93×10^{-4}	1.23×10^{-4}	3.08×10^{-5}
$\varepsilon_n/\varepsilon_{n/2}$		3.998068	3.999997	4.000000	4.000000

Table 3. Numerical results for $f(x) = x^2$, $x \in [-1, 1]$, nonuniform mesh, n odd.

n	5	15	45	135	405
ε_n	3.95×10^{-2}	9.44×10^{-4}	1.05×10^{-4}	1.16×10^{-5}	1.29×10^{-6}
$\varepsilon_n/\varepsilon_{n/3}$		41.790849	9.012392	9.003344	9.001046

Table 4. Numerical results for $f(x) = x^3$, $x \in [-1, 1]$, uniform mesh, n even.

n	8	16	32	64	128
ε_n	3.70×10^{-3}	5.23×10^{-4}	6.81×10^{-5}	8.66×10^{-6}	1.09×10^{-6}
$\varepsilon_n/\varepsilon_{n/2}$		7.086933	7.681549	7.856177	7.931600

Table 5. Numerical results for $f(x) = |x^3|$, $x \in [-1, 1]$, uniform mesh, n odd.

n	5	15	45	135	405
ε_n	1.27×10^{-2}	6.32×10^{-4}	2.47×10^{-5}	9.31×10^{-7}	3.47×10^{-8}
$\varepsilon_n/\varepsilon_{n/3}$		20.130665	25.566869	26.557522	26.858469

Table 6. Numerical results for $f(x) = x^2 \operatorname{sgn} x$, $x \in [-1, 1]$, uniform mesh, n even.

n	8	16	32	64	128
ε_n	4.39×10^{-3}	1.10×10^{-3}	2.75×10^{-4}	6.86×10^{-5}	1.72×10^{-5}
$\varepsilon_n/\varepsilon_{n/2}$		3.998591	3.999998	4.000000	4.000000

We used the boundary conditions $\alpha = f'(x_0)$ and $\beta = f'(x_n)$. The approximates to the errors $\|S - f\|_\infty$ were calculated on ten times refined grid as

$$\varepsilon_n = \max_{1 \leq i \leq n} \max_{1 \leq k \leq 10} \left| S\left(x_{i-1} + \frac{kh}{10}\right) - f\left(x_{i-1} + \frac{kh}{10}\right) \right|.$$

Nonlinear system of m_i , $i = 0, \dots, n$, was solved by Newton's method. Results of numerical tests are presented in Tables 1–6.

Numerical results are completely in concordance with theoretical ones.

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