SOLUTION OF TWO-DIMENSIONAL MULTI-INTERFACE STEFAN PROBLEM BY THE METHOD OF DYNAMIC ADAPTATION

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ABSTRACT

In the present work a method of numerical solution of multi-interface two-dimensional Stefan problem with explicit tracking of the interfaces in the domains of arbitrary form is considered. The method is based on the idea of dynamic adaptation of the calculated grid by means of transition to an arbitrary non-stationary coordinate system. The coordinate system transformation is controlled by the solution. The method is described by using the example of the problem that is typical for treatment of materials with concentrated energy fluxes.

INTRODUCTION

The interest to the Stefan problems is caused by their important physical and technological applications dealing with at the influence of concentrated energy fluxes on metals and ceramics [1]. Principal complexity of mathematical studies of Stefan-like problems is due to presence of moving boundaries that leads to essential nonlinearity. Analytic solution of such problems can be obtained only under strong simplifying assumptions. There are two widely used approaches to the numerical solution: explicit tracking of moving surface and using of smoothing procedures. For problems of pulse action of high energy fluxes on materials the non-equilibrium of fast phase transformation can play

a dominant role. In this case it is necessary to locate explicitly the phase boundaries and to take into account the related processes.

In the present work the application of a dynamic adaptation method [2] for the solution of a multi-interface Stefan problem in arbitrary two-dimensional areas with explicit tracking of the interfaces is considered. This method is widely used for solution of one-dimensional problems of mathematical physics [3]. The dynamic adaptation method for solution of two-dimensional problems is based on the idea of transition to non-stationary curvilinear coordinate system. In this coordinate system the problem is described by an extended system of the differential equations, one part of which describes the physical phenomenon, and the second one deals with the movement of computational mesh nodes.

1. MATHEMATICAL MODEL

The mathematical formulation of the classical version of the two-dimensional Stefan problem, describing the melting and crystallization processes, is reduced to quasi-linear heat transfer equation in an arbitrary region Ω_{xy} with a priori unknown moving boundary $\Gamma_{sl}(t)$ which separates solid $\Omega_s(t)$ and liquid $\Omega_l(t)$ phases:

$$\left[\frac{\partial H}{\partial t} = -\frac{\partial W_1}{\partial x} - \frac{\partial W_2}{\partial y} + g\right]_m, \ m = s, l, \tag{1.1}$$

$$H_m = c_p\rho_m T, \quad \left(W_1\right)_m = -\lambda_m(T)\frac{\partial T}{\partial x}, \quad \left(W_2\right)_m = -\lambda_m(T)\frac{\partial T}{\partial y}.$$

On the $\Gamma_{sl}(t)$ the differential Stefan condition is fulfilled

$$W_l^n - W_s^n = L_m\rho v_{sl}^n, \quad W_s^\tau = W_l^\tau \tag{1.2}$$

and the temperature is continuous and equal to the equilibrium transition temperature

$$T_s = T_l = T_m. \tag{1.3}$$

Here n and τ denote normal and tangent components, the s and l refer to solid and liquid phases, T_m, L_m are the temperature and latent heat of melting/crystallization, v_{sl} is the velocity of motion of the interface. On the boundary $\partial\Omega_{xy}$ the boundary conditions are specified

$$(\vec{W}, \vec{n})|_{\partial\Omega_{xy}} = f,$$

where $\vec{W} = (W_1, W_2)$ is the vector of heat flow, $\vec{n}$ is the external normal to $\partial\Omega_{xy}$, f is a given function.

The account of evaporation results to appearance of the second mobile interface $\Gamma_{lv}(t)$ in area Ω_{xy}. The process of the advanced surface evaporation on this boundary is described by three conservation laws (mass, momentum and energy)

$$\rho_l v_{lv}^n = \rho_v(u - v_{lv}^n), \tag{1.4}$$

$$P_l + \rho_l(v_{lv}^n)^2 = P_v + \rho_v(u - v_{lv}^n)^2, \tag{1.5}$$

$$-\lambda\frac{\partial T}{\partial n} = G^n - L_v\rho_l v_{lv}^n \tag{1.6}$$

and two additional relations describing the kinetics of phase transformations, they are determined from the Knudsen layer approximation [4]

$$T_v = T_v(T_l, M), \quad \rho_v = \rho_v(\rho_H, M).$$

Here the index v denotes vapor, G is energy source intensity, u is hydrodynamical velocity of vapor, P is a pressure, M is the Mach number and ρ_H denotes the saturated vapor density.

2. PROBLEM STATEMENT IN AN ARBITRARY CURVILINEAR UNSTEADY COORDINATE SYSTEM

The solution of this problem consists of determination of temperature fields and position of phase fronts $\Gamma_{sl}(t)$, $\Gamma_{lv}(t)$. To map the physical space with coordinates (x, y, t) into computational one with (ξ, η, τ) we shall apply general transformation $\xi = \xi(x, y, t), \eta = \eta(x, y, t), \tau = t$. The differential problem $(1.1) - (1.6)$ in the arbitrary non-stationary curvilinear coordinate system (ξ, η, τ) can be written in the form:

$$\left[\frac{\partial}{\partial\tau}(\psi H) = -\frac{\partial}{\partial\xi}\left[(\rho W_1 + HQ_1)\frac{\partial y}{\partial\eta} - (\rho W_2 + HQ_2)\frac{\partial x}{\partial\eta}\right] - \right. \tag{2.1}$$

$$\left. -\frac{\partial}{\partial\eta}\left[-(\rho W_1 + HQ_1)\frac{\partial y}{\partial\xi} + (\rho W_2 + HQ_2)\frac{\partial x}{\partial\xi}\right] + \psi g\right]_m,$$

$$\left[\frac{\partial x}{\partial\tau} = -\frac{Q_1}{\rho}\right]_m, \quad \left[\frac{\partial y}{\partial\tau} = -\frac{Q_2}{\rho}\right]_m, \quad m = s, l \tag{2.2}$$

with the corresponding boundary conditions on the lines of phase transitions

$(\xi, \eta = \eta_{sl}) \in \Gamma_{sl}$:

$$\left[\left(-W_1 \frac{\partial y}{\partial \xi} + W_2 \frac{\partial x}{\partial \xi}\right)_l - \left(-W_1 \frac{\partial y}{\partial \xi} + W_2 \frac{\partial x}{\partial \xi}\right)_s\right]\gamma^{\frac{1}{2}} = -L_m Q_{sl}^n, \qquad (2.3)$$

$(\xi, \eta) \in \Gamma_{lv}$:

$$Q_{lv}^n = -\rho_v\left(u + \frac{Q_{lv}^n}{\rho_l}\right), \qquad (2.4)$$

$$P_l + \frac{(Q_{lv}^n)^2}{\rho_l} = P_v + \rho_v\left(u + \frac{Q_{lv}^n}{\rho_l}\right)^2, \qquad (2.5)$$

$$\left(-W_1 \frac{\partial y}{\partial \xi} + W_2 \frac{\partial x}{\partial \xi}\right)_l \gamma^{\frac{1}{2}} = G^n + L_v Q_{lv}^n, \quad \eta = const, \qquad (2.6)$$

$$\left(W_1 \frac{\partial y}{\partial \eta} - W_2 \frac{\partial x}{\partial \eta}\right)_l \alpha^{\frac{1}{2}} = G^n + L_v Q_{lv}^n, \quad \xi = const, \qquad (2.7)$$

where

$$W_1 = -\frac{\lambda\rho}{\psi}\left(\frac{\partial y}{\partial \eta}\frac{\partial T}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial T}{\partial \eta}\right), \quad W_2 = -\frac{\lambda\rho}{\psi}\left(-\frac{\partial x}{\partial \eta}\frac{\partial T}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial T}{\partial \eta}\right),$$

$$\psi = \rho J^{-1} = \rho\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right), \quad \alpha = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2, \quad \gamma = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2.$$

Here Q_{sl} and Q_{lv} are the flows of substance through the interfaces Γ_{sl} and Γ_{lv} respectively, Q_1, Q_2 are the arbitrary transformation functions and J^{-1} is the Jacobian of inverse transformation.

3. ALGORITHM OF SOLUTION

For the finite difference approximation of the problem $(2.1) - (2.7)$ in the domain $\Omega_{\xi\eta} \times [0, t_0]$;

$$\Omega_{\xi\eta} = \{(\xi, \eta) : 0 \le \xi \le 1, \quad 0 \le \eta \le 1\}$$

we shall introduce the rectangular grid ω with the steps $h_\xi, h_\eta, \Delta\tau^j$ respectively.

The functions $x_{i,k}^j, y_{i,k}^j, Q_{1,i,k}^j, Q_{2,i,k}^j$ are determined in the grid nodes, while the functions $T_{i+1/2,k+1/2}^j, \psi_{i+1/2,k+1/2}^j, H_{1,i+1/2,k+1/2}^j$ are evaluated at the

cell centers. Variables $W^j_{1,i+1/2,k}$, $W^j_{2,i+1/2,k}$, $W^j_{1,i,k+1/2}$, $W^j_{2,i,k+1/2}$ are evaluated at the centers of cell edges. Using the integro-interpolational method [5] the initial differential problem is approximated by means of the implicit difference scheme

$$
\frac{(\psi H)^{j+1}_{i+1/2,k+1/2} - (\psi H)^{j}_{i+1/2,k+1/2}}{\Delta \tau^j} = -\frac{1}{h_\xi h_\eta} \Big\{ (\rho W_1 + HQ_1)_{i+1,k+1/2}
$$

$$
\times (y_{i+1,k+1} - y_{i+1,k}) - (\rho W_1 + HQ_1)_{i,k+1/2} (y_{i,k+1} - y_{i,k})
$$

$$
- (\rho W_1 + HQ_1)_{i+1/2,k+1} (y_{i+1,k+1} - y_{i,k+1}) + (\rho W_1 + HQ_1)_{i+1/2,k}
$$

$$
\times (y_{i+1,k} - y_{i,k}) - (\rho W_2 + HQ_2)_{i+1,k+1/2} (x_{i+1,k+1} - x_{i+1,k})
$$

$$
- (\rho W_2 + HQ_2)_{i,k+1/2} (x_{i,k+1} - x_{i,k}) - (\rho W_2 + HQ_2)_{i+1/2,k+1}
$$

$$
\times (x_{i+1,k+1} - x_{i,k+1}) + (\rho W_2 + HQ_2)_{i+1/2,k} (x_{i+1,k} - x_{i,k}) \Big\}^{j+1}
$$

$$
- (\psi g)^{j+1}_{i+1/2,k+1/2}.
$$

Here

$$
W_{1,i+1/2,k} = -\frac{\lambda \rho}{\psi_{i+1/2,k} h_\xi h_\eta} \Big\{ \big(y_{i+1/2,k+1/2} - y_{i+1/2,k-1/2} \big) (T_{i+1,k} - T_{i,k})
$$

$$
- (y_{i+1,k} - y_{i,k}) \big(T_{i+1/2,k+1/2} - T_{i+1/2,k-1/2} \big) \Big\},
$$

$$
W_{1,i,k+1/2} = -\frac{\lambda \rho}{\psi_{i,k+1/2} h_\xi h_\eta} \Big\{ (y_{i,k+1} - y_{i,k}) \big(T_{i+1/2,k+1/2} - T_{i-1/2,k+1/2} \big)
$$

$$
- \big(y_{i+1/2,k+1/2} - y_{i-1/2,k+1/2} \big) (T_{i,k+1} - T_{i,k}) \Big\},
$$

$$
W_{2,i+1/2,k} = -\frac{\lambda \rho}{\psi_{i+1/2,k} h_\xi h_\eta} \Big\{ - \big(x_{i+1/2,k+1/2} - x_{i+1/2,k-1/2} \big) (T_{i+1,k} - T_{i,k})
$$

$$
+ (x_{i+1,k} - x_{i,k}) \big(T_{i+1/2,k+1/2} - T_{i+1/2,k-1/2} \big) \Big\},
$$

$$
W_{2,i,k+1/2} = -\frac{\lambda \rho}{\psi_{i,k+1/2} h_\xi h_\eta} \Big\{ - (x_{i,k+1} - x_{i,k}) \big(T_{i+1/2,k+1/2} - T_{i-1/2,k+1/2} \big)
$$

$$
+ \big(x_{i+1/2,k+1/2} - x_{i-1/2,k+1/2} \big) (T_{i,k+1} - T_{i,k}) \Big\}.
$$

The required interpolations are performed by means of the following formulas:

$$
H_{i,k+1/2} = 0.5(H_{i+1/2,k+1/2} + H_{i-1/2,k+1/2}),
$$

$$
H_{i+1/2,k} = 0.5(H_{i+1/2,k+1/2} + H_{i+1/2,k-1/2}),
$$

$$Q_{i,k+1/2} = 0.5(Q_{i,k+1} + Q_{i,k}),$$

$$Q_{i+1/2,k} = 0.5(Q_{i+1,k} + Q_{i,k}),$$

$$x_{i+1/2,k+1/2} = 0.25(x_{i,k} + x_{i,k+1} + x_{i+1,k} + x_{i+1,k+1}),$$

$$y_{i+1/2,k+1/2} = 0.25(y_{i,k} + y_{i,k+1} + y_{i+1,k} + y_{i+1,k+1}).$$

The algorithm based on two iteration cycles have been used on the each time step $\Delta\tau^j$ to solve the difference scheme. In the first cycle the $Q_{i,k}$ velocities of grid nodes and their positions have been determined and in the second cycle the distribution of temperatures in the domains Ω_l, Ω_s have been calculated. The obtained systems of the linear algebraic equations have been solved by means of the method refereed in [6]. The value of normal component of the mass flow on the line of phase front Γ_{sl} in the edge centers of cells is determined from the Stefan condition:

$$Q^n_{i+1/2,k} = L_m^{-1}\left[(W^n_{i+1/2,k})_s - (W^n_{i+1/2,k})_l\right], i = 0,\ldots,I-1,\ k = k_{sl}.$$

Here $(\cdot)^n$ is the normal component of corresponding vector with respect to the line of phase front. To determine velocity components for the nodes of the phase fronts Γ_{sl}, Γ_{lv} the mass conservation law have been used. To provide more equidistant nodes distribution on the line of phase front the tangential component of the mass flow $Q^\tau_{i,k}$ also have been determined. Finally, we obtain the following expression for the determination of $\vec{Q}_{i,k} = (Q_{1,i,k}, Q_{2,i,k})$

$$\vec{Q}_{i,k} = Q^n_{i,k} \cdot \vec{n} + Q^\tau_{i,k} \cdot \vec{\tau},$$

where $\vec{n}$ is the unit vector of normal and $\vec{\tau}$ is the unit vector in the tangential direction to the phase front line.

Verification of the dynamic adaptation method have been carried out on the freezing problem referred in [7]. The obtained results [8] testify a high accuracy of this method for solution of the Stefan problem with explicit phase-boundary tracking.

4. NUMERICAL EXPERIMENT

Let us consider application of the suggested dynamic adaptation algorithm to solve the problem of action of high-energy flux on metal target. Rectangular energy pulse of diameter $3.8 \cdot 10^{-4}$m and intensity 10^5W/cm^2 incidents on the surface of the elliptical domain Ω_{xy}. Thermophysical parameters close to ones of lead have been chosen.

The solution algorithm consists of two stages. At the first stage the entire domain is equal to solid sub-domain $\Omega_{xy} = \Omega_s$ and the process is described by the heat transfer equation. The fixed computational grid (19×19 nodes)

with maximum nodes concentration in the energy release zone have been constructed before the beginning of calculations Fig. 1. This grid have been used for temperature field calculation until $T_{max} < T_m$ condition is fulfilled on the irradiated surface.
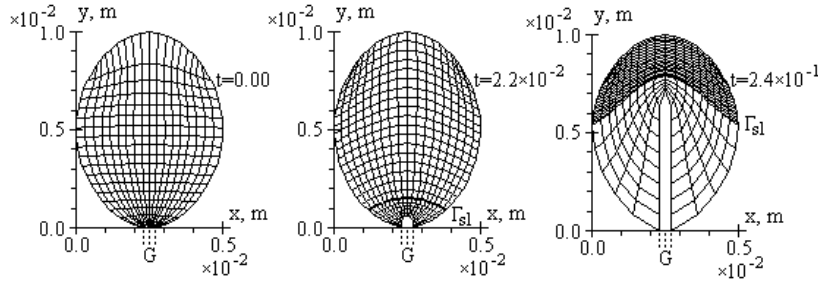
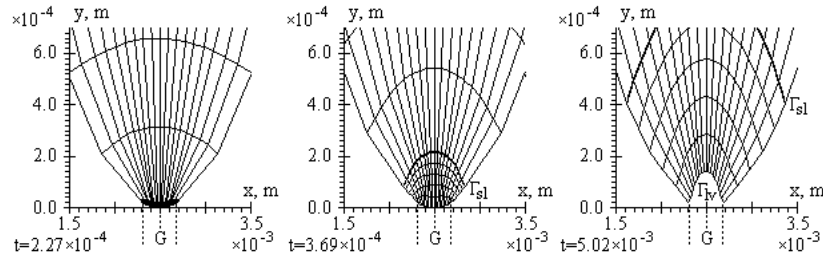

Figure 1. The computational grid in the physical space.

Figure 2. Physical space fragments with introduced liquid phase.

At the second stage, that begins when equilibrium melting temperature T_m is reached on the surface, the processes are described by the two-phase Stefan model. To introduce the new (liquid) phase the overheating of irradiated surface by 0.1 K is supposed. From the relation of overheating energy and latent heat L_m initial thickness of the liquid phase is determined (about 10^{-8} m) and the new subregion Ω_l, $\Omega_{xy} = \Omega_s \cup \Omega_l$ is introduced. A computational grid with 19 × 6 nodes is defined in the new phase (total number of nodes is equal to 19 × 24). From this time moment the grid is reconstructed on each time layer. The numerical solution is accompanied by radical reorganization of a computational grid Fig. 1. Fragments of physical space immediately after the new phase introduction are shown in Fig. 2 in an expanded scale.

The peculiarities of the problem are a high speed of the interfaces movement and an essential deformation of the initial area. The change of maximum temperature in Ω_{xy} and maximum velocities of interface boundaries Γ_{sl} and Γ_{lv} are presented in Fig. 3. The maximum velocity v_{sl} reaches ~18 m/s immediately after the liquid phase introduction and then it decreases to ~0.03

m/s. The distribution of temperature fields in the region Ω_{xy} at different time instants is shown in Fig. 4.
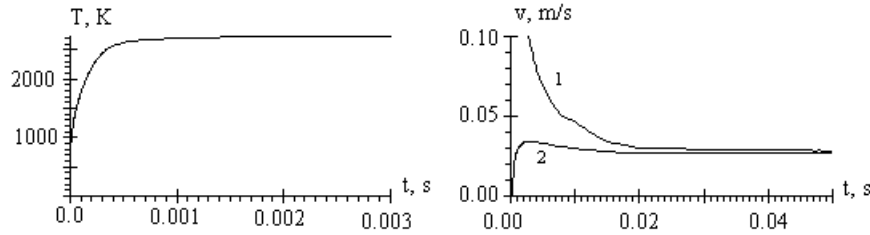

Figure 3. The change of maximum temperature and maximum velocities (1-v_{sl}, 2-v_{lv}).
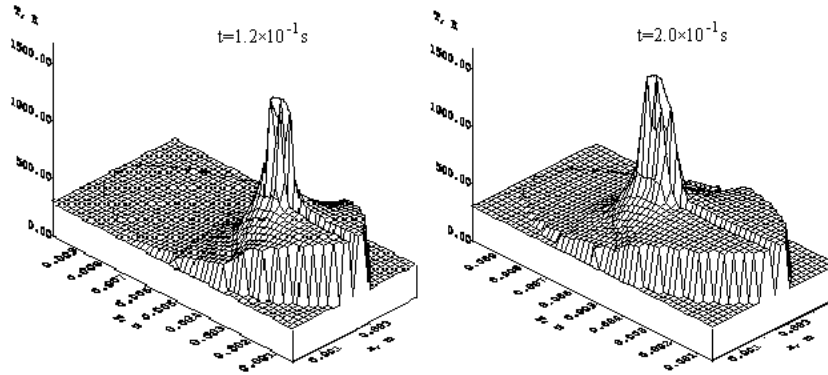

Figure 4. The distribution of temperature fields in the region Ω_{xy} at different time instants.

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DVIMAČIO DAUGIASLUOKSNIO STEFANO UŽDAVINIO SPRENDIMAS DINAMINIO ADAPTAVIMO METODU

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Straipsnyje nagrinėjamas dvimačio daugiasluoksnio Stefano uždavinio adaptyvus sprendimo algoritmas. Naudojamas išreikštinis fronto skaičiavimo algoritmas, leidžiantis rasti bet kokios formos kontūrus. Remiantis dinaminiu adaptavimo algoritmu, fizinė sritis yra atvaizduojama į fiksuotą skaičiavimo sritį. Transformacijos atvaizdavimas adaptyviai priklauso nuo pačio uždavinio sprendinio. Bendrasis metodas pateiktas dviejų medžiagų sąveikos uždaviniui. Naujojo algoritmo efektyvumas iliustruojamas skaičiavimo eksperimento rezultatais.